EPQ model with scrap and backordering under Vendor managed inventory policy

Maryam Akbarzadeh¹, Maryam Esmaeili¹*, Ata Allah Taleizadeh²

¹Department of Industrial Engineering, Alzahra University, Tehran, Iran
m.akbarzadeh66@yahoo.com, esmaeili_m@alzahra.ac.ir

²School of Industrial Engineering, College of Engineering, University of Tehran, Tehran, Iran
taleizadeh@ut.ac.ir

Abstract

This paper presents the economic production quantity (EPQ) models for the imperfect quality items produced with/without the presence of shortage condition. The models are presented in a two-level supply chain composed of a single manufacturer and a single buyer to investigate the performance of vendor-managed inventory (VMI) policy. The total costs are minimized to obtain the optimal production lot size and the allowable backorder level before and after applying VMI policy. Numerical examples and sensitivity analysis based on certain parameters are performed to show the capability of the proposed supply chain model enhanced with VMI policy.

Keywords: Economic production quantity (EPQ), Vendor-managed inventory (VMI), Supply chain, Defective items

1. Introduction and literature review

Economic Production Quantity (EPQ) has been applied widely in the manufacturing sector to optimize the production quantity or lot-sizing considering the capacities and limitations. Although, all EPQ models minimize the total costs or maximize the manufacturer’s profit apparently, they are different mainly regarding various assumptions. One of the impractical assumptions of the EPQ models is that all units have perfect quality. However, in practice, most of the manufacturing processes are not free of defect and cause non-conforming items. Generally, these non-conforming items are categorized into two groups: the repairable and the scrap (the latter is discarded).

¹* Corresponding Author

ISSN: 1735-8272, Copyright © 2015 JISE. All rights reserved
Recent developments in information technology have facilitated the advent of new supply chain initiatives such as vendor managed inventory (VMI) policy. VMI is a supply chain initiative where the vendor decides on the appropriate inventory levels of each of the products and the appropriate inventory policies to sustain those levels. It is also known as continuous replenishment or supplier-managed inventory. VMI becomes more critical for the EPQ models with imperfect quality production. In this paper, the EPQ model based on VMI policy is considered for the defective items produced under the presence and absence of the shortage. The models are developed for a two-level supply chain consisting of a single manufacturer and a single buyer and examine the inventory management practices before and after implementing VMI. Therefore, the literature reviews are organized for both EPQ models with scrap and VMI policy.

Many researchers introduced EPQ models with scrap such as Salameh and Jaber (2000). They developed an inventory model which explained imperfect quality items using the EPQ/EOQ formulates. They supposed that the defective items are sold as a single batch at the end of the total screening process. Cardenas-Barron (2000) also revised Salameh and Jaber's (2000) model and showed that the error only affects the optimal order size value. Later, Goyal and Cardenas-Barron (2002) developed Salameh and Jaber's (2000) model and proposed a practical approach to determine EPQ for imperfect quality items. Goyal et al. (2003) studied the model of Goyal and Cardenas-Barron (2002), regarding vendor-buyer integration. Konstantaras et al. (2007) presented a production-inventory model for defective items regarding two conditions. The first one was to sell the items to a secondary shop as a single batch at a lower price comparing to new ones and the second one was to rework them at some cost and return to its original quality. Wee et al. (2007) and Eroglu and Ozdemir (2007) extended the model of Salameh and Jaber (2000) independently regarding the shortages permit. Cardenas-Barron (2008) proposed a simple derivation to find an optimal manufacturing batch size with rework process in a single stage production system. Khan et al. (2011) extended the Salameh and Jaber (2000) EOQ model for imperfect items and presented a detailed review of EOQ/EPQ models with imperfect quality. Recently, Hsu and Hsu (2013) built two optimal production quantity models with imperfect production processes, inspection errors, planned backorders, and sales returns. Taheri-Tolgari et al. (2012) presented a discounted cash-flow approach for an inventory model for imperfect items under inflationary conditions with considering inspection errors and related defect sales return issues. Sarkar and Moon (2011) developed a production inventory model in an imperfect production system considering stochastic demand and the effect of inflation. Pal et al. (2013) derived a mathematical model on EPQ with stochastic demand in an imperfect production system. Cárdenas-Barrón et al. (2012) derived jointly both the optimal replenishment lot size and the optimal number of shipments for the inventory models.

