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EPQ model with scrap and backordering under Vendor managed inventory policy

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Abstract

This paper presents the economic production quantity (EPQ) models for the imperfect quality items produced with/without the presence of shortage condition. The models are presented in a two-level supply chain composed of a single manufacturer and a single buyer to investigate the performance of vendor-managed inventory (VMI) policy. The total costs are minimized to obtain the optimal production lot size and the allowable backorder level before and after applying VMI policy. Numerical examples and sensitivity analysis based on certain parameters are performed to show the capability of the proposed supply chain model enhanced with VMI policy.

Keywords: Economic production quantity (EPQ), Vendor-managed inventory (VMI), Supply chain, Defective items

1. Introduction and literature review

Economic Production Quantity (EPQ) has been applied widely in the manufacturing sector to optimize the production quantity or lot-sizing considering the capacities and limitations. Although, all EPQ models minimize the total costs or maximize the manufacturer’s profit apparently, they are different mainly regarding various assumptions. One of the impractical assumptions of the EPQ models is that all units have perfect quality. However, in practice, most of the manufacturing processes are not free of defect and cause non-conforming items. Generally, these non-conforming items are categorized into two groups: the repairable and the scrap (the latter is discarded).

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Recent developments in information technology have facilitated the advent of new supply chain initiatives such as vendor managed inventory (VMI) policy. VMI is a supply chain initiative where the vendor decides on the appropriate inventory levels of each of the products and the appropriate inventory policies to sustain those levels. It is also known as continuous replenishment or supplier-managed inventory. VMI becomes more critical for the EPQ models with imperfect quality production. In this paper, the EPQ model based on VMI policy is considered for the defective items produced under the presence and absence of the shortage. The models are developed for a two-level supply chain consisting of a single manufacturer and a single buyer and examine the inventory management practices before and after implementing VMI. Therefore, the literature reviews are organized for both EPQ models with scrap and VMI policy.

Many researchers introduced EPQ models with scrap such as Salameh and Jaber (2000). They developed an inventory model which explained imperfect quality items using the EPQ/EOQ formulates. They supposed that the defective items are sold as a single batch at the end of the total screening process. Cardenas-Barron (2000) also revised Salameh and Jaber's (2000) model and showed that the error only affects the optimal order size value. Later, Goyal and Cardenas-Barron (2002) developed Salameh and Jaber's (2000) model and proposed a practical approach to determine EPQ for imperfect quality items. Goyal et al. (2003) studied the model of Goyal and Cardenas-Barron (2002), regarding vendor-buyer integration. Konstantaras et al. (2007) presented a production-inventory model for defective items regarding two conditions. The first one was to sell the items to a secondary shop as a single batch at a lower price comparing to new ones and the second one was to rework them at some cost and return to its original quality. Wee et al. (2007) and Eroglu and Ozdemir (2007) extended the model of Salameh and Jaber (2000) independently regarding the shortages permit. Cardenas-Barron (2008) proposed a simple derivation to find an optimal manufacturing batch size with rework process in a single stage production system. Khan et al. (2011) extended the Salameh and Jaber (2000) EOQ model for imperfect items and presented a detailed review of EOQ/EPQ models with imperfect quality. Recently, Hsu and Hsu (2013) built two optimal production quantity models with imperfect production processes, inspection errors, planned backorders, and sales returns. Taheri-Tolgari et al. (2012) presented a discounted cash-flow approach for an inventory model for imperfect items under inflationary conditions with considering inspection errors and related defect sales return issues. Sarkar and Moon (2011) developed a production inventory model in an imperfect production system considering stochastic demand and the effect of inflation. Pal et al. (2013) derived a mathematical model on EPQ with stochastic demand in an imperfect production system. Cárdenas-Barrón et al. (2012) derived jointly both the optimal replenishment lot size and the optimal number of shipments for the inventory models.

The success of a firm will depend on its ability to integrate with other participants in the chain responsible for physical, financial and information flows. Organizations are striving to achieve joint total effectiveness across the entire chain. This has given rise to a need for various coordination mechanisms. Vendor Managed Inventory (VMI) is one of the practices which has gained a lot of attention in recent times. Many companies such as HP, Shell and Walmart have reportedly adopted VMI. Researchers are actively engaged in studying issues related to VMI, including, but not limited to, replenishment decisions, contracts, relationships as well as the strategic implications of such a mechanism (Marques et al., 2010). The costs of VMI policy with the traditional one are highly compared in the literature (Waller et al. (1999), Cetinkaya and Lee (2000)). Moreover, having a deterministic demand is one of the usual assumptions (Valentini and Zavanella (2003), Shah and Goh (2006), Woo et al. (2001)). They considered a supply chain consisting of a single manufacturer and multiple buyers or retailers. Kleywegt et al. (2002) presented an inventory routing problem of a manufacturer who owns the inventory at the retailers. An approximation method is used to find the minimum-cost routing policy. Later Zhang et al. (2007) extended Woo et al.'s model by omitting the common replenishment cycle assumption and having different ordering cycles for the retailers. A number of papers analyzed the VMI policy in a supply chain as a centralized or a coordination channel. For instance, Bernstein et al. (2006) studied the constant wholesale price and quantity discount contracts. They showed the role of VMI in a complete coordination in a supply chain with multiple competing retailers. It is shown that the performance of the overall system is improved in VMI regarding certain conditions (Nagarajan and Rajagopalan (2008)). Pasandideh et al. (2010) considered the retailer-supplier partnership through a VMI. In addition, they developed an economic order quantity (EOQ) with shortage to explore the effects of important supply chain parameters on the cost savings realized from VMI. Ma et al. (2013) developed a VMI model for a supply chain with one vendor and one retailer including a VMI warehouse near the retailer and with a freshness clause. They showed that the freshness clause had a significant impact on the decisions of both vendor and retailer. Hariga et al. (2013a&b) analyzed a VMI system with different replenishment cycles for different retailers. Kannan et al. (2013) studied the benefits of a VMI arrangement when the same organization owns the vendor and the retailers. Some of these studies, like Darwish and Odah (2010) and Kannan (2013), also showed that adopting VMI leads to system wide cost reductions.

