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A Queueing-Inventory System with Repair Center for Defective Items and One-for-One Ordering Policy

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ABSTRACT

In this paper we consider a system consisting of a supplier with a single processing unit, a repair center, and a retailer with Poisson demand. We assume that the retailer applies one-for-one ordering policy with backorders for his inventory control. The retailer's orders form a queue in the supplier processing unit. We also assume that a certain fraction of the products produced by the supplier are defective and they must be repaired in the repair center before going to the retailer. Further, we assume that the processing time of each unit at the supplier and the service time of each defective item in the repair center are exponentially distributed random variables with known means.
The purpose of this paper is to obtain the optimal value of the inventory position of the retailer which minimizes the total cost of the system. To achieve this purpose we consider two cases, Case (1) the ratio of the arrival rate to service rate, at the supplier and at the repair center are not equal and Case (2) these ratios are equal. For both cases, we first derive the long run probability distribution of the number of outstanding orders of the retailer. Then we obtain the average on hand inventory and backorders of the retailer, and derive the long run unit total cost of the system. We also investigate the convexity of this total system cost function and obtain the optimal value of the inventory position of the retailer and present a numerical example.

***Keywords*:** One-for-one ordering, Defectives, Inventory control, Poisson demand.

1. INTRODUCTION

Over the last decade research on complex integrated production-inventory systems or service-inventory systems has attracted the attention of the researchers, often in connection with the research on integrated supply chain management. Haji *et al*.(2011) considered a two echelon inventory system with one supplier and one retailer with Poisson demand. Both supplier and retailer apply base stock policy. The retailer's order joins the queue at the service unit. They assumed that the demand will be lost if the supplier has no on-hand inventory. They showed that the steady state joint distributions of the "queue length" and "on-hand inventory of the supplier has a product form

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and obtainedthe optimal value of the inventory position of the retailer which minimizes the total system cost. He *et al.*, (2002) derived optimal inventory policy for a make-to-order inventory-production system with Poisson arrival process of demands, exponentially distributed processing times and zero replenishment lead times of raw material. Wang *et al*., (2000) considered a two-echelon repairable inventory system consisting of a central depot and multiple stocking centers In their model they applied one for one ordering policy for inventory control of centers, They assumed that centers receive defective items and pass them to the depot for repair. They also assumed that depot replenishment lead times are different across stocking centers. They investigated the impact of such assumption on system performance and derived probability distributions of the random delays at the depot experienced by center replenishment orders. Berman and Kim (2001) considered a service system with an attached inventory, with Poisson demand, exponential service times, and Erlang distribution of replenishment lead times. They formulated model as a Markov decision problem to characterize an optimal inventory policy as a monotonic threshold structure which minimizes system costs.

Schwarz et al, (2006) considered various M/M/1-systems with inventory under different continuous review inventory management policies. For the case of lost sales, Poisson demand, and exponentially distributed service and lead times they derived stationary distributions of joint queue length and inventory processes in explicit product form and calculated performance measures of the respective systems. A similar model for the backordering case was considered by Schwarz and Daduna (2006).They computed performance measures, presented an approximation scheme for it, and derived optimality conditions under different order policies. Zhao and Lian (2011) considered a queueing-inventory system in which two classes of customers arrive at a service facility according to Poisson processes and service times and supplier lead times follow exponential distributions. They used a priority service rule to minimize the long-run expected waiting cost by dynamic programming method and obtained the necessary and sufficient condition for stability of the priority queueing- inventory system. Liu *et. al.*, (2004) considered a multistage queueing-inventory model and they decomposed the model into multiple single-stage inventory queues. Using this approach they provide accurate performance estimates and solved an optimization problem that minimizes the total inventory cost subject to a required service level. Olsson and Hill (2006) considered a two-echelon inventory system with one supplier, M independent retailers, and. Poisson demand. They assumed that each retailer applies base stock policy with backorders and the supplier's production time is fixed. They obtained the performance characteristics at each retailer and proposed two alternative approximation procedures based on order lead time distribution.