The success of a firm will depend on its ability to integrate with other participants in the chain responsible for physical, financial and information flows. Organizations are striving to achieve joint total effectiveness across the entire chain. This has given rise to a need for various coordination mechanisms. Vendor Managed Inventory (VMI) is one of the practices which has gained a lot of attention in recent times. Many companies such as HP, Shell and Walmart have reportedly adopted VMI. Researchers are actively engaged in studying issues related to VMI, including, but not limited to, replenishment decisions, contracts, relationships as well as the strategic implications of such a mechanism (Marques et al., 2010). The costs of VMI policy with the traditional one are highly compared in the literature (Waller et al. (1999),
Cetinkaya and Lee (2000)). Moreover, having a deterministic demand is one of the usual assumptions (Valentini and Zavanella (2003), Shah and Goh (2006), Woo et al. (2001)). They considered a supply chain consisting of a single manufacturer and multiple buyers or retailers. Kleywegt et al. (2002) presented an inventory routing problem of a manufacturer who owns the inventory at the retailers. An approximation method is used to find the minimum-cost routing policy. Later Zhang et al. (2007) extended Woo et al.’s model by omitting the common replenishment cycle assumption and having different ordering cycles for the retailers. A number of papers analyzed the VMI policy in a supply chain as a centralized or a coordination channel. For instance, Bernstein et al. (2006) studied the constant wholesale price and quantity discount contracts. They showed the role of VMI in a complete coordination in a supply chain with multiple competing retailers. It is shown that the performance of the overall system is improved in VMI regarding certain conditions (Nagarajan and Rajagopalan (2008)). Pasandideh et al. (2010) considered the retailer-supplier partnership through a VMI. In addition, they developed an economic order quantity (EOQ) with shortage to explore the effects of important supply chain parameters on the cost savings realized from VMI. Ma et al. (2013) developed a VMI model for a supply chain with one vendor and one retailer including a VMI warehouse near the retailer and with a freshness clause. They showed that the freshness clause had a significant impact on the decisions of both vendor and retailer. Hariga et al. (2013a&b) analyzed a VMI system with different replenishment cycles for different retailers. Kannan et al. (2013) studied the benefits of a VMI arrangement when the same organization owns the vendor and the retailers. Some of these studies, like Darwish and Odah (2010) and Kannan (2013), also showed that adopting VMI leads to system wide cost reductions.

Regarding the effectiveness of VMI, we studied the integration of VMI policy in EPQ models producing defective items in this paper. To the best our knowledge, it has been ignored in the literature. Therefore, in this paper, we present the EPQ model to investigate the performance of VMI in a two-level supply chain composed of a manufacturer and a buyer. The manufacturer produces defective items under the presence and absence of the shortage costs. The production process generates randomly imperfect quality items at a constant production rate. The manufacturer and the buyer are faced with the setup cost, production cost, screening cost, disposal cost per scrap item, holding cost and ordering cost without shortage. The optimal production lot size is obtained by minimizing the total expected inventory costs before and after implementing VMI. The backordering cost per item per time unit is also considered regarding shortage permission in addition to the costs mentioned. The total expected inventory costs are minimized to obtain the optimal maximum allowable backorder level and production lot size before and after implementing VMI. The numerical example is presented in this paper, including sensitivity analysis of certain parameters. It shows the capability of supply chain regarding implementing VMI with and without shortage. Although a simple supply chain is used for computational ease, the obtained results can be generalized to more complex supply chains.

The rest of this paper is organized as follows. Section 2 reviews the notation and problem formulation. Section 3 presents the model formulations. In Section 4, the numerical example and the sensitivity analysis are presented to analyze the effect of different parameters on the optimal economic production quantity and the total cost before and after implementing VMI. The final section is dedicated to conclusions and recommendations for future studies.
2. Notation and problem formulation

This section introduces the notation and formulation used in the models. Specifically, all decision variables, input parameters and assumptions underlying the models will be stated.

2.1. Input parameters

The following notations are used in the paper:

- **Q**: The production lot size
- **Q_{VMI}**: The production lot size in VMI
- **B**: The maximum allowable backorder level
- **B_{VMI}**: The maximum allowable backorder level in VMI
- **P**: The production rate
- **λ**: The demand rate
- **x**: The defective rate, a random variable with known probability density function
- **d**: The production rate of defective items (during the regular production process), in units per time unit
- **H**: The maximum level of on-hand inventory in units, when the rework process ends
- **b**: The backordering cost per item per time unit
- **K_m**: The manufacturer setup cost
- **K_b**: The buyer ordering cost
- **C**: The production cost (screening cost for each item is included)
- **C_r**: The repair cost of each imperfect quality item reworked (screening cost for each rework item is also included)
- **C_s**: The disposal cost per scrap item
- **h**: The holding cost
- **TC(Q)**: The total cost per time unit
- **E(TCU_{m_{0i}})**: The manufacturer’s expected total cost before VMI in case i (i=1,2)
- **E(TCU_{b_{0i}})**: The buyer’s expected total cost before VMI in case i
- **E(TCU_{m_{1i}})**: The manufacturer’s expected total cost in VMI in case i
- **E(TCU_{b_{1i}})**: The buyer’s expected total cost in VMI in case i
- **E(TCU_{0i})**: The expected total cost before VMI in case i
- **E(TCU_{1i})**: The expected total cost of VMI in case i

2.2. Assumptions

The mathematical models are developed based on the following assumptions.

(a) A single-manufacturer-single-buyer supply chain with defective items is considered.
(b) The production rate P is a constant and is greater than the demand rate λ.
(c) The EPQ model is imperfect and production process generates randomly x percent of imperfect quality items at production rate d.
(d) The defective production rate d is expressed by d=Px.
3. Model formulations

The models are presented in two cases; 1: shortage is not allowed, 2: shortage is allowed and is completely backordered.