Regarding the effectiveness of VMI, we studied the integration of VMI policy in EPQ models producing defective items in this paper. To the best our knowledge, it has been ignored in the literature. Therefore, in this paper, we present the EPQ model to investigate the performance of VMI in a two-level supply chain composed of a manufacturer and a buyer. The manufacturer produces defective items under the presence and absence of the shortage costs. The production process generates randomly imperfect quality items at a constant production rate. The manufacturer and the buyer are faced with the setup cost, production cost, screening cost, disposal cost per scrap item, holding cost and ordering cost without shortage. The optimal production lot size is obtained by minimizing the total expected inventory costs before and after implementing VMI. The backordering cost per item per time unit is also considered regarding shortage permission in addition to the costs mentioned. The total expected inventory costs are minimized to obtain the optimal maximum allowable backorder level and production lot size before and after implementing VMI. The numerical example is presented in this paper, including sensitivity analysis of certain parameters. It shows the capability of supply chain regarding implementing VMI with and without shortage. Although a simple supply chain is used for computational ease, the obtained results can be generalized to more complex supply chains.

The rest of this paper is organized as follows. Section 2 reviews the notation and problem formulation. Section 3 presents the model formulations. In Section 4, the numerical example and the sensitivity analysis are presented to analyze the effect of different parameters on the optimal economic production quantity and the total cost before and after implementing VMI. The final section is dedicated to conclusions and recommendations for future studies.

2. Notation and problem formulation

This section introduces the notation and formulation used in the models. Specifically, all decision variables, input parameters and assumptions underlying the models will be stated.

2.1. Input parameters

The following notations are used in the paper:

Q:	The production lot size
Q_{VMI}:	The production lot size in VMI
B:	The maximum allowable backorder level
B_{VMI}:	The maximum allowable backorder level in VMI
P:	The production rate
λ:	The demand rate
x:	The defective rate, a random variable with known probability density function
d:	The production rate of defective items (during the regular production process), in units per time unit
H:	The maximum level of on-hand inventory in units, when the rework process ends
b:	The backordering cost per item per time unit
K_m:	The manufacturer setup cost
K_b:	The buyer ordering cost
C:	The production cost (screening cost for each item is included)
C_r:	The repair cost of each imperfect quality item reworked (screening cost for each rework item is also included)
C_s:	The disposal cost per scrap item,
h:	The holding cost
TC(Q):	The total cost per time unit
$E(TCU_{m_{0i}})$:	The manufacturer's expected total cost before VMI in case i (i=1,2)
$E(TCU_{b_{0i}})$:	The buyer's expected total cost before VMI in case i
$E(TCU_{m_{1i}})$:	The manufacturer's expected total cost in VMI in case i
$E(TCU_{b_{1i}})$:	The buyer's expected total cost in VMI in case i
$E(TCU_{0i})$:	The expected total cost before VMI in case i
$E(TCU_{1i})$:	The expected total cost of VMI in case i

2.2. Assumptions

The mathematical models are developed based on the following assumptions.

(a) A single-manufacturer-single-buyer supply chain with defective items is considered.
(b) The production rate P is a constant and is greater than the demand rate λ.
(c) The EPQ model is imperfect and production process generates randomly x percent of imperfect quality items at production rate d.
(d) The defective production rate d is expressed by d=Px.

3. Model formulations

The models are presented in two cases; 1: shortage is not allowed, 2: shortage is allowed and is completely backordered.

3.1. Case 1. Shortage is not permitted

The model with the random defective rate and an imperfect rework process without shortage is depicted in Figure 1. According to assumption b, P is greater than or equal to the sum of the demand rate λ and also $P-d-\lambda \geq 0$ or $0 \leq x \leq \left(1-\frac{\lambda}{P}\right)$. Therefore, based on Figure (1), the production cycle length (T) is

$$T = t_1 + t_2 = \frac{Q(1-x)}{\lambda} \tag{1}$$

Since $t_1 = \frac{Q}{P} = \frac{H}{P-d-\lambda}$

$$H = (P-d-\lambda)t_1 = Q\,(1-x-\frac{\lambda}{P}) \tag{2}$$

The total defective items produced during the regular production uptime "t_1" are:

$$dt_1 = xQ \tag{3}$$

The production downtime time "t_2" needed for consuming the maximum on-hand inventory H is computed in Eq. (4).

