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Developing EOQ model with instantaneous deteriorating items for a vendor-managed inventory (VMI) system

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Abstract

This paper studies the economic-order-quantity model (EOQ) for deteriorating items in two cases (with and without shortages) to evaluate how vendor managed inventory (VMI) affects supply chain. We consider two-level supply chain (single supplier and a single retailer) with one instantaneous deteriorating item. A numerical example and sensitivity analysis are provided to illustrate the effect of related parameters on total cost and optimal order quantity of two systems. The results show that VMI works better and delivers lower cost in all conditions than traditional supply chain (the system before implementation of VMI).

Keywords: Vendor-managed inventory, Supply chain, Economic order quantity model (EOQ), Deterioration.

1. Introduction

The effect of deterioration is very important in many inventory systems. Deterioration is explained as decay or damage such that the item cannot be used for its original purpose. Most of the commodities experience decay or deterioration over time. In supply chain management, it is too difficult to protect highly volatile liquids, food stuff, gasoline, liquid medicines, etc., for all business sectors. Owing to this fact, how to control and preserve inventories of deteriorating items becomes a significant problem for decision makers in modern organization.

VMI is an inventory cooperation initiative in supply chain system. Under a VMI system, the vendor decides on the appropriate inventory levels for each product of itself and its retailers, and the proper inventory policies to maintain these levels (Simchi-Livi,Kaminsky and Simchi-Livi, 2008).

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The retailer makes its real-time inventory level accessible for the vendor. In fact in VMI system, the retailer’s role shifts from managing inventory to simply renting retailing space.

We organize the literature review section for both inventory models for deteriorating items and those for VMI systems. Ghare and Schrader (1963) were the pioneers of deteriorating inventory studies who examined the classical no-shortage inventory model with a constant rate of decay. Covert and Philip (1973) extended Ghare and Schrader’s constant deterioration rate to a two-parameter Weibull distribution. Philip (1974) then developed the inventory model with a three-parameter Weibull distribution rate and no shortages. Bahari-Kashani (1989) considered a replenishment schedule for deteriorating items with time-dependent demand. Goswami and Chaudhuri (1992) proposed a deterministic model in which they assumed deterioration is time-proportional and the replenishment rate is directly proportional to the time-dependent demand rate. Kim (1995) presented an inventory replenishment policy for deteriorating items with linearly increasing demand. Bhunia and Maiti (1998) presented an inventory model that assumed deterioration is a linearly increasing function of time. A detailed review of deteriorating inventory literatures is given in Raafat (1991) Goyal and Giri (2001). Lin, Tan and Lee (2000) examined the property of deterioration in EOQ model. They investigated the inventory replenishment policies for the cases with time-varying demand, linearly increasing deterioration rate, partial back-ordering, constant service level and equal replenishment intervals over a fixed planning horizon. Lin and Lin (2004) studied the joint inventory model between supplier and retailer relying on mutual cooperation, for dealing with more general cases they considered the deteriorated rate and partial back-ordering in their assumptions. Ghosh and Chaudhuri (2006) studied an EOQ model over a finite time horizon for a deteriorating item with a quadratic time-dependent demand, considering shortages in inventory. Sana (2010) investigated an EOQ model over an infinite time horizon for deteriorating items while the demand is price-sensitive, allowing partial backordering and time dependent deterioration rate. Khanra, Ghosh and Chuadhuri (2011) developed an EOQ model for a deteriorating item having time dependent demand when delay in payment is permissible. In their model the deterioration rate is assumed to be constant and the time varying demand rate is taken to be a quadratic function of time. Sicilia et al. (2014) studied a deterministic inventory system for items with a constant deterioration rate. In their model demand varies in time and it is assumed that it follows a power pattern. Shortages are allowed and backlogged. The ordering cost, the holding cost, the backlogging cost, the deteriorating cost, and the purchasing cost are considered in the inventory management. An approach is proposed to minimize the total cost per inventory cycle. Guchhait, Maiti and Maiti (2014) developed an inventory model of a deteriorating item with stock and selling price dependent demand under two-level credit period. The model is formulated as retailer’s profit maximization problem for both crisp and fuzzy inventory costs and solved using a modified Genetic Algorithm (MGA).

As a new concept, VMI can be traced back to the classical contribution of Magee(Magee and John, 1958). In recent years; various VMI models were widely studied by researchers. Dong and Xu (2002) presented an analytical model to evaluate the short-term and long term impact of VMI on supply chain profitability by analyzing the inventory systems of the parties involved. Yao, Evers and Dresner (2007) using the same assumptions as Dong with an additional assumption (the order quantity for the supplier is likely to be an integer multiple of the buyer's replenishment quantity) and explored how important supply chain parameters affect the cost savings to be realized from collaborative initiatives such as vendor-managed inventory (VMI). They then determined how the benefits were likely to be distributed between a buyer and a supplier in a supply chain. Ji, Shen and Wei (2008) focused on VMI's role as a strategy of integrated supply chain. This study helps to provide a better understanding of how important supply chain parameters, namely ordering costs and carrying charges, affect the inventory cost savings to be realized from VMI and the distribution of these savings between buyers and suppliers. Darwish and Odah (2010) develop a model for a supply chain with single vendor and multiple retailers under VMI mode of operation. Pasandideh, Akhavan