3.1. Case 1. Shortage is not permitted

The model with the random defective rate and an imperfect rework process without shortage is depicted in Figure 1. According to assumption b, $P$ is greater than or equal to the sum of the demand rate $\lambda$ and also $P - d - \lambda \geq 0$ or $0 \leq x \leq \left(1 - \frac{\lambda}{P}\right)$. Therefore, based on Figure (1), the production cycle length ($T$) is

$$T = t_1 + t_2 = \frac{Q(1-x)}{\lambda}$$  \hspace{1cm} (1)

Since $t_1 = \frac{Q}{P} = \frac{H}{P - d - \lambda}$

$$H = (P - d - \lambda)t_1 = Q \left(1 - x - \frac{\lambda}{P}\right)$$  \hspace{1cm} (2)

The total defective items produced during the regular production uptime “$t_1$” are:

$$dt_1 = xQ$$  \hspace{1cm} (3)

The production downtime time “$t_2$” needed for consuming the maximum on-hand inventory $H$ is computed in Eq. (4).

$$t_2 = \frac{H}{\lambda} = Q \left(\frac{1}{\lambda} - \frac{x}{\lambda} - \frac{1}{P}\right)$$  \hspace{1cm} (4)

![Figure 1](image.png)

Figure 1. The EPQ model with the random defective rate without shortage
3.1.1. Supply chain without VMI policy

Regardless of implementing VMI, the inventory cost of the manufacturer, the buyer and the total inventory cost of the supply chain are as follows:

\[ TC_{m_{si}}(Q) = CQ + C_s(xQ) + K_m + h \left( \frac{H}{2} (t_1 + t_2) \right) + h \frac{xQ}{2} t_1 \]  
\[ TC_{b_{si}} = K_b \]  
\[ TC(Q) = CQ + C_s(xQ) + K_m + h \left( \frac{H}{2} (t_1 + t_2) \right) + h \frac{xQ}{2} t_1 + K_b \]

The proportion of defective items \( x \) is considered to be a random variable with a known probability density function. Thus, the expected value of \( x \) is used in the inventory cost analysis. Since the production cycle length is not a constant, by using the renewal theorem approach (Zipkin, 2000) we have:

\[ E[TCU(Q)] = E[TC(Q)] \]  
\[ E[T] = \frac{Q}{E[x]} \]  

Where,

\[ E[T] = \frac{Q}{\lambda} (1 - E[x]) \]  

Then,

\[ E[TCU_{m_{si}}(Q)] = \lambda \left[ \frac{C}{1 - E[x]} + \frac{C_s E[x]}{1 - E[x]} + \frac{K_m}{Q} \frac{1}{1 - E[x]} \right] + h \left( \frac{1 - \lambda}{P} \right) \frac{1}{1 - E[x]} - hQ \left( \frac{1 - \lambda}{P} \right) \frac{E[x]}{1 - E[x]} + hQ \frac{E[x^2]}{2} \frac{1}{1 - E[x]} \]  

And also,

\[ E[TCU_{b_{si}}(Q)] = \frac{K_b \lambda}{Q} \frac{1}{1 - E[x]} \]  

If we let \( E_0 = \frac{1}{1 - E[x]} \), \( E_1 = \frac{E[x]}{1 - E[x]} \) and \( E_2 = \frac{E[x^2]}{1 - E[x]} \) for some simplifications we have:

\[ E[TCU_{m_{si}}(Q)] = \lambda \left[ C.E_0 + C_s.E_1 \right] + K_m \lambda \frac{E_0}{Q} \]  
\[ + \frac{hQ}{2} \left( \frac{1 - \lambda}{P} \right) E_0 - hQ \left( \frac{1 - \lambda}{P} \right) E_1 + \frac{hQ}{2} E_2 \]  
\[ E[TCU_{b_{si}}(Q)] = \frac{K_b \lambda}{Q} . E_0 \]
\[ E[TCU_{01}(Q)] = \lambda [CE_0 + C_S E_1] + \frac{(K_m + K_b)\lambda}{Q} E_0 \]

\[ + \frac{hQ}{2} \left(1 - \frac{\lambda}{P}\right) E_0 - hQ \left(1 - \frac{\lambda}{P}\right) E_1 + \frac{hQ}{2} E_2 \]

(14)

Since the second derivative of \( E[TCU_{01}(Q)] \) respect to \( Q \) is positive, \( \frac{2K_m\lambda}{Q} E_0 > 0 \), the total objective cost is convex. Then the optimal production quantity is obtained by setting the first derivative of \( E[TCU_{m_{01}}(Q)] \) respect to \( Q \) equal to zero yielding;

\[ Q^* = \sqrt{\frac{2K_m\lambda}{h \left(1 - \frac{\lambda}{P}\right) - 2h \left(1 - \frac{\lambda}{P}\right) E[x] + hE[x^2]}} \]

(15)