$$t_2 = \frac{H}{\lambda} = Q\left(\frac{1}{\lambda} - \frac{x}{\lambda} - \frac{1}{P}\right) \tag{4}$$

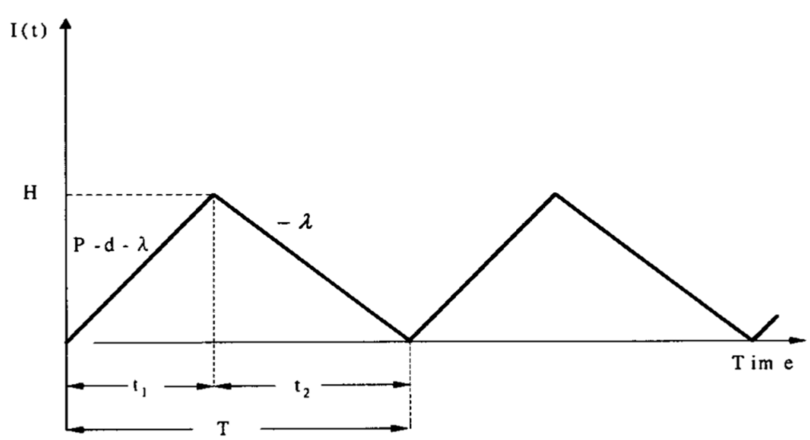

Figure1.The EPQ model with the random defective rate without shortage

3.1.1. Supply chain without VMI policy

Regardless of implementing VMI, the inventory cost of the manufacturer, the buyer and the total inventory cost of the supply chain are as follows:

$$TC_{m_{01}}(Q)=CQ+C_S(xQ)+K_m+h\left[\frac{H}{2}(t_1+t_2)\right]+h\frac{xQ}{2}t_1 \tag{5}$$

$$TC_{b_{01}}=K_b \tag{6}$$

$$TC(Q)=CQ+C_S(xQ)+K_m+h\left[\frac{H}{2}(t_1+t_2)\right]+h\frac{xQ}{2}t_1+K_b \tag{7}$$

The proportion of defective items x is considered to be a random variable with a known probability density function. Thus, the expected value of x is used in the inventory cost analysis. Since the production cycle length is not a constant, by using the renewal theorem approach (Zipkin, 2000) we have;

$$E[TCU(Q)]=\frac{E[TC(Q)]}{E[T]} \tag{8}$$

Where,

$$E[T]=\frac{Q}{\lambda}(1-E[x]) \tag{9}$$

Then,

$$E[TCU_{m_{01}}(Q)]=\lambda\left[\frac{C}{1-E[x]}+\frac{C_S E[x]}{1-E[x]}+\frac{K_m}{Q}\frac{1}{1-E[x]}\right]+\frac{h}{2}\left(1-\frac{\lambda}{P}\right)\frac{1}{1-E[x]}$$
$$-hQ\left(1-\frac{\lambda}{P}\right)\frac{E[x]}{1-E[x]}+\frac{hQ}{2}\frac{E[x^2]}{1-E[x]} \tag{10}$$

And also,

$$E[TCU_{b_{01}}(Q)]=\frac{K_b\lambda}{Q}\frac{1}{1-E[x]} \tag{11}$$

If we let $E_0=\frac{1}{1-E[x]}$, $E_1=\frac{E[x]}{1-E[x]}$ and $E_2=\frac{E[x^2]}{1-E[x]}$ for some simplifications we have:

$$E[TCU_{m_{01}}(Q)]=\lambda[C.E_0+C_S.E_1]+\frac{K_m\lambda}{Q}.E_0$$
$$+\frac{hQ}{2}\left(1-\frac{\lambda}{P}\right)E_0-h.Q.\left(1-\frac{\lambda}{P}\right)E_1+\frac{hQ}{2}E_2 \tag{12}$$

$$E[TCU_{b_{01}}(Q)]=\frac{K_b\lambda}{Q}.E_0 \tag{13}$$

$$E\left[TCU_{01}\left(Q\right)\right]=\lambda\left[CE_0+C_S E_1\right]+\frac{(K_m+K_b)\lambda}{Q}E_0$$
$$+\frac{hQ}{2}\left(1-\frac{\lambda}{P}\right)E_0-hQ\left(1-\frac{\lambda}{P}\right)E_1+\frac{hQ}{2}E_2 \tag{14}$$

Since the second derivative of $E\left[TCU_{01}\left(Q\right)\right]$ respect to Q is positive, $\frac{2K_m\lambda}{Q^3}E_0>0$, the total objective cost is convex. Then the optimal production quantity is obtained by setting the first derivative of $E\left[TCU_{m_{01}}\left(Q\right)\right]$ respect to Q equal to zero yielding;

$$Q^*=\sqrt{\frac{2K_m\lambda}{h\left(1-\frac{\lambda}{P}\right)-2h\left(1-\frac{\lambda}{P}\right)E\left[x\right]+h.E\left[x^2\right]}} \tag{15}$$