Niaki and Roozbehnia (2010)'s research is the most related one to our paper, they considered the retailer–supplier partnership through a VMI system and developed an analytical model to explore the effect of important supply chain parameters on the cost savings realized from collaborative initiatives. However, their model was an EOQ model with shortage, while our model assumes additional assumption of deterioration rate. In 2011 they developed an economic order quantity (EOQ) model first for a two-level supply chain system consisting of several products, one supplier and one-retailer, in which shortages are backordered, the supplier's warehouse has limited capacity and there is an upper bound on the number of orders. Since the model of the problem is of a non-linear integer-programming type, a genetic algorithm is then proposed to find the order quantities and the maximum backorder levels such that the total inventory cost of the supply chain is minimized(Pasandideh, Akhavan Niaki and Roozbehnia, 2011). C´ardenas-Barr'on, Trevi˜no-Graza and Wee (2012) presented an alternative heuristic algorithm to solve a multi-product, multi-constraint VMI system based on EOQ with backorders considering two classical backorder costs: linear and fixed. The literature review for deteriorating items and VMI system are summarized in the Tables 1 and 2.

Since the combination of the two respective research streams (inventory models for deteriorating items and those for VMI systems) is scarce, this paper aims to fill the gap and propose vendor-managed inventory (VMI) as one of the new and effective policies for managing inventories in supply chains for deteriorating items.

This paper is organized as follows: Section 2 presents assumption, notation, and the models with the optimal solution regarding shortage and without shortage. Numerical examples are presented in section 3 to analyze the influence of different parameters on the optimal economic order quantity and the total cost before and after implementation of VMI. Finally, conclusions and future research topics are presented in section 4.

Table 1. Deteriorating Items

Shortage				**Type of model**		**Time horizon**		**Deterioration rate**		**Demand rate**			**Paper**	**Reference Number**
Non-permissible	**Permissible**			**Production**	**Purchase**	**Infinite**	**Finite**	**Variable**	**Constant**	**Probabilistic**	**Deterministic**			
	Partial Backlogging	**Lost Sale**	**Backlogged**								**Variable**	**Constant**		
*					*		*		*			*	**Ghare and Schrader (1963)**	**1**
*					*		*	*				*	**Covert and Philip (1973)**	**2**
*					*		*	*				*	**Philip (1974)**	**3**
			*		*		*		*		*		**Bahari-Kashani (1989)**	**4**
			*		*		*	*			*		**Goswami and Chaudhuri (1992)**	**5**
			*		*		*		*		*		**Kim (1995)**	**6**
			*		*		*	*			*		**Bhunia and Maiti (1998)**	**7**
	*				*		*	*			*		**Lin, Tan and Lee (2000)**	**8**
	*				*		*		*		*		**Lin and Lin (2004)**	**9**
			*		*		*		*		*		**Ghosh and Chaudhuri (2006)**	**10**
	*				*		*	*			*		**Sana (2010)**	**11**
					*		*		*		*		**Khanra, Ghosh and Chuadhuri (2011)**	**12**
			*		*		*		*		*		**Et al. Sicilia (2014)**	**13**
			*		*		*		*		*		**Guchhait, Maiti and Maiti (2014)**	**14**

Table 2. VMI Systems

Shortage				Number of Production		Type of model		Time Horizon		Number of SC level			Lead Time			Demand Rate			Paper	Reference Number
	Permissible													Deterministic			Deterministic			
Non-Permissible	Partial Backlogging	Lost Sale	Backlogged	Multi	One	Production	Purchase	Infinite	Finite	Multi	Two	One	Probabilistic	Positive	Zero	Probabilistic	Variable	Constant		
*					*		*	*			*				*			*	Dong and Xu (2002)	1
*					*		*	*			*				*			*	Yao et al.(2007)	2
*					*		*	*		*					*			*	Odah and Darwish,(2010)	3
			*		*		*	*			*				*			*	Pasandideh, Akhavan Niaki and Roozbehnia,(2010)	4
			*	*			*	*			*				*			*	Pasandideh, Akhavan Niaki and Roozbehnia,(2011)	5
			*	*			*	*		*					*		*		C´ardenas-Barr'on, Trevi˜no-Graza and Wee, (2012)	6

2. Model structure

In this research, the problem of a single instantaneous deteriorating product under VMI policy is studied. We construct a two-level supply chain consisting of a single supplier and single retailer and examine the inventory management practices before and after implementation of VMI. We assume that the retailer faces external demand from consumers and investigate the model in two cases including a) shortage is not permitted and b) shortage is permitted and will be fully backordered.

The mathematical models are developed based on the following assumptions:

a) A single- supplier- single-buyer supply chain with one instantaneous deteriorating item is considered.
b) Deliveries of orders are assumed to be instantaneous, that is, the lead time is zero.
c) The retailer faces external demand from consumers; Costumer's demand is deterministic.
d) The production rate is infinite.
e) The product will be deteriorated with the fixed rate.
f) There is no repair or replacement of the deteriorated inventory during the period under consideration.

In addition, the following notations are used in model.