3.1.2. Supply chain with VMI policy

After implementing VMI, the total cost of the buyer, the manufacturer and the chain are calculated as below;

\[ TC_{m_{11}}(Q_{VM1}) = CQ_{VM1} + C_S (xQ_{VM1}) + K_m + h \left[\frac{H}{2}(t_1 + t_2)\right] + h \frac{xQ_{VM1}}{2} t_1 + K_b \]

(16)

\[ TC_{b_{11}} = 0 \]

(17)

\[ TC_{VM1}(Q_{VM1}) = TC_{m_{11}}(Q_{VM1}) + TC_{b_{11}}(Q_{VM1}) \]

\[ = CQ_{VM1} + C_S (xQ_{VM1}) + K_m + h \left[\frac{H}{2}(t_1 + t_2)\right] + h \frac{xQ_{VM1}}{2} t_1 + K_b \]

(18)

Using \( E_0, E_1, E_2 \) defined in the previous section, we have,

\[ E[TCU_{m_{11}}(Q_{VM1})] = \lambda [CE_0 + C_S E_1] + \frac{(K_m + K_b)\lambda}{Q_{VM1}} E_0 + hQ_{VM1} \left(1 - \frac{\lambda}{P}\right) E_0 \]

\[ - hQ_{VM1} \left(1 - \frac{\lambda}{P}\right) E_1 + \frac{hQ_{VM1}}{2} E_2 \]

(19)

\[ E[TCU_{b_{11}}] = 0 \]

(20)

\[ E[TCU_{11}(Q_{VM1})] = E[TCU_{m_{11}}(Q_{VM1})] + E[TCU_{b_{11}}(Q_{VM1})] \]

\[ = \lambda [CE_0 + C_S E_1] + \frac{(K_m + K_b)\lambda}{Q_{VM1}} E_0 + hQ_{VM1} \left(1 - \frac{\lambda}{P}\right) E_0 \]

\[ - hQ_{VM1} \left(1 - \frac{\lambda}{P}\right) E_1 + \frac{hQ_{VM1}}{2} E_2 \]

(21)
Since the second derivative of \( E[TCU_{11}(Q_{VMI})] \) respect to \( Q_{VMI} \) is positive, \( \frac{2(K_m + K_b)\lambda}{Q_{VMI}^3} E_0 > 0 \), the total objective cost is convex. Then the optimal production quantity is obtained by setting the first derivative of \( E[TCU_{11}(Q_{VMI})] \) respect to \( Q_{VMI} \) equal to zero yielding:

\[
Q_{VMI}^* = \sqrt{\frac{2(K_m + K_b)\lambda}{h\left(1 - \frac{\lambda}{P}\right) - 2h\left(1 - \frac{\lambda}{P}\right)E[x] + hE[x^2]}}
\] (22)

### 3.2. Case 2. Shortage is permitted

The inventory control in this case is shown in Figure 2. According to the definitions in case 1, we have:

\[
t_1 = \frac{H}{P - d - \lambda}
\] (23)

\[
t_2 = \frac{H}{\lambda}
\] (24)

\[
t_3 = \frac{B}{\lambda}
\] (25)

\[
t_4 = \frac{B}{P - d - \lambda}
\] (26)

\[H = (P - d - \lambda)\frac{Q}{P} - B\] (27)

\[
t_1 + t_4 = \frac{Q}{P}
\] (28)

The cycle length \( T \) is:

![Figure 2](image_url)

**Figure 2.** The EPQ model with the random defective rate under presence of the shortage
\[ T = \sum_{i=1}^{4} t_i = \frac{(1-x)Q}{\lambda} \] (29)

### 3.2.1. Supply chain without VMI policy

In this case, the inventory cost of the manufacturer, the buyer and the total inventory costs of the chain are as below:

\[
TC_{m_{w}}(Q,B) = CQ + C_S x Q + K_m +\frac{H}{2}(t_1 + t_2) + b \left(\frac{B}{2}(t_3 + t_4) + h \left[\frac{d(t_1 + t_4)}{2}(t_1 + t_4)\right]\right)
\] (30)

\[ TC_{b_{w}} = K_b \] (31)

\[
TC_{noVMI}(Q,B) = (C + C_S x)Q + K_m +\frac{H}{2}(t_1 + t_2) + \frac{d(t_1 + t_4)}{2}(t_1 + t_4) + \frac{bB}{2}(t_3 + t_4) + K_b
\] (32)

Similar to the first case, by using the renewal theorem approach (Zipkin, 2000), we have:

\[
E\left[TCU_{m_{w}}(Q,B)\right] = \lambda \left(\frac{C + C_S E[x]}{1-E[x]}\right) + \frac{1}{Q} \left\{K_m \lambda + h \left[\frac{1}{1 - \frac{\lambda}{P}} Q - 2B\right]\right\} \frac{1}{1-E[x]}
\]

\[ + \frac{B^2}{2Q} \left(\frac{1-x}{1-E[x]}\right) + h \left[\lambda \left(\frac{1}{1-E[x]}\right) + \frac{Q}{1-E[x]} + hQ \frac{E[x^2]}{1-E[x]}\right] \] (33)