3.1.2. Supply chain with VMI policy

After implementing VMI, the total cost of the buyer, the manufacturer and the chain are calculated as below;

$$TC_{m_{11}}\left(Q_{VMI}\right)=CQ_{VMI}+C_S\left(xQ_{VMI}\right)+K_m+h\left[\frac{H}{2}\left(t_1+t_2\right)\right]+h\frac{xQ_{VMI}}{2}t_1+K_b \tag{16}$$

$$TC_{b_{11}}=0 \tag{17}$$

$$TC_{VMI}\left(Q_{VMI}\right)=TC_{m_{11}}\left(Q_{VMI}\right)+TC_{b_{11}}\left(Q_{VMI}\right)$$
$$=CQ_{VMI}+C_S\left(xQ_{VMI}\right)+K_m+h\left[\frac{H}{2}\left(t_1+t_2\right)\right]+h\frac{xQ_{VMI}}{2}t_1+K_b \tag{18}$$

Using E_0, E_1, E_2 defined in the previous section, we have,

$$E\left[TCU_{m_{11}}\left(Q_{VMI}\right)\right]=\lambda\left[CE_0+C_S E_1\right]+\frac{(K_m+K_b)\lambda}{Q_{VMI}}E_0+\frac{hQ_{VMI}}{2}\left(1-\frac{\lambda}{P}\right)E_0$$
$$-hQ_{VMI}\left(1-\frac{\lambda}{P}\right)E_1+\frac{hQ_{VMI}}{2}E_2 \tag{19}$$

$$E\left[TCU_{b_{11}}\right]=0 \tag{20}$$

$$E\left[TCU_{11}\left(Q_{VMI}\right)\right]=E\left[TCU_{m_{11}}\left(Q_{VMI}\right)\right]+E\left[TCU_{b_{11}}\left(Q_{VMI}\right)\right]$$
$$=\lambda\left[CE_0+C_S E_1\right]+\frac{(K_m+K_b)\lambda}{Q_{VMI}}E_0+\frac{hQ_{VMI}}{2}\left(1-\frac{\lambda}{P}\right)E_0$$
$$-hQ_{VMI}\left(1-\frac{\lambda}{P}\right)E_1+\frac{hQ_{VMI}}{2}E_2 \tag{21}$$

Since the second derivative of $E\left[TCU_{11}(Q_{VMI})\right]$ respect to Q_{VMI} is positive, $\frac{2(K_m+K_b)\lambda}{Q_{VMI}^3}E_0>0$, the total objective cost is convex. Then the optimal production quantity is obtained by setting the first derivative of $E\left[TCU_{m_{01}}(Q_{VMI})\right]$ respect to Q_{VMI} equal to zero yielding;

$$Q_{VMI}^{*}=\sqrt{\frac{2(K_m+K_b)\lambda}{h\left(1-\frac{\lambda}{P}\right)-2h\left(1-\frac{\lambda}{P}\right)E[x]+h.E\left[x^2\right]}} \tag{22}$$

3.2. Case 2 .Shortage is permitted

The inventory control in this case is shown in Figure 2. According to the definitions in case 1, we have:

$$t_1=\frac{H}{P-d-\lambda} \tag{23}$$

$$t_2=\frac{H}{\lambda} \tag{24}$$

$$t_3=\frac{B}{\lambda} \tag{25}$$

$$t_4=\frac{B}{P-d-\lambda} \tag{26}$$

$$H=(P-d-\lambda)\frac{Q}{P}-B \tag{27}$$

$$t_1+t_4=\frac{Q}{P} \tag{28}$$

The cycle length T is:

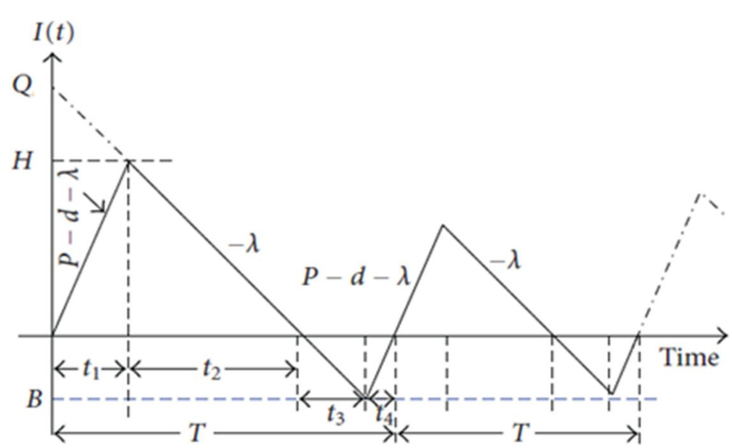

Figure2. The EPQ model with the random defective rate under presence of the shortage

$$T=\sum_{i=1}^{4}t_i=\frac{(1-x)Q}{\lambda} \tag{29}$$

3.2.1. Supply chain without VMI policy

In this case, the inventory cost of the manufacturer, the buyer and the total inventory costs of the chain are as below;

$$TC_{m_{02}}(Q,B)=CQ+C_S.x.Q+K_m+h\left[\frac{H}{2}(t_1+t_2)\right]+b\,\frac{B}{2}(t_3+t_4)+h\left[\frac{d\,(t_1+t_4)}{2}(t_1+t_4)\right] \tag{30}$$

$$TC_{b_{02}}=K_b \tag{31}$$

$$TC_{noVMI}(Q,B)=(C+C_S x)Q+K_m+h\left[\frac{H}{2}(t_1+t_2)+\frac{d\,(t_1+t_4)}{2}(t_1+t_4)\right]+\frac{bB}{2}(t_3+t_4)+K_b \tag{32}$$