Q	The order quantity
Q_{VMI}	The order quantity in VMI policy
q	The constant rate of instantaneous deterioration
A_s	The supplier's ordering cost per order
A_B	The buyer's ordering cost per order
C	The deterioration cost per unit
D	The buyer's constant demand rate
h_B	The inventory holding cost held in buyer's store in a period per unit time
i	The holding cost rate ($h_b = iC < \beta C$)
b	The maximum level of backordering shortage
b_{VMI}	The maximum level of backordering in VMI system
p	The fixed cost of shortage per unit
$\hat{p}$	The cost of shortage per unit per time
T	The time cycle before VMI
T_{VMI}	The time cycle after VMI
F	The percentage of cycle length in which inventory is positive
KB_{0i}	The buyer's inventory cost before VMI in case i
KB_{1i}	The buyer's inventory cost after VMI in case i
KS_{0i}	The supplier's inventory cost before VMI in case i
Ks_{1i}	The supplier's inventory cost afterVMI in case i
TC	The total cost before VMI
TC_{VMI}	The total cost of VMI system

2.1. Case 1. Shortage is not permitted

A pictorial description of the inventory policy without shortage is given in Figure 1.

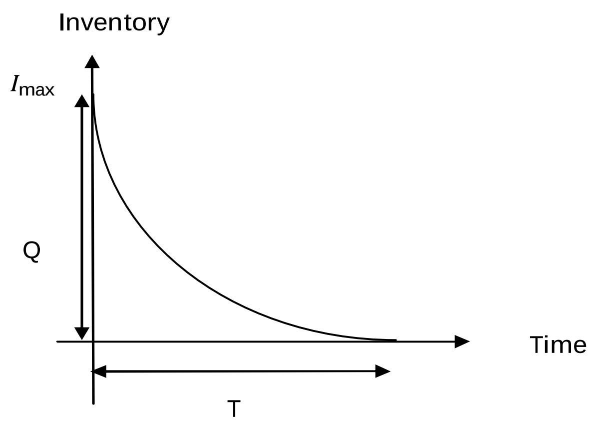

Figure 1. The EOQ model for instantaneous deteriorating items

The inventory level is dropping to zero because of demand and deterioration. So differential equation shown in equation (1) shows the changing the inventory level during $[0,T]$.

$$\frac{dI_1(t)}{dt}=-\theta(t)I_1(t)-D \quad 0\le t\le T \tag{1}$$

$$I_1(t)=e^{-\int_0^t \beta dt}\left[\int_t^T D e^{\int_0^t \beta dt}\, dt\right]=\frac{D}{\beta}\left(e^{\beta(T-t)}-1\right) \tag{2}$$

From the Figure 1, since $I(T)=0$ & $I(0)=I_{max}=Q$ we have;

$$\left(I(T)=0, I(0)=I_{max}=Q\right)\Rightarrow Q=\frac{D\left(e^{\beta T}-1\right)}{\beta} \tag{3}$$

The total inventory system cost for the cycle time T is made up of the buyer’s ordering cost, supplier’s ordering cost, product’s carrying cost that held in buyer’s store in a period and deterioration cost. The buyer’s holding cost, is:

$$h_B\int_0^T I_1(t)dt=h_B\int_0^T \frac{D\left(e^{\beta(T-t)}-1\right)}{\beta}dt$$
$$=h_B\frac{D}{\beta^2}\left(e^{\beta T}-1\right)-h_B\frac{DT}{\beta} \tag{4}$$

Moreover the fixed cost of buyer is A_B and the deterioration cost will be;

$$C(Q-DT)=CD\left(\frac{\left(e^{\beta T}-1\right)}{\beta}-T\right) \tag{5}$$

2.1.1. Analysis of inventory costs

So prior to implementing VMI, the total cost of the buyer and the total cost of supplier are respectively shown in equation (6) and (7). Then the total cost of the chain is shown in equation (8).

$$KB_{01}=\frac{1}{T}\left(A_B+h_B\int_0^T I_1(t)dt+CD\left(\frac{\left(e^{\beta T}-1\right)}{\beta}-T\right)\right) \tag{6}$$

$$KS_{01}=\frac{A_S}{T} \tag{7}$$

$$TC=KB_{01}+KS_{01}=\frac{1}{T}\left(\begin{array}{l}A_B+A_S+CD\left(\frac{\left(e^{\beta T}-1\right)}{\beta}-T\right)\\+h_B\frac{D}{\beta^2}\left(e^{\beta T}-1-\beta T\right)\end{array}\right) \tag{8}$$

Using approximation of the Taylor series expansion $e^{\beta T}=1+\beta T+\frac{\beta^2T^2}{2}$ we have;

$$KB_{01}=\frac{1}{T}\left(A_B+h_B\frac{D}{\beta^2}\left(\frac{\beta^2T^2}{2}\right)+\frac{CD\beta T^2}{2}\right) \tag{9}$$

$$KS_{01}=\frac{A_S}{T} \tag{10}$$

$$TC=KB_{01}+KS_{01}$$

$$=\frac{1}{T}\left(A_S+A_B+h_B\frac{D}{\beta^2}\left(\frac{\beta^2T^2}{2}\right)+\frac{CD\beta T^2}{2}\right) \tag{11}$$

$$=\frac{\left(A_S+A_B\right)}{T}+\frac{h_BDT}{2}+\frac{CD\beta T}{2}$$

Since $\frac{\partial^2 KB_{01}}{\partial T^2}=\frac{2A_B}{T^3}>0$, the total cost function is convex. So the optimal value of T can be obtained by setting the first derivative of total cost function respect to T equal to zero yielding;

$$T=\sqrt{\frac{2A_B}{h_BD+CD\beta}} \tag{12}$$

Therefore;