And also:

\[
E\left[TCU_{b_{w}}(Q)\right] = \frac{K_b \lambda}{Q} \frac{1}{1-E[x]}
\] (34)

By considering

\[ E_0 = \frac{1}{1-E[x]}, \quad E_1 = \frac{E[x]}{1-E[x]}, \quad E_2 = \frac{E[x^2]}{1-E[x]} \]

and

\[ E_3 = E \left(\frac{1-x}{1-x-\frac{\lambda}{P}}\right) \frac{1}{1-E[x]} \]

for some simplifications, we have:
\[ E[TCU_{m_1}(Q,B)] = \lambda(CE_0 + C_s E_1) + \frac{K_m \lambda}{Q} E_0 + \frac{h}{2} \left[ 1 - \frac{\lambda}{P} \right] Q - 2B \right] E_0 \\
+ \frac{B^2}{2Q} (b+h) E_3 + h \left[ B - \left( 1 - \frac{\lambda}{P} \right) Q \right] E_1 + \frac{hQ}{2} E_2 \]

(35)

\[ E[TCU_{h_2}(Q)] = \frac{K_b \lambda}{Q} E_0 \]

(36)

Therefore, the total expected inventory costs are,

\[ E[TCU_{02}(Q,B)] = \lambda(CE_0 + C_s E_1) + \frac{(K_m + K_b) \lambda}{Q} E_0 + \frac{h}{2} \left[ 1 - \frac{\lambda}{P} \right] Q - 2B \right] E_0 \\
+ \frac{B^2}{2Q} (b+h) E_3 + h \left[ B - \left( 1 - \frac{\lambda}{P} \right) Q \right] E_1 + \frac{hQ}{2} E_2 \]

(37)

To prove the convexity of \( E[TCU_{m_2}(Q,B)] \) with respect to Q and B, the Hessian matrix is utilized (Rardin, 1998). Therefore:

\[
\begin{bmatrix}
\frac{\partial^2 E[TCU_{m_2}(Q,B)]}{\partial Q^2} & \frac{\partial^2 E[TCU_{m_2}(Q,B)]}{\partial Q \partial B} \\
\frac{\partial^2 E[TCU_{m_2}(Q,B)]}{\partial B \partial Q} & \frac{\partial^2 E[TCU_{m_2}(Q,B)]}{\partial B^2}
\end{bmatrix}
\begin{bmatrix}
Q \\
B
\end{bmatrix}
= \frac{2K_m \lambda}{Q} E_0 > 0
\]

(38)

Since (38) is strictly positive, equating the first derivative of \( E[TCU_{m_2}(Q,B)] \) with respect to \( Q \) and \( B \) to zero yields the optimal values as below.

\[ Q^* = \frac{2K_m \lambda}{h \left( 1 - \frac{\lambda}{P} \right) - \frac{h^2}{b+h} \left( 1 - E[x] \right)^2 \left[ E \left( \frac{1-x}{1-x-\frac{\lambda}{P}} \right) \right]^{-1} - 2h \left( 1 - \frac{\lambda}{P} \right) E[x] + hE[x^2]} \]

(39)

\[ B^* = \left( \frac{h}{b+h} \right) (1 - E[x]) \left[ E \left( \frac{1-x}{1-x-\frac{\lambda}{P}} \right) \right]^{-1} Q^* \]

(40)
3.2.2. Supply chain with VMI policy

After implementing VMI, the total cost of the buyer, the manufacturer and the chain are calculated as below:

\[
TC_{m,12}(Q_{VMI}, B_{VMI}) = CQ_{VMI} + C_5 xQ_{VMI} + K_m + h \left[ \frac{H}{2}(t_1 + t_2) \right] + b \frac{B_{VMI}}{2}(t_3 + t_4) + h b \left[ \frac{d (t_1 + t_4)}{2} \right] + K_b
\]  

(41)

\[
TC_{b,12} = 0
\]  

(42)

\[
TC_{VMI}(Q_{VMI}, B_{VMI}) = CQ_{VMI} + C_5 xQ_{VMI} + K_m + h \left[ \frac{H}{2}(t_1 + t_2) \right] + b \frac{B_{VMI}}{2}(t_3 + t_4) + h b \left[ \frac{d (t_1 + t_4)}{2} \right] + K_b
\]  

(43)

For some simplifications, \( E_2 = E \left[ \frac{x^2}{1 - E[x]} \right] \) and \( E_3 = E \left[ \frac{1-x}{1-x \frac{\lambda}{P}} \right] \), and then:

\[
E \left[ TCU_{m,12}(Q_{VMI}, B_{VMI}) \right] = \frac{(K_m + K_b)\lambda}{Q_{VMI}} E_0 + h \left[ \frac{1 - \frac{\lambda}{P}}{2} \right] Q_{VMI} - 2B_{VMI} E_0 + \frac{hQ_{VMI}^2 \varphi^2}{2} E_2 + \frac{B_{VMI}^2}{2Q_{VMI}} (b + h) E_3 + h B_{VMI} - \left( \frac{1 - \frac{\lambda}{P}}{2} \right) Q_{VMI} E_1 + \lambda \left[ CE_0 + C_5 E_1 \right]
\]  