Similar to the first case, by using the renewal theorem approach (Zipkin, 2000), we have:

$$\begin{aligned}E\left[TCU_{m_{02}}(Q,B)\right]&=\lambda\left(\frac{C+C_S E[x]}{1-E[x]}\right)+\frac{1}{Q}\frac{K_m\lambda}{1-E[x]}+\frac{h}{2}\left[\left(1-\frac{\lambda}{P}\right)Q-2B\right]\frac{1}{1-E[x]}\\&+\frac{B^2}{2Q}\frac{(b+h)}{1-E[x]}E\left(\frac{1-x}{1-x-\frac{\lambda}{P}}\right)+h\left[B-\left(1-\frac{\lambda}{P}\right)Q\right]\frac{E[x]}{1-E[x]}+\frac{hQ}{2}\frac{E[x^2]}{1-E[x]}\end{aligned} \tag{33}$$

And also:

$$E\left[TCU_{b_{02}}(Q)\right]=\frac{K_b\lambda}{Q}.\frac{1}{1-E[x]} \tag{34}$$

By considering

$$E_0=\frac{1}{1-E[x]},\ E_1=\frac{E[x]}{1-E[x]},E_2=\frac{E[x^2]}{1-E[x]}$$

and

$E_3=E\left(\dfrac{1-x}{1-x-\frac{\lambda}{P}}\right)\dfrac{1}{1-E[x]}$ for some simplifications, we have;

$$E\left[TCU_{m_{02}}(Q,B)\right]=\lambda\left(CE_0+C_SE_1\right)+\frac{K_m\lambda}{Q}E_0+\frac{h}{2}\left[\left(1-\frac{\lambda}{P}\right)Q-2B\right]E_0 \\ +\frac{B^2}{2Q}(b+h)E_3+h\left[B-\left(1-\frac{\lambda}{P}\right)Q\right]E_1+\frac{hQ}{2}E_2 \tag{35}$$

$$E\left[TCU_{b_{02}}(Q)\right]=\frac{K_b\lambda}{Q}E_0 \tag{36}$$

Therefore, the total expected inventory costs are,

$$E\left[TCU_{02}(Q,B)\right]=\lambda\left(CE_0+C_SE_1\right)+\frac{(K_m+K_b)\lambda}{Q}E_0+\frac{h}{2}\left[\left(1-\frac{\lambda}{P}\right)Q-2B\right]E_0 \\ +\frac{B^2}{2Q}(b+h)E_3+h\left[B-\left(1-\frac{\lambda}{P}\right)Q\right]E_1+\frac{hQ}{2}E_2 \tag{37}$$

To prove the convexity of $E\left[TCU_{m_{02}}(Q,B)\right]$ with respect to Q and B, the Hessian matrix is utilized (Rardin, 1998). Therefore:

$$\begin{bmatrix} Q & B \end{bmatrix}\begin{bmatrix} \frac{\partial^2 E\left[TCU_{m_{02}}(Q,B)\right]}{\partial Q^2} & \frac{\partial^2 E\left[TCU_{m_{02}}(Q,B)\right]}{\partial Q\partial B} \\ \frac{\partial^2 E\left[TCU_{m_{02}}(Q,B)\right]}{\partial B\partial Q} & \frac{\partial^2 E\left[TCU_{m_{02}}(Q,B)\right]}{\partial B^2} \end{bmatrix}\begin{bmatrix} Q \\ B \end{bmatrix}=\frac{2K_m\lambda}{Q}E_0>0 \tag{38}$$

Since (38) is strictly positive, equating the first derivative of $E\left[TCU_{m_{02}}(Q,B)\right]$ with respect to Q and B to zero yields the optimal values as below.

$$Q^*=\sqrt{\frac{2K_m\lambda}{h\left(1-\frac{\lambda}{P}\right)--\frac{h^2}{b+h}\left\{1-E[x]\right\}^2\left(E\left(\frac{1-x}{1-x-\frac{\lambda}{P}}\right)\right)^{-1}-2h\left(1-\frac{\lambda}{P}\right)E[x]+hE\left[x^2\right]}} \tag{39}$$

$$B^*=\left(\frac{h}{b+h}\right)\left(1-E[x]\right)\left(E\left(\frac{1-x}{1-x-\frac{\lambda}{P}}\right)\right)^{-1}Q^* \tag{40}$$

3.2.2. Supply chain with VMI policy

After implementing VMI, the total cost of the buyer, the manufacturer and the chain are calculated as below;

$$TC_{m_{12}}(Q_{VMI},B_{VMI}) = CQ_{VMI} + C_S xQ_{VMI} + K_m + h\left[\frac{H}{2}(t_1+t_2)\right] + b\,\frac{B_{VMI}}{2}(t_3+t_4) + h\left[\frac{d\,(t_1+t_4)}{2}(t_1+t_4)\right] + K_b \tag{41}$$

$$TC_{b_{12}} = 0 \tag{42}$$

$$TC_{VMI}(Q_{VMI},B_{VMI}) = CQ_{VMI} + C_S xQ_{VMI} + K_m + h\left[\frac{H}{2}(t_1+t_2)\right] + b\,\frac{B_{VMI}}{2}(t_3+t_4) + h\left[\frac{d\,(t_1+t_4)}{2}(t_1+t_4)\right] + K_b \tag{43}$$