In this study we consider a queueing-inventory system consisting of a supplier with a single processing unit, one repair center with a single server, one retailer, and Poisson demand. The retailer applies one for one ordering policy for his inventory control. We also assume that a certain fraction of the produced items are defective and must be repaired at the repair center before going to the retailer. Further, we assume that the processing time at both supplier and repair center are exponential and unsatisfied demand will be backordered. We consider two cases for the traffic intensities, the ratio of the arrival rate to service rate, at the supplier and at the repair center: Case (1) traffic intensities are unequal and Case (2) they are equal. For both cases we derive the long run probability of total outstanding orders of the retailer. We also obtain the average number of backorders of the retailer, and derive the long run unit total cost of the system consisting of holding and backordering costs of retailer, processing cost of supplier, waiting time cost of defective items in the repair center, and repair cost of the repair center. We then investigate the convexity of the total system cost function and obtain the optimal value of the inventory position of the retailer which minimizes this total system cost.

2. THE MODEL

This paper deals with a system consisting of one supplier with a single processing unit, a retailer, and a repair center with a single server. The retailer faces a Poisson demand with rate λ and his ordering policy is one for one, (R-1, R), policy, that is, as soon as a demand arrives he orders a unit to the supplier. When a demand arrives and the retailer is out of stock this demand will be backordered.Supplier processing time and repair center service time are both exponentially distributed with parameter μ_2 .and μ_1 respectively. A certain fraction,α, of the items produced by the supplier are defective. Upon arrival of a defective item to the retailer, he sends it to the repair center for repair. Each item after repair will go immediately to the retailer (see Figure 1).

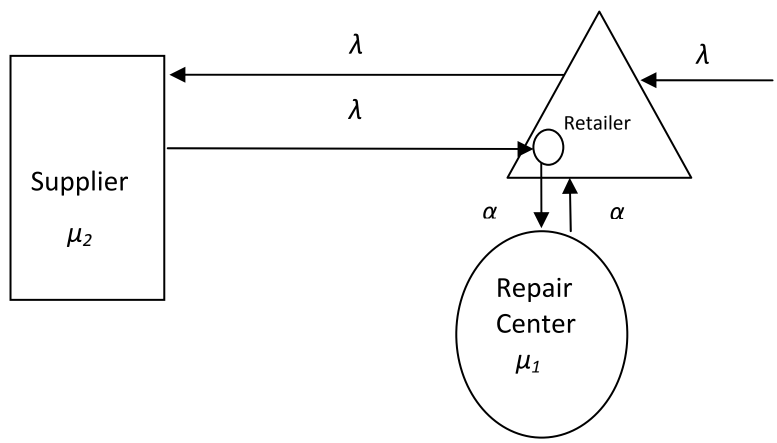

Figure1 Queueing-inventory system with one retailer, one supplier and a repair center

Clearly, one can see that we have a queueing-inventory system with two queueing stations; the supplier station and the repair center station. Thus the total number of outstanding orders of the retailer is the sum of the queue sizes of the supplier and the repair center. For this system first we derive the long run probability of the number of outstanding orders of the retailer. In this derivation we consider two cases: Case (1) The ratio of the arrival rate to service rate, at the supplier and at the repair center are not equal and Case (2) These ratios are equal. We then obtain the average value of on hand inventory and backorders of the retailer and derive the long run unit total cost of the system. Further, for each case we prove that the total system cost function is convex and obtain the relation which gives us the optimal value of the inventory position of the retailer which minimizes the total system cost.

The following notations are used in this paper:

λ : Demand rate at the retailer

μ_1: Service rate at repair center

μ_2 : Service rate at supplier

$\rho_1 = \alpha\lambda/\mu_1$: Traffic intensity at repair center

$\rho_2 = \lambda/\mu_2$: Traffic intensity at supplier

α : Percentage of defective items

h : Unit holding cost per unit time at the retailer

$\hat{\pi}$: Unit backorder cost per unit time at the retailer

C_1: Unit repair cost at repair center

C_2: Unit production cost at the supplier

C_3: Unit waiting cost of a defective item at repair center

R : Inventory position of the retailer

$\bar{I}(R)$: Expected number of on-hand inventory of the retailer

$\bar{b}(R)$: Expected number of backorders of the retailer

O_1: Number of outstanding orders of retailer in the supplier center

$\bar{O}_1$: Expected value of O_1

O_2 : Number of outstanding orders of retailer in the repair center

$\bar{O}_2$: Expected value of O_2

O : Total number of outstanding orders of the retailer = $O_1 + O_2$

ν : The expected number of outstanding orders of retailer

$TC(R)$: Expected total cost of the system per unit time

3. PROBLEM FORMULATION

The expected total cost function of the system, consisting of inventory holding and shortage costs at the retailer, waiting cost of defective items at the repair center, processing costs at the supplier, and repair costs at the repair center, is

$$TC(R) = h\bar{I}(R) + \hat{\pi}\bar{b}(R) + C_3\bar{O}_1 + C_2\lambda + C_1\alpha\lambda \tag{1}$$

Where

$C_3\bar{O}_1$ = the long run unitwaiting cost of defective items at the repair center,
$C_2\lambda$ = the long run average processing cost of supplier, and
$C_1\alpha\lambda$ = the unit long run repair cost of retailer.

$C_2\lambda$ and $C_1\alpha\lambda$ in the right hand side of relation (1) are constant. Therefore for investigating the convexity of $TC(R)$ we only need to consider the expected sum of holding and backordering costs of the retailer, and waiting cost of defective items at the repair center, $K_1(R)$. That is,

$$K_1(R) = h\bar{I}(R) + \hat{\pi}\bar{b}(R) + C_3\bar{O}_1$$

As explained before, ν denotes the expected number of demands during the lead time. Thus, we can write $\bar{I}(R) = R - \nu + \bar{b}(R)$ (see Hadley and Whitin, 1963) and rewrite $K_1(R)$ as follows

$$K_1(R) = h\left[R - \nu + \bar{b}(R)\right] + \hat{\pi}\bar{b}(R) + C_3\bar{O}_1$$

or,

$$K_1(R) = h(R - \nu) + (h + \hat{\pi})\bar{b}(R) + C_3\bar{O}_1$$

Let

$$K_1(R) = K(R) + C_3\bar{O}_1$$

where

$$K(R) = h(R - \nu) + (h + \hat{\pi})\bar{b}(R) \qquad (2)$$

To find $K(R)$ we need to compute $\bar{b}(R)$. For this purpose we first find the probability distribution of outstanding orders of the retailer: Let O_1 and O_2 be the outstanding orders of retailer waiting at the repair center and the supplier respectively. Then, $O=O_1+O_2$ and the probability distribution of outstanding orders of retailer, $P(O=j)$, is obtained by conditioning on the value of the random variable O_1. Thus,

$$P(O = j) = P(O_1 + O_2 = j) = \sum_{k=0}^{j} P(O_2 = j - k \mid O_1 = k) \times P(O_1 = k) \qquad (3)$$

Since the queue of orders in the supplier is an $M/M/1$ model with arrival rate λ and service rate μ_2. The departure process from the supplier is also a Poisson process with rate λ (Ross, 2010). Further, since each item is defective with probability α, the departure process of defective items from the supplier is also a Poisson process with rate $\lambda\alpha$ (see Figure 1). Thus, the queuing modelof repair center is an $M/M/1$ model with arrival rate $\lambda\alpha$ and service rate μ_1. Therefore, we can write (Ross, 2010)

$$P(O_2 = j - k) = (1-\rho_2)\rho_2^{j-k},$$

$$P(O_1 = k) = (1-\rho_1)\rho_1^{k}, \text{ and}$$

$$\bar{O}_1 = \frac{\rho_1}{1-\rho_1},$$

Since $\bar{O}_1$ is a constant value, thus the minimization of $K_1(R)$ is equivalent to minimization of $K(R)$ given in relation (2).