(44)

\[
E \left[ TCU_{b,12} \right] = 0
\]  

(45)

\[
E \left[ TCU_{12}(Q_{VMI}, B_{VMI}) \right] = E \left[ TCU_{m,12}(Q_{VMI}, B_{VMI}) \right] + E \left[ TCU_{b,12} \right] = \lambda \left[ CE_0 + C_5 E_1 \right] + \frac{(K_m + K_b)\lambda}{Q_{VMI}} E_0 + h \left[ \frac{1 - \frac{\lambda}{P}}{2} \right] Q_{VMI} - 2B_{VMI} E_0
\]  

\[
+ \frac{B_{VMI}^2}{2Q_{VMI}} (b + h) E_3 + h B_{VMI} - \left( \frac{1 - \frac{\lambda}{P}}{2} \right) Q_{VMI} E_1 + \frac{hQ_{VMI}^2 \varphi^2}{2} E_2
\]  

(46)

To prove the convexity of \( E \left[ TCU_{12}(Q_{VMI}, B_{VMI}) \right] \) with respect to \( Q_{M^*} \) and \( B_{M^*} \), the Hessian matrix is used (Rardin, 1998) as shown in Eq. (47).
\[
\begin{bmatrix}
\partial^2 E\left[TCU_u\left(Q_{\text{VMI}}, B_{\text{VMI}}\right)\right] & \partial E\left[TCU_u\left(Q_{\text{VMI}}, B_{\text{VMI}}\right)\right] \\
\partial Q_{\text{VMI}} & \partial B_{\text{VMI}} \\
\partial E\left[TCU_u\left(Q_{\text{VMI}}, B_{\text{VMI}}\right)\right] & \partial^2 E\left[TCU_u\left(Q_{\text{VMI}}, B_{\text{VMI}}\right)\right]
\end{bmatrix}
\begin{bmatrix}
Q_{\text{VMI}} \\
B_{\text{VMI}}
\end{bmatrix}
= \frac{2(K_m + K_s)\lambda}{Q_{\text{VMI}}}E_o > 0
\] (47)

Since the result is strictly positive, equating the first derivative of \(E\left[TCU_u\left(Q_{\text{VMI}}, B_{\text{VMI}}\right)\right]\) with respect to \(Q_{\text{VMI}}\) and \(B_{\text{VMI}}\) to zero yields the optimal values as below.

\[
Q_{\text{VMI}}^* = \sqrt{\frac{2(K_m + K_s)\lambda}{h\left(1 - \frac{\lambda}{P}\right) - \frac{h^2}{b + h} \{1 - E\left[x\right]\}^2}} \left\{ E\left[\frac{1 - x}{1 - x - \frac{\lambda}{P}}\right]\right\}^{-1} - 2h\left(1 - \frac{\lambda}{P}\right)E\left[x\right] + hE\left[x^2\right]
\] (48)

\[
B_{\text{VMI}}^* = \left(\frac{h}{b + h}\right)\left\{ E\left[\frac{1 - x}{1 - x - \frac{\lambda}{P}}\right]\right\}^{-1} Q_{\text{VMI}}^*
\] (49)

4. **Numerical example and sensitivity analysis**

Consider a manufacturer who produces aluminum doors in different sizes. Because of the especial manufacturing process and according to the different types of orders, both machine and laborers have important roles in the production process. So, producing doors in different sizes without defective is inevitable, but using rework process, this problem can be removed from the products. Also, because of humans’ errors which are different in all laborers, the defective production rate is a stochastic variable during the production process and depends on the experiments of laborers. In addition, because of specifications of products the items are produced according to a contract and are held by the manufacturer which means vendor managed inventory is being used by the manufacturer.

Assume that the demand rate of products is 4,000 units per year. Items are produced at a rate of 10,000 units per year. The financial department has estimated that it costs 450$ to initiate a production run, each unit costs the company 2$ to manufacture, the holding cost is 0.6$ per item per year, and the disposal cost is 0.3$ for each scrap item. The production rate of defective items is uniformly distributed over the interval [0, 0.41]. The inventory data are shown in Table 1 and the optimal solutions for both models are shown in Table 2.
To consider the effect of parameters on the optimal solution in both models, sensitivity analysis is performed in this paper. At each stage, one parameter is changed by +75%, +50%, +25%, -25%, -50% and -75% and the rest of the parameters remain at their original levels. The results are shown in Tables 3 and 4.
Table 3. Continue

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% Changes</th>
<th>Q</th>
<th>Q_{VMI}</th>
<th>E[TCU]</th>
<th>E[TCU]_{VMI}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$K_b$</td>
<td>-75</td>
<td>3825.3</td>
<td>4,131.8</td>
<td>12119</td>
<td>12115</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>3825.3</td>
<td>4,417.1</td>
<td>12218</td>
<td>12203</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>3825.3</td>
<td>4,685</td>
<td>12316</td>
<td>12286</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>3825.3</td>
<td>5,179.5</td>
<td>12513</td>
<td>12439</td>
</tr>
<tr>
<td></td>
<td>+50</td>
<td>3825.3</td>
<td>5,409.8</td>
<td>12612</td>
<td>12511</td>
</tr>
<tr>
<td></td>
<td>+75</td>
<td>3825.3</td>
<td>5,630.7</td>
<td>12711</td>
<td>12579</td>
</tr>
</tbody>
</table>

$E[x]$ is increasing cumulatively at the rate of 0.05 in the interval $[0.005, 0.5]$.