For some simplifications, $E_2 = \frac{E\left[x^2\right]}{1-E\left[x\right]}$ and $E_3 = E\left(\frac{1-x}{1-x-\frac{\lambda}{P}}\right)\frac{1}{1-E\left[x\right]}$,and then:

$$E\left[TCU_{m_{12}}(Q_{VMI},B_{VMI})\right] = \frac{(K_m+K_b)\lambda}{Q_{VMI}}E_0 + \frac{h}{2}\left[\left(1-\frac{\lambda}{P}\right)Q_{VMI} - 2B_{VMI}\right]E_0 + \frac{hQ_{VMI}\varphi^2}{2}E_2 + \frac{B_{VMI}{}^2}{2Q_{VMI}}(b+h)E_3 + h\left[B_{VMI} - \left(1-\frac{\lambda}{P}\right)Q_{VMI}\right]E_1 + \lambda\left[CE_0 + C_S E_1\right] \tag{44}$$

$$E\left[TCU_{b_{12}}\right] = 0 \tag{45}$$

$$\begin{aligned} E\left[TCU_{12}(Q_{VMI},B_{VMI})\right] &= E\left[TCU_{m_{12}}(Q_{VMI},B_{VMI})\right] + E\left[TCU_{b_{12}}\right] \\ &= \lambda\left[CE_0 + C_S E_1\right] + \frac{(K_m+K_b)\lambda}{Q_{VMI}}E_0 + \frac{h}{2}\left[\left(1-\lambda/P\right)Q_{VMI} - 2B_{VMI}\right]E_0 \\ &\quad + \frac{B_{VMI}{}^2}{2Q_{VMI}}(b+h)E_3 + h\left[B_{VMI} - \left(1-\lambda/P\right)Q_{VMI}\right]E_1 + \frac{hQ_{VMI}\varphi^2}{2}E_2 \end{aligned} \tag{46}$$

To prove the convexity of $E\left[TCU_{12}(Q_{VMI},B_{VMI})\right]$ with respect to Q_{VMI} and B_{VMI} the Hessian matrix is used (Rardin, 1998) as shown in Eq. (47).

$$\begin{bmatrix} Q_{VMI} & B_{VMI} \end{bmatrix} \begin{bmatrix} \dfrac{\partial^2 E\left[TCU_{12}\left(Q_{VMI}, B_{VMI}\right)\right]}{\partial Q_{VMI}^2} & \dfrac{\partial^2 E\left[TCU_{12}\left(Q_{VMI}, B_{VMI}\right)\right]}{\partial Q_{VMI}\partial B_{VMI}} \\ \dfrac{\partial^2 E\left[TCU_{12}\left(Q_{VMI}, B_{VMI}\right)\right]}{\partial B_{VMI}\partial Q_{VMI}} & \dfrac{\partial^2 E\left[TCU_{12}\left(Q_{VMI}, B_{VMI}\right)\right]}{\partial B_{VMI}^2} \end{bmatrix} \begin{bmatrix} Q_{VMI} \\ B_{VMI} \end{bmatrix} = \frac{2\left(K_m + K_b\right)\lambda}{Q_{VMI}} E_0 > 0 \tag{47}$$

Since the result is strictly positive, equating the first derivative of $E\left[TCU_{12}\left(Q_{VMI}, B_{VMI}\right)\right]$ with respect to Q_{VMI} and B_{VMI} to zero yields the optimal values as below.

$$Q_{VMI}^* = \sqrt{\frac{2\left(K_m + K_b\right)\lambda}{h\left(1-\dfrac{\lambda}{P}\right) - \dfrac{h^2}{b+h}\left\{1-E[x]\right\}^2 \left(E\left(\dfrac{1-x}{1-x-\dfrac{\lambda}{P}}\right)\right)^{-1} - 2h\left(1-\dfrac{\lambda}{P}\right)E[x] + hE\left[x^2\right]}} \tag{48}$$

$$B_{VMI}^* = \left(\frac{h}{b+h}\right)\left(E\left(\frac{1-x}{1-x-\dfrac{\lambda}{P}}\right)\right)^{-1} Q_{VMI}^* \tag{49}$$

4. Numerical example and sensitivity analysis

Consider a manufacturer who produces aluminum doors in different sizes. Because of the especial manufacturing process and according to the different types of orders, both machine and laborers have important roles in the production process. So, producing doors in different sizes without defective is inevitable, but using rework process, this problem can be removed from the products. Also, because of humans' errors which are different in all laborers, the defective production rate is a stochastic variable during the production process and depends on the experiments of laborers. In addition, because of specifications of products the items are produced according to a contract and are held by the manufacturer which means vendor managed inventory is being used by the manufacturer.