$$Q=DT\left(1+\frac{\beta T}{2}\right) \tag{13}$$

But under VMI policy, since supplier should pay the buyer costs, then equation (6) to (8) will change to;

$$KB_{11}=0 \tag{14}$$

$$KS_{11}=\frac{1}{T_{VMI}}\left(\begin{array}{l}A_B+A_S+h_B\frac{D}{\beta^2}\left(e^{\beta T_{VMI}}-1-\beta T_{VMI}\right)\\+CD\left(\frac{\left(e^{\beta T_{VMI}}-1\right)}{\beta}-T_{VMI}\right)\end{array}\right) \tag{15}$$

$$TC_{VMI}=KB_{11}+KS_{11}$$

$$=\frac{1}{T_{VMI}}\left(\begin{array}{l}A_B+A_S+h_B\frac{D}{\beta^2}\left(e^{\beta T_{VMI}}-1-\beta T_{VMI}\right)\\+CD\left(\frac{\left(e^{\beta T_{VMI}}-1\right)}{\beta}-T_{VMI}\right)\end{array}\right) \tag{16}$$

In this section we use again from approximation of the Taylor series expansion for computational ease.
We can rewrite above formulas as follow:

$$KB_{11}=0 \tag{17}$$

$$KS_{11}=\frac{1}{T_{VMI}}\left(A_S+A_B+h_B\frac{D}{\beta^2}\left(\frac{\beta^2T_{VMI}^2}{2}\right)+\frac{CD\beta T_{VMI}^2}{2}\right) \tag{18}$$

$$\begin{aligned}TC_{VMI}&=KB_{11}+KS_{11}\\&=\frac{(A_S+A_B)}{T_{VMI}}+\frac{h_BDT_{VMI}}{2}+\frac{CD\beta T_{VMI}}{2}\end{aligned} \tag{19}$$

$\frac{\partial^2 TC_{VMI}}{\partial T_{VMI}^2}=\frac{2(A_S+A_B)}{T_{VMI}^3}>0$ Once again, since $\frac{\partial^2 TC_{VMI}}{\partial T_{VMI}^2}=\frac{2(A_S+A_B)}{T_{VMI}^3}>0$, the total inventory cost of the integrated supply chain shown in equation (19) is convex. So the optimal value of T can be obtained by setting the first derivative of equation (19) respect to T equal to zero yielding;

$$T_{VMI}=\sqrt{\frac{2(A_S+A_B)}{h_BD+CD\beta}} \tag{20}$$

And the order quantity in VMI policy can be determined as below.

$$Q_{VMI}=DT_{VMI}\left(1+\frac{\beta T_{VMI}}{2}\right) \tag{21}$$

2.2. Case 2. Shortage is permitted

In this section, we develop the previous model with an additional assumption that shortage is permitted and will be fully backordered $(\hat{\pi}\neq 0,\pi=0)$. A pictorial description of the inventory policy with shortage is given in Figure 2.

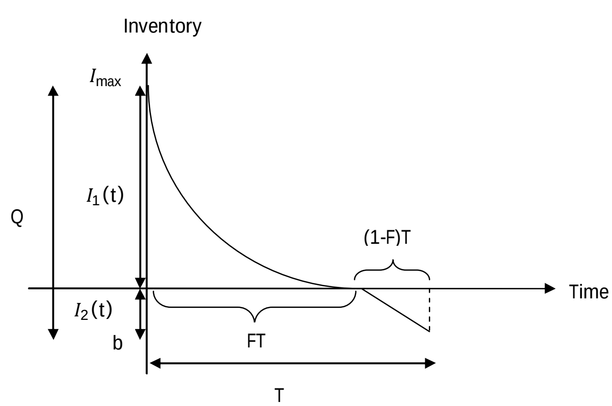

Figure 2. The EOQ model for instantaneous deteriorating items with shortage

During the time interval $[0,FT]$ the inventory level is dropping to zero because of demand and deterioration. So differential equation shown in equation (22) shows the changing the inventory level during $[0,FT]$.

$$\frac{dI_1(t)}{dt}=-\theta(t)I_1(t)-D \quad 0 \le t \le FT \tag{22}$$

Therefore we have;

$$\frac{D}{\beta}\left(e^{\beta(FT-t)}-1\right) \tag{23}$$

Furthermore, at time FT, shortage occurs and the inventory level starts dropping below 0.

$$FT \le t \le T \quad I_2(t)=\frac{b(1-F)T}{2} \tag{24}$$

$$(1-F)T=\frac{b}{D}Þb=D(1-F)T \tag{25}$$

Therefore we have;

$$I_2(t)=\frac{D(1-F)^2T^2}{2} \tag{26}$$

From the Figure 2, since $I(FT)=0$ and $I(0)=I_{max}$, we have;

$$\left(I(FT)=0, I(0)=I_{max}\right) \Rightarrow I_{max}=\frac{D\left(e^{\beta FT}-1\right)}{\beta} \tag{27}$$

From the Figure 2, since $Q=I_{max}+b$ we have;

$$Q=I_{max}+b=\frac{D\left(e^{\beta FT}-1\right)}{\beta}+(1-F)TD \tag{28}$$

The total system cost for the cycle time T is made up of the buyer's ordering cost, supplier's ordering cost, product's carrying cost that held in buyer's store in a period, deterioration cost and cost of shortage. The buyer's holding cost is:

$$h_B\int_0^{FT} I_1(t)dt=h_B\int_0^{FT}\frac{D\left(e^{\beta(FT-t)}-1\right)}{\beta}dt \\ =h_B\frac{D}{\beta^2}\left(e^{\beta FT}-1-\beta FT\right) \tag{29}$$