$E[x]$ | 0.05 | 3178.100 | 4,102.900 | 9564.100 | 9515.900 |
|       | 0.055 | 3339.400 | 4,311.200 | 10056.000 | 10008.000 |
|       | 0.105 | 3503.900 | 4,523.500 | 10610.000 | 10561.000 |
|       | 0.155 | 3667.700 | 4,735.000 | 11236.000 | 11187.000 |
|       | 0.205 | 3825.300 | 4,938.400 | 11951.000 | 11901.000 |
|       | 0.255 | 3969.900 | 5,125.200 | 12772.000 | 12720.000 |
|       | 0.305 | 4093.700 | 5,284.900 | 13725.000 | 13671.000 |
|       | 0.355 | 4188.300 | 5,407.100 | 14840.000 | 14784.000 |
|       | 0.405 | 4246.700 | 5,482.400 | 16162.000 | 16102.000 |
|       | 0.455 | 4263.800 | 5,504.500 | 17746.000 | 17681.000 |

Table 4. The effect of changes of certain parameters on the optimal solution in the second model (with shortage)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% Changes</th>
<th>Q</th>
<th>Q_{VMI}</th>
<th>B</th>
<th>B_{VMI}</th>
<th>E[TCU]</th>
<th>E[TCU]_{VMI}</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C$</td>
<td>-75</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>3569.966</td>
<td>3546.316</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>6085.69</td>
<td>6062.039</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>8601.413</td>
<td>8577.762</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>13632.86</td>
<td>13609.21</td>
</tr>
<tr>
<td></td>
<td>+50</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>16148.58</td>
<td>16124.93</td>
</tr>
<tr>
<td></td>
<td>+75</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>18664.31</td>
<td>18640.66</td>
</tr>
<tr>
<td>$C_s$</td>
<td>-75</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>10885.06</td>
<td>10861.41</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>10962.42</td>
<td>10938.77</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>11039.78</td>
<td>11016.13</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>11194.94</td>
<td>11170.84</td>
</tr>
<tr>
<td></td>
<td>+50</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>11271.85</td>
<td>11248.2</td>
</tr>
<tr>
<td></td>
<td>+75</td>
<td>8106.42</td>
<td>10465.34</td>
<td>3249.998</td>
<td>4195.73</td>
<td>11349.21</td>
<td>11325.56</td>
</tr>
<tr>
<td>$h$</td>
<td>-75</td>
<td>10261.974</td>
<td>13248.152</td>
<td>2350.969</td>
<td>3035.088</td>
<td>10960.687</td>
<td>10942.004</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>8797.397</td>
<td>11357.390</td>
<td>2821.618</td>
<td>3642.693</td>
<td>11058.636</td>
<td>11036.843</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>8310.917</td>
<td>10729.348</td>
<td>3075.678</td>
<td>3970.683</td>
<td>11098.810</td>
<td>11075.741</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>8025.213</td>
<td>10360.505</td>
<td>3386.780</td>
<td>4372.314</td>
<td>11124.673</td>
<td>11100.783</td>
</tr>
<tr>
<td></td>
<td>+50</td>
<td>8010.945</td>
<td>10342.085</td>
<td>3503.695</td>
<td>4523.251</td>
<td>11126.013</td>
<td>11102.080</td>
</tr>
<tr>
<td></td>
<td>+75</td>
<td>8038.672</td>
<td>10377.881</td>
<td>3609.577</td>
<td>4659.944</td>
<td>11123.413</td>
<td>11099.563</td>
</tr>
<tr>
<td>$K_m$</td>
<td>-75</td>
<td>4053.210</td>
<td>7761.307</td>
<td>11024.035</td>
<td>7761.307</td>
<td>11024.035</td>
<td>10907.152</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>5732.105</td>
<td>8755.935</td>
<td>11030.652</td>
<td>8755.935</td>
<td>11030.652</td>
<td>10975.691</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>7020.366</td>
<td>9648.570</td>
<td>11071.030</td>
<td>9648.570</td>
<td>11071.030</td>
<td>11037.202</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>9063.253</td>
<td>11222.831</td>
<td>11163.413</td>
<td>11222.831</td>
<td>11163.413</td>
<td>11145.683</td>
</tr>
<tr>
<td></td>
<td>+50</td>
<td>9928.297</td>
<td>11932.328</td>
<td>11208.511</td>
<td>11932.328</td>
<td>11208.511</td>
<td>11194.574</td>
</tr>
<tr>
<td></td>
<td>+75</td>
<td>10723.786</td>
<td>12601.943</td>
<td>11252.050</td>
<td>12601.943</td>
<td>11252.050</td>
<td>11240.717</td>
</tr>
</tbody>
</table>
Table 4. continue