Assume that the demand rate of products is 4,000 units per year. Items are produced at a rate of 10,000 units per year. The financial department has estimated that it costs 450$ to initiate a production run, each unit costs the company 2$ to manufacture, the holding cost is 0.6$ per item per year, and the disposal cost is 0.3$ for each scrap item. The production rate of defective items is uniformly distributed over the interval [0, 0.41]. The inventory data are shown in Table 1 and the optimal solutions for both models are shown in Table 2.

Table 1. Inventory data

P	λ	C	C_s	h	K_m	K_b	b	E[x]
10000	4000	2	0.3	0.6	450	300	0.2	0.205

Table2. The optimal solution for both models

Case	Q	Q_{VMI}	B	B_{VMI}	E[TCU]	$E[TCU]_{VMI}$
1	3825.3	4938.4	-	-	11951	11901
2	8106.4	10465.3	3250	4195.7	11117.1	11093.5

To consider the effect of parameters on the optimal solution in both models, sensitivity analysis is performed in this paper. At each stage, one parameter is changed by +75%, +50%, +25%, -25%, -50% and -75% and the rest of the parameters remain at their original levels. The results are shown in Tables 3 and 4.

Table 3. The effect of changes of certain parameters on the optimal solution in the first model (without shortage)

Parameter	%Changes	Q	Q_{VMI}	E[TCU]	$E[TCU]_{VMI}$
C	-75	3825.300	4,938.400	4403.500	4353.400
	-50	3825.300	4,938.400	6919.200	6869.100
	-25	3825.300	4,938.400	9435.000	9384.900
	+25	3825.300	4,938.400	14466.000	14416.000
	+50	3825.300	4,938.400	16982.000	16932.000
	+75	3825.300	4,938.400	19498.000	19448.000
C_s	-75	3825.3	4,938.4	11719	11668
	-50	3825.3	4,938.4	11796	11746
	-25	3825.3	4,938.4	11873	11823
	+25	3825.3	4,938.4	11719	11978
	+50	3825.3	4,938.4	11796	12055
	+75	3825.3	4,938.4	11873	12133
h	-75	7650.6	9,876.9	11626	11601
	-50	5409.8	6,984	11953	11917
	-25	4417.1	5,702.4	12203	12160
	+25	3421.5	4,417.1	12601	12545
	+50	3123.3	4,032.2	12770	12708
	+75	2891.7	3,733.1	12924	12858
K_m	-75	1912.7	3,662.5	11753	11506
	-50	2704.9	4,131.8	11767	11651
	-25	3312.8	4,553	11853	11781
	+25	4276.8	5,295.9	12049	12011
	+50	4685	5,630.7	12144	12115
	+75	5060.4	5,946.7	12237	12213

Table 3. Continue

Parameter	%Changes	Q	Q_{VMI}	E[TCU]	E[TCU] $_{VMI}$
K_b	-75	3825.3	4,131.8	12119	12115
	-50	3825.3	4,417.1	12218	12203
	-25	3825.3	4,685	12316	12286
	+25	3825.3	5,179.5	12513	12439
	+50	3825.3	5,409.8	12612	12511
	+75	3825.3	5,630.7	12711	12579
$E[x]$ is increasing cumulatively at the rate of 0.05 in the interval [0.005,0.5]					
$E[x]$	0.05	3178.100	4,102.900	9564.100	9515.900
	0.055	3339.400	4,311.200	10056.000	10008.000
	0.105	3503.900	4,523.500	10610.000	10561.000
	0.155	3667.700	4,735.000	11236.000	11187.000
	0.205	3825.300	4,938.400	11951.000	11901.000
	0.255	3969.900	5,125.200	12772.000	12720.000
	0.305	4093.700	5,284.900	13725.000	13671.000
	0.355	4188.300	5,407.100	14840.000	14784.000
	0.405	4246.700	5,482.400	16162.000	16102.000
	0.455	4263.800	5,504.500	17746.000	17681.000

Table 4. The effect of changes of certain parameters on the optimal solution in the second model (with shortage)

Parameter	%Changes	Q	Q_{VMI}	B	B_{VMI}	E[TCU]	$E[TCU]_{VMI}$
C	-75	8106.42	10465.34	3249.998	4195.73	3569.966	3546.316
	-50	8106.42	10465.34	3249.998	4195.73	6085.69	6062.039
	-25	8106.42	10465.34	3249.998	4195.73	8601.413	8577.762
	+25	8106.42	10465.34	3249.998	4195.73	13632.86	13609.21
	+50	8106.42	10465.34	3249.998	4195.73	16148.58	16124.93
	+75	8106.42	10465.34	3249.998	4195.73	18664.31	18640.66
C_s	-75	8106.42	10465.34	3249.998	4195.73	10885.06	10861.41
	-50	8106.42	10465.34	3249.998	4195.73	10962.42	10938.77
	-25	8106.42	10465.34	3249.998	4195.73	11039.78	11016.13
	+25	8106.42	10465.34	3249.998	4195.73	11194.49	11170.84
	+50	8106.42	10465.34	3249.998	4195.73	11271.85	11248.2
	+75	8106.42	10465.34	3249.998	4195.73	11349.21	11325.56
h	-75	10261.974	13248.152	2350.969	3035.088	10960.687	10942.004
	-50	8797.397	11357.390	2821.618	3642.693	11058.636	11036.843
	-25	8310.917	10729.348	3075.678	3970.683	11098.810	11075.741
	+25	8025.213	10360.505	3386.780	4372.314	11124.673	11100.783
	+50	8010.945	10342.085	3503.695	4523.251	11126.013	11102.080
	+75	8038.672	10377.881	3609.577	4659.944	11123.413	11099.563
K_m	-75	4053.210	7761.307	11024.035	7761.307	11024.035	10907.152
	-50	5732.105	8755.935	11030.652	8755.935	11030.652	10975.691
	-25	7020.366	9648.570	11071.103	9648.570	11071.103	11037.202
	+25	9063.253	11222.831	11163.413	11222.831	11163.413	11145.683
	+50	9928.297	11932.328	11208.511	11932.328	11208.511	11194.574
	+75	10723.786	12601.943	11252.050	12601.943	11252.050	11240.717