Moreover the fixed cost of buyer is A_B and the deterioration cost will be;

$$C\left(I_{max}-DFT\right)=C\frac{D\left(e^{\beta FT}-1-FT\beta\right)}{\beta} \tag{30}$$

And the Shortage cost is:

$$\hat{\pi}\times I_2(t)=\frac{\hat{\pi}D(1-F)^2T^2}{2} \tag{31}$$

2.2.1. Analysis of inventory costs

So prior to implementing VMI, the total cost of the buyer and the total cost of supplier are respectively shown in equation (32) and (33). Then the total cost of the chain is shown in equation (34).

$$KB_{02}=\frac{1}{T}\left(\begin{array}{l} A_B+h_B\int_0^{FT} I_1(t)dt+\frac{\hat{\pi}D(1-F)^2T^2}{2} \\ +\frac{CD\left(e^{\beta FT}-1-FT\beta\right)}{\beta}\end{array}\right) \tag{32}$$

$$KS_{02}=\frac{A_S}{T} \tag{33}$$

$$TC=KB_{02}+KS_{02} \\ =\frac{1}{T}\left(\begin{array}{l} A_B+A_S+h_B\frac{D}{\beta^2}\left(e^{\beta FT}-1-\beta FT\right)+ \\ \frac{CD\left(e^{\beta FT}-1-FT\beta\right)}{\beta}+\frac{\hat{\pi}D(1-F)^2T^2}{2}\end{array}\right) \tag{34}$$

Using approximation of the Taylor series expansion $e^{\beta FT}=1+\beta FT+\frac{\beta^2F^2T^2}{2}$ we have;

$$KB_{02}=\frac{1}{T}\left(\begin{array}{l}A_B+\frac{h_B DF^2T^2}{2}+\frac{CD\beta F^2T^2}{2}\\+\frac{\hat{\pi}D(1-F)^2T^2}{2}\end{array}\right) \tag{35}$$

$$KS_{02}=\frac{A_S}{T} \tag{36}$$

$$TC=KB_{02}+KS_{02}=\frac{1}{T}\left(\begin{array}{l}A_S+A_B+\frac{h_B DF^2T^2}{2}+\\\frac{CD\beta F^2T^2}{2}+\frac{\hat{\pi}D(1-F)^2T^2}{2}\end{array}\right) \tag{37}$$

The buyer's inventory cost in Eq.35 is a function of T and F. So, global optimal value of T and F can be obtained by taking the partial derivative of Eq.35 respect to T and F, then setting them equal to zero (in Appendix we prove that T* and F* give a global optimal solution for the EOQ with instantaneous deteriorating items and shortage).

$$\frac{\partial KB_{02}}{\partial T}=\frac{-A_B}{T^2}+\frac{h_B DF^2}{2}+\frac{CD\beta F^2}{2}+\frac{\hat{\pi}D(1-F)^2}{2} \tag{38}$$

$$\frac{\partial KB_{02}}{\partial T}=0\Rightarrow$$
$$T^*(F)=\sqrt{\frac{2A_B}{DF^2(h_B+C\beta)+\hat{\pi}D(1-F)^2}} \tag{39}$$

$$\frac{\partial KB_{02}}{\partial F}=h_B DFT+CD\beta FT-\hat{\pi}D(1-F)T \tag{40}$$

$$\frac{\partial KB_{02}}{\partial F}=0\Rightarrow F^*=\frac{\hat{\pi}}{h_B+C\beta+\hat{\pi}} \tag{41}$$

Substituting F^* into $T^*(F)$, yields;

$$T^*=\sqrt{\frac{2A_B}{h_B D+C\beta D}}\sqrt{\frac{h_B+C\beta+\hat{\pi}}{\hat{\pi}}} \tag{42}$$

Considering Eq. 25, amount of shortage before VMI can be calculated as follow:

$$b=(1-F)TD \tag{43}$$

Considering Eq.35 and Eq.42, the ordering quantity over the cycle before VMI can be determined as:

$$Q=I_{max}+b=D\left(FT+\frac{\beta F^2T^2}{2}\right)+(1-F)TD \tag{44}$$

But under VMI policy, since supplier should pay the buyer costs, then equation (32) to (34) will change to;

$$KB_{12}=0 \tag{45}$$

$$KS_{12}=\frac{1}{T_{VMI}}\left(\begin{array}{l}A_B+A_S+h_B\frac{D}{\beta^2}\left(e^{\beta FT_{VMI}}-1-\beta FT_{VMI}\right)\\+\frac{CD\left(e^{\beta FT_{VMI}}-1-FT_{VMI}\beta\right)}{\beta}\\+\frac{\hat{\pi}D(1-F)^2T_{VMI}^2}{2}\end{array}\right) \tag{46}$$

$$TC_{VMI}=KB_{12}+KS_{12}=\frac{1}{T_{VMI}}\left(\begin{array}{l}A_B+A_S+\frac{\hat{\pi}D(1-F)^2T_{VMI}^2}{2}+\\h_B\frac{D}{\beta^2}\left(e^{\beta FT_{VMI}}-1-\beta FT_{VMI}\right)\\+\frac{CD\left(e^{\beta FT_{VMI}}-1-FT_{VMI}\beta\right)}{\beta}\end{array}\right) \tag{47}$$