<table>
<thead>
<tr>
<th>Parameter</th>
<th>% Changes</th>
<th>Q</th>
<th>Q_VMI</th>
<th>B</th>
<th>B_VMI</th>
<th>E[TCU]</th>
<th>E[TCU]_VMI</th>
</tr>
</thead>
<tbody>
<tr>
<td>K_b</td>
<td>-75</td>
<td>8106.420</td>
<td>8755.935</td>
<td>3249.998</td>
<td>3510.399</td>
<td>10977.484</td>
<td>10975.691</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>8106.420</td>
<td>9360.488</td>
<td>3249.998</td>
<td>3752.775</td>
<td>11024.035</td>
<td>11017.351</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>8106.420</td>
<td>9928.297</td>
<td>3249.998</td>
<td>3980.419</td>
<td>11070.586</td>
<td>11056.478</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>8106.420</td>
<td>10976.145</td>
<td>3249.998</td>
<td>4400.518</td>
<td>11163.687</td>
<td>11128.684</td>
</tr>
<tr>
<td></td>
<td>+50</td>
<td>8106.420</td>
<td>11464.210</td>
<td>3249.998</td>
<td>4596.191</td>
<td>11210.237</td>
<td>11162.316</td>
</tr>
<tr>
<td></td>
<td>+75</td>
<td>8106.420</td>
<td>11932.328</td>
<td>3249.998</td>
<td>4783.868</td>
<td>11256.788</td>
<td>11194.574</td>
</tr>
<tr>
<td>b</td>
<td>-75</td>
<td>18384.555</td>
<td>23734.358</td>
<td>9071.597</td>
<td>11711.381</td>
<td>10700.741</td>
<td>10690.312</td>
</tr>
<tr>
<td></td>
<td>-50</td>
<td>11449.213</td>
<td>14780.871</td>
<td>5245.919</td>
<td>6772.452</td>
<td>10899.676</td>
<td>10882.931</td>
</tr>
<tr>
<td></td>
<td>-25</td>
<td>9254.494</td>
<td>11947.500</td>
<td>3957.632</td>
<td>5109.280</td>
<td>11024.738</td>
<td>11004.022</td>
</tr>
<tr>
<td></td>
<td>+25</td>
<td>7383.697</td>
<td>9532.312</td>
<td>2786.114</td>
<td>3596.858</td>
<td>11190.039</td>
<td>11164.073</td>
</tr>
<tr>
<td></td>
<td>+50</td>
<td>6881.052</td>
<td>8833.399</td>
<td>2452.202</td>
<td>3165.779</td>
<td>11249.771</td>
<td>11221.908</td>
</tr>
<tr>
<td></td>
<td>+75</td>
<td>6508.750</td>
<td>8402.761</td>
<td>2197.445</td>
<td>2836.889</td>
<td>11299.961</td>
<td>11270.504</td>
</tr>
</tbody>
</table>

E\[x\] is increasing cumulatively at the rate of 0.02 in the interval [0.025,0.205]

According to the obtained results from Tables 3 and 4, the following achievements are shown in Table 5.

Table 5. The effects of changing parameters on optimal solutions

<table>
<thead>
<tr>
<th>Parameter change (From -75% to +75)</th>
<th>Effects of parameter change on optimal solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q</td>
<td>Case1</td>
</tr>
<tr>
<td></td>
<td>Q</td>
</tr>
<tr>
<td>C</td>
<td>None</td>
</tr>
<tr>
<td>C_s</td>
<td>None</td>
</tr>
<tr>
<td>h</td>
<td>Decreas</td>
</tr>
<tr>
<td>K_m</td>
<td>Increase</td>
</tr>
<tr>
<td>K_b</td>
<td>None</td>
</tr>
<tr>
<td>b</td>
<td>-</td>
</tr>
<tr>
<td>E(x)</td>
<td>Decreas</td>
</tr>
</tbody>
</table>
Considering the above mentioned results, VMI policy is more beneficial for both parties in the both models that is to say the expected total cost after implementation of VMI is less than the expected total cost in traditional supply chain. Besides, the optimal production quantity in VMI is greater than its quantity in the traditional policy.

5. Conclusions and future research

In this paper, the performance of the VMI policy in a two-echelon supply chain consisting of a manufacturer and a buyer is investigated regarding two conditions, with and without shortage. The production process generates randomly imperfect quality items at a constant production rate. The optimal production quantity and backordered level are obtained by minimizing total cost of the supply chain. The numerical example and sensitivity analysis have been performed to illustrate the differences in total cost and an optimal solution of both models. It is demonstrated that the VMI policy is more beneficial for both parties in the both models. Moreover, the optimal production quantity in VMI is greater than its quantity in the traditional policy.

There is much scope in extending the paper. For example, full backordering could consider partial backordering. Also, the levels of supply chain and the number of buyers could increase. In this case, we could investigate the impact of non-cooperative and cooperative relationships between the buyers.

References


EPQ model with scrap and backordering under…