Table 4. continue

Parameter	%Changes	Q	Q_{VMI}	B	B_{VMI}	E[TCU]	$E[TCU]_{VMI}$
K_b	-75	8106.420	8755.935	3249.998	3510.399	10977.484	10975.691
	-50	8106.420	9360.488	3249.998	3752.775	11024.035	11017.351
	-25	8106.420	9928.297	3249.998	3980.419	11070.586	11056.478
	+25	8106.420	10976.145	3249.998	4400.518	11163.687	11128.684
	+50	8106.420	11464.210	3249.998	4596.191	11210.237	11162.316
	+75	8106.420	11932.328	3249.998	4783.868	11256.788	11194.574
b	-75	18384.555	23734.358	9071.597	11711.381	10700.741	10690.312
	-50	11449.213	14780.871	5245.919	6772.452	10899.676	10882.931
	-25	9254.494	11947.500	3957.632	5109.280	11024.738	11004.022
	+25	7383.697	9532.312	2786.114	3596.858	11190.039	11164.073
	+50	6881.052	8883.399	2452.202	3165.779	11249.771	11221.908
	+75	6508.750	8402.761	2197.445	2836.889	11299.961	11270.504
	$E[x]$ is increasing cumulatively at the rate of 0.02 in the interval [0.025,0.205]						
$E[x]$	0.025	16023.78	20686.61	8997.397	11615.59	8543.133	8533.377
	0.045	14634.22	18892.69	7978.845	10300.64	8776.962	8766.056
	0.065	13427	17334.18	7099.112	9164.914	9021.913	9009.772
	0.085	12373.75	17334.18	6334.895	8178.314	9278.599	9265.137
	0.105	11449.3	14780.98	5666.171	7314.995	9547.754	9532.879
	0.125	10632.1	13725.98	5076.134	6553.261	9830.244	9813.86
	0.145	9903.889	12785.87	4550.729	5874.966	10127.09	10109.09
	0.165	9249.108	11940.55	4078.051	5264.742	10439.48	10419.75
	0.185	8654.139	11172.45	3647.696	4709.155	10768.89	10747.28
	0.205	8106.42	10465.34	3249.998	4195.73	11117.14	11093.49

According to the obtained results from Tables 3 and 4, the following achievements are shown in Table 5.

Table 5. The effects of changing parameters on optimal solutions

Parameter change (From -75% to +75)	Effects of parameter change on optimal solutions											
	Q		Q_{VMI}		B		B_{VMI}		E[TCU]		$E[TCU]_{VMI}$	
	Case1	Case2	Case1	Case2	Case1	Case2	Case1	Case2	Case1	Case2	Case1	Case2
C	None	None	None	None	-	None	-	None	Increase	Increase	Increase	Increase
C_s	None	None	None	None	-	None	-	None	Increase	Increase	Increase	Increase
h	Decrease	Decrease	Decrease	Decrease	-	Increase	-	Increase	Increase	Increase	Increase	Increase
K_m	Increase	Increase	Increase	Increase	-	Increase	-	Increase	Increase	Increase	Increase	Increase
K_b	None	None	None	None	-	None	-	None	Increase	Increase	Increase	Increase
b	-	Decrease	-	Decrease	-	Decrease	-	Decrease	-	Increase	-	Increase
E(x)	Decrease	Decrease	Decrease	Decrease	-	Decrease	-	Decrease	Increase	Increase	Increase	Increase

Considering the above mentioned results, VMI policy is more beneficial for both parties in the both models that is to say the expected total cost after implementation of VMI is less than the expected total cost in traditional supply chain. Besides, the optimal production quantity in VMI is greater than its quantity in the traditional policy.

5. Conclusions and future research

In this paper, the performance of the VMI policy in a two-echelon supply chain consisting of a manufacturer and a buyer is investigated regarding two conditions, with and without shortage. The production process generates randomly imperfect quality items at a constant production rate. The optimal production quantity and backordered level are obtained by minimizing total cost of the supply chain. The numerical example and sensitivity analysis have been performed to illustrate the differences in total cost and an optimal solution of both models. It is demonstrated that the VMI policy is more beneficial for both parties in the both models. Moreover, the optimal production quantity in VMI is greater than its quantity in the traditional policy.

There is much scope in extending the paper. For example, full backordering could consider partial backordering. Also, the levels of supply chain and the number of buyers could increase. In this case, we could investigate the impact of non-cooperative and cooperative relationships between the buyers.

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