Using approximation of the Taylor series expansion equations (46) and (47) will change to;

$$KS_{12}=\frac{1}{T_{VMI}}\left(\begin{array}{l}A_S+A_B+\frac{h_BDF^2T_{VMI}^2}{2}+\\\frac{CD\beta F^2T_{VMI}^2}{2}+\frac{\hat{\pi}D(1-F)^2T_{VMI}^2}{2}\end{array}\right) \tag{48}$$

$$TC_{VMI}=KB_{12}+KS_{12}=\frac{1}{T_{VMI}}\left(\begin{array}{l}A_S+A_B+\frac{\hat{\pi}D(1-F)^2T_{VMI}^2}{2}\\+\frac{h_BDF^2T_{VMI}^2}{2}+\frac{CD\beta F^2T_{VMI}^2}{2}\end{array}\right) \tag{49}$$

Once again, since the total inventory cost in Eq.49 is a function of T and F. So, global optimal value of T and F can be obtained by taking the partial derivative of equation 49 respect to T and F, then setting them equal to zero (as we mentioned in pervious section in Appendix we prove that T* and F* give a global optimal solution for the EOQ with non-instantaneous deteriorating items and shortage).

$$\frac{\partial TC_{VMI}}{\partial T_{VMI}}=\frac{-(A_B+A_S)}{T_{VMI}^2}+\frac{(h_B+C\beta)}{2}DF^2+\frac{\hat{\pi}D(1-F)^2}{2} \quad (50)$$

$$\frac{\partial TC_{VMI}}{\partial T_{VMI}}=0 \Rightarrow T_{VMI}^*(F)=\sqrt{\frac{2(A_B+A_S)}{DF^2(h_B+C\beta)+\hat{\pi}D(1-F)^2}} \quad (51)$$

$$\frac{\partial C_{VMI}}{\partial F}=\left(h_BF+C\beta F-\hat{\pi}(1-F)\right)DT_{VMI} \quad (52)$$

$$\frac{\partial TC_{VMI}}{\partial F}=0 \Rightarrow F^*=\frac{\hat{\pi}}{h_B+C\beta+\hat{\pi}} \quad (53)$$

Substituting F_{VMI}^* into $T_{VMI}^*(F)$, yields;

$$T_{VMI}^*=\sqrt{\frac{2(A_B+A_S)}{h_BD+C\beta D}}\sqrt{\frac{h_B+C\beta+\hat{\pi}}{\hat{\pi}}} \quad (54)$$

Amount of shortage after VMI can be calculated as follow:

$$b_{VMI}=(1-F_{VMI})T_{VMI}D \quad (55)$$

The ordering quantity over the cycle after VMI can be determined as:

$$Q_{VMI}=I_{max}+b_{VMI}=D\left(F_{VMI}T_{VMI}+\frac{\beta F_{VMI}^2T_{VMI}^2}{2}\right)+(1-F_{VMI})T_{VMI}D \quad (56)$$

3. Numerical example and sensitivity analysis

In order to illustrate above solution procedure, let us consider an inventory system with the following data:

Table 3. General data

A_S	A_B	h_B	D	C	β
150	45	90	10000	1000	0.005

We consider this example for both models. However, in the second model shortage is allowed and completely backlogged $(\hat{\pi}=80, \pi=0)$. We solve the example by MATLAB and get following results.

Table 4. The optimal values of decision variables

case	Q_{NoVMI}	Q_{VMI}	TC_{NoVMI}	TC_{VMI}
1	97.3352	202.6248	24658	19248
2	143.9583	299.6756	16672	13014

We now study the effect of changes in the system parameters $A_S, A_B, h_B, D, C, \beta$ for both models and $\hat{\pi}$ just for second model on the optimal order quantity per cycle Q and the total relevant inventory cost per unite before and after implementation of VMI policy. The sensitivity analysis is performed by changing the parameters A_S, A_B, h_B, D, C by +75%, +50%, +25%, -25%, -50% and -75% , and increasing β cumulatively at the rate of 0.05 in the interval [0.005,0.5], taking one parameter at a time and keeping the remaining parameters unchanged. The results are shown in Tables 5 and 6.

Table 5. Effect of changes in various parameters of the model in case 1

Item	% changes	Q_{NoVMI}	Q_{VMI}	TC_{NoVMI}	TC_{VMI}
A_S	-75	97.3352	131.7936	13099	12520
	-50	97.3352	158.9502	16952	15100
	-25	97.3352	182.1014	20805	17299
	+25	97.3352	221.2528	28510	21018
	+50	97.3352	238.4300	32363	22650
	+75	97.3352	254.4505	36216	24171
A_B	-75	48.6670	184.2566	35445	17504
	-50	68.8259	190.5761	28333	18104
	-25	84.2945	196.6927	25803	18685
	+25	108.8244	208.3880	24122	19796
	+50	119.2115	213.9962	23908	20329
	+75	128.7634	219.4611	23882	20848
h_B	-75	180.9150	376.6230	13266	10356
	-50	134.1686	279.3043	17889	13964
	-25	111.4203	231.9469	21541	16815
	+25	87.5209	182.1936	27423	21407
	+50	80.1800	166.9116	29933	23367
	+75	74.4222	154.9253	32249	25174
C	-75	99.3152	206.7466	24166	18865
	-50	98.6418	205.3448	24331	18993
	-25	97.9820	203.9712	24495	19121
	+25	96.7011	201.3046	24819	19375
	+50	96.0792	200.0100	24980	19500
	+75	95.4692	198.7400	25140	19625
β is increasing cumulatively at the rate of 0.05 in the interval [0.005,0.5].					
β	0.055	78.8009	164.0756	30463	23780
	0.105	67.9609	141.5264	35327	27577
	0.155	60.6376	126.2914	39598	30911
	0.205	55.2657	115.1152	43451	33919
	0.255	51.1087	106.4660	46989	36681
	0.305	47.7682	99.5156	50279	39249
	0.355	45.0078	93.7720	53367	41659
	0.405	42.6770	88.9221	56285	43937
	0.455	40.6747	84.7557	59059	46103

Table 6. Effect of changes in various parameters of the model in case 2

Item	% Changes	Q_{NoVMI}	Q_{VMI}	TC_{NoVMI}	TC_{VMI}
A_S	-75	143.9583	194.9210	8856.8	8465.1
	-50	143.9583	235.0841	11462	10209
	-25	143.9583	269.3231	14067	11696
	+25	143.9583	327.2247	19277	14211
	+50	143.9583	352.6283	21881	15314
	+75	143.9583	376.3209	24486	16343
A_B	-75	71.9789	272.5105	23965	11835
	-50	101.7937	281.8566	19156	12240
	-25	124.6714	290.9025	17446	12633
	+25	160.9504	308.1990	16310	13384
	+50	176.3125	316.4929	16165	13745
	+75	190.4394	324.5751	16147	14096
h_B	-75	209.7137	436.5675	11445	8933.9
	-50	171.0291	356.0317	14033	10954
	-25	153.8320	320.2306	15602	12179
	+25	137.5129	286.2576	17453	13624
	+50	132.9614	276.7825	18050	14091
	+75	129.5708	269.7241	18523	14459
C	-75	145.3043	302.4777	16517	12894
	-50	144.8449	301.5213	16570	12935
	-25	144.3964	300.5876	16621	12975
	+25	143.5303	298.7846	16721	13053
	+50	143.1121	297.9139	16770	13091
	+75	142.7032	297.0628	16818	13129
$\hat{\pi}$	-75	233.3964	485.8543	10283	8027.1
	-50	178.8126	372.2296	13422	10477
	-25	156.4416	325.6612	15341	11976
	+25	135.9191	282.9410	17658	13784
	+50	130.2844	271.2116	18421	14380
	+75	126.1056	262.5128	19032	14857
β is increasing cumulatively at the rate of 0.05 in the interval [0.005,0.5].					
β	0.055	132.1306	275.0655	18165	14180
	0.105	125.9649	262.2328	19054	14874
	0.155	122.1687	254.3301	19646	15336
	0.205	119.5928	248.9673	20069	15667
	0.255	117.7292	245.0870	20387	15914
	0.305	116.3179	242.1481	20634	16108
	0.355	115.2117	239.8446	20832	16262
	0.405	114.3213	237.9902	20994	16389
	0.455	113.5890	236.4653	21130	16494

Regarding the results obtained from tables 3 and 4, the following analysis are fulfilled:

(a) Increasing supplier’s ordering cost causes no effects on optimal order quantity in both models before implementation of VMI. But, it leads to rise in the optimal order quantity after implementation of VMI. Moreover, increasing supplier’s ordering cost leads to increase of inventory total costs having sharp increase in traditional supply chain. Inventory total costs before and after VMI are close to each other for lower order quantities. But in general, inventory total costs after implementation of

VMI are lower than traditional supply chain and this gap will grow by increasing supplier’s ordering cost.

(b) Increasing buyer’s ordering cost in both models leads to increase in optimal order quantity having greater slope in traditional supply chain without VMI. In general, optimal order quantity after VMI is greater than it’s quantity before VMI. Moreover, inventory total cost before VMI has downward trend increasing buyer’s ordering cost, make this trend slighter. However, increasing buyer’s ordering cost leads to increase in inventory total cost after VMI implementation. In general, inventory total costs after implementation of VMI are lower than traditional supply chain.

(c) Increasing buyer’s holding cost in both models eventuate decline in optimal order quantity. But, optimal order quantity is greater after implementing VMI policy. Furthermore, increasing buyer’s holding cost leads to increase in inventory total cost before and after VMI policy. It is obvious that inventory total costs in traditional supply chain are greater than VMI supply chain and this deviation will grow by increasing buyer’s holding cost.

(d) Increasing buyer’s demand rate in both before and after VMI policy, leads to increase in optimal order quantity having greater slope in VMI supply chain. However, optimal order quantity after implementing VMI policy is higher than before it. This distance will grow by increasing demand rate. Moreover, increasing buyer’s demand rate in both before and after VMI policy, leads to increase in inventory total costs. As it observed in those tables, inventory total costs after implementation of VMI are lower than traditional supply chain.

(e) Increasing deterioration cost has a slight effect on optimal order quantity in traditional supply chain and VMI supply chain. As it obvious in the tables, optimal order quantity in VMI supply chain is significantly greater than traditional supply chain. Furthermore, increasing deterioration cost in both before and after VMI policy, eventuates increase in inventory total costs. Inventory total costs after VMI are much less than before VMI implementation.

(f) Increasing deterioration rate leads to decline in optimal order quantity of both models. However, optimal order quantity after VMI policy is greater than the quantity of before it. Furthermore, increasing deterioration rate leads to increase in inventory total cost before and after VMI policy. It is clear that inventory total costs in VMI supply chain are lower than traditional supply chain and this distance will grow with increasing deterioration rate.

(g) According to Table 6, increasing shortage cost eventuate decline in optimal order quantity. But, optimal order quantity after VMI implementation is higher than the before VMI quantity. Furthermore, increasing shortage cost leads to increase in inventory total cost before and after VMI policy. In general, inventory total costs in traditional supply chain are higher than VMI supply chain and this distance will grow with increasing shortage cost.

4. Conclusions and future research

In this paper, we have considered a two-level supply chain for the EOQ model with single instantaneous deteriorating item to evaluate the performance of the VMI system. The total inventory costs and optimal order quantities have been derived as the performance measures. A numerical example and sensitivity analysis have been provided to illustrate the difference in total cost and optimal order quantity of both systems. It has been demonstrated that the VMI system is more beneficial for the coordination system and delivers lower cost in all conditions including back order. Furthermore, optimal order quantity in all conditions in VMI supply chain is greater than it’s quantity in traditional supply chain.

There are a number of directions for future research. For instance, in this study we assumed two-level supply chain with a single deterioration item, while the new model in which one supplier faces two or more buyers could be focused. The model in which the lost sales are considered and also shortage is partial backordering rather than completely backordering could be also investigated.

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Appendix A. proof of the optimality of the solution (41), (42), (53) and (54)

Although the cost function in (35), (49) is not convex, we can prove that (41), (42), (53) and (54) are global optimal using Pentico and Drake (2009)'s method. We can rewrite the cost function as follow:

$$TC=\frac{G_0}{T}+T\left(\left(G_1+G_2+G_3\right)F^2-2G_3F+G_3\right) \tag{A1}$$

Where:

$$A=G_0;(NoVMI(35):A=A_B;\ VMI(49):A=A_B+A_S) \tag{A2}$$

$$\frac{h_B D}{2}=G_1 \tag{A3}$$

$$\frac{CD\beta}{2}=G_2 \tag{A4}$$

$$\frac{\hat{\pi}D}{2}=G_3 \tag{A5}$$

Note that all the G_i s are positive and $\left(G_2>G_1, C\beta>h_B\right)$
For ease of notation, we can rewrite (A1) as:

$$TC=TC(F,T)=\frac{G_0}{T}+Tr(F) \tag{A6}$$

Where:

$$r(F)=\left(G_1+G_2+G_3\right)F^2-2G_3F+G_3 \tag{A7}$$

Our objective is to establish the condition under which equation (A6) has a unique interior minimize. Differentiating (A6) with respect to T yields:

$$\frac{\partial TC}{\partial T}=-\frac{G_0}{T^2}+r(F) \tag{A8}$$

Which equals zero if and only if T satisfies:

$$T=T^*(F)=\sqrt{\frac{G_0}{r(F)}} \tag{A9}$$

Note that this is the same result, with appropriate change of notation, gives in (39) and (51).
Since the discriminant of $r(F)$ is negative, $r(F)$ has no roots. Thus, $r(F)$ is either all positive or all negative. Since $r(0)=G_3>0$, $r(F)$ is strictly positive in $[0,1]$. Thus, (A9) gives, for each F, a unique $T^*=T^*(F)$ that minimizes the cost function given by (A6). Substituting the expression for $T^*(F)$ in (A9) into $TC(F,T)$ given by (A6) gives:

$$TC(F) \equiv TC\left(T^*(F),F\right)=2\sqrt{G_0 r(F)} \tag{A10}$$

Which represent that minimal possible cost for each value of F.
Note that $TC(F)$ is continuous, so on the compact interval $[0,1]$ it has one or more local minima, the smallest of which will be the global minimum of the cost function. To find these minima, take the first and second derivatives of $TC(F)$ with respect to F, yielding:

$$TC'(F)=\sqrt{G_0}\,\frac{r'(F)}{\sqrt{r(F)}} \tag{A11}$$

$$TC''(F)=\frac{\sqrt{G_0}\left[2r''(F)r(F)-\left(r'(F)\right)^2\right]}{2r(F)^{\frac{3}{2}}} \tag{A12}$$

Note that $TC'(F)$, which is, with the change in notation, the same as $\frac{\partial TC(F,T)}{\partial F}$ as given in (40) and (52) is continuous and satisfies $TC'(0)<0$:

$$r'(F)=2\left(-G_1+G_2+G_3\right)F-2G_3 \tag{A13}$$

$$r''(F)=2\left(-G_1+G_2+G_3\right) \tag{A14}$$

$$TC'(0)=\sqrt{G_0}\,\frac{-2G_3}{\sqrt{G_3}}<0 \tag{A15}$$

The second derivative $TC''(F)$ given in (A12) factors into:

$$TC''(F)=\frac{2G_3\sqrt{G_0}\left[G_2-G_1\right]}{r(F)^{\frac{3}{2}}} \tag{A16}$$

Which is positive for all F.