Considering the fact that the number of items in the supplier and the number of items in the repair center are independent, by replacing the values of $P(O_1 = k)$ and $P(O_2 = j - k)$ in relation (3) we can rewrite (3) as follows

$$P(O=j)=\sum_{k=0}^{j}(1-\rho_1)(1-\rho_2)\rho_1^k\,\rho_2^{j-k}$$

Or equivalently,

$$P(O=j)=(1-\rho_1)(1-\rho_2)\rho_2^j\sum_{k=0}^{j}(\rho_1/\rho_2)^k \tag{4}$$

Now we will consider two different cases: Case (1) $\rho_1\neq\rho_2$ and Case (2) $\rho_1=\rho_2$.

3.1. Case 1: $\boldsymbol{\rho_1\neq\rho_2}$

For this case equation (4) can be written as follows:

$$P(O=j)=(1-\rho_1)(1-\rho_2)\rho_2^j\left[1-(\rho_1/\rho_2)^{j+1}\right]\Big/\left[1-(\rho_1/\rho_2)\right] \tag{5}$$

Further, the expected total number of backorders of the retailer is

$$\bar{b}(R)=\sum_{j=R+1}^{\infty}(j-R)P(O=j) \tag{6}$$

Using relation (5) we can rewrite (6) as follows:

$$\bar{b}(R)=\sum_{j=R+1}^{\infty}(j-R)(1-\rho_1)(1-\rho_2)\rho_2^j\left[1-(\rho_1/\rho_2)^{j+1}\right]\Big/\left[1-(\rho_1/\rho_2)\right]$$

With some algebra we can write the above relation as

$$\bar{b}(R)=\left[(1-\rho_1)(1-\rho_2)/(\rho_2-\rho_1)\right]\left[\left[\rho_2^{R+2}/(1-\rho_2)^2\right]-\left[\rho_1^{R+2}/(1-\rho_1)^2\right]\right] \tag{7}$$

Substituting (7) in (2) we have

$$K(R)=h(R-\nu)+(h+\widehat{\pi})(1-\rho_1)(1-\rho_2)/(\rho_2-\rho_1)\left[\left(\rho_2^{R+2}/(1-\rho_2)^2\right)-\left(\rho_1^{R+2}/(1-\rho_1)^2\right)\right] \tag{8}$$

Clearly, ν, the expected number of outstanding orders of retailer, is a constant number.

To obtain the optimal solution we first investigate the convexity of *TC(R)* or equivalently the convexity of *K(R)*. For this purpose, let

$$\Delta K(R)=K(R+1)-K(R).$$

Thus from (8) with some algebra one can show that

$$\Delta K(R)=h+(h+\widehat{\pi})\big((1-\rho_1)(1-\rho_2)/(\rho_2-\rho_1)\big)\left[\frac{\rho_2^2}{(1-\rho_2)^2}\rho_2^R(\rho_2-1)-\frac{\rho_1^2}{(1-\rho_1)^2}\rho_1^R(\rho_1-1)\right]$$

Or equivalently,

$$\Delta K(R) = h + (h+\widehat{\pi})\left[\rho_1^{R+2}(1-\rho_2) - (1-\rho_1)\rho_2^{R+2}\right]\Big/(\rho_2-\rho_1) \tag{9}$$

Now we can find $\Delta^2 K(R) = \Delta K(R+1) - \Delta K(R)$. From (9) we can write

$$\Delta^2 K(R) = (h+\widehat{\pi})\left((1-\rho_1)(1-\rho_2)/\rho_2-\rho_1\right)\left[-\left(\rho_2^{R+2}(\rho_2-1)/(1-\rho_2)\right)+\left(\rho_1^{R+2}(\rho_1-1)/(1-\rho_1)\right)\right]$$

or equivalently,

$$\Delta^2 K(R) = (h+\widehat{\pi})\left((1-\rho_1)(1-\rho_2)/(\rho_2-\rho_1)\right)\left[\rho_2^{R+2}-\rho_1^{R+2}\right]$$

Clearly, $\Delta^2 K$ for all permissible values of ρ_1 and ρ_2 is greater than zero. Thus, *K(R)* is convex and the optimal solution, R^*,is the smallest integer value of *R* which satisfies the relation

$$\Delta K(R) \geq 0$$

Or from (9) the optimal solution, R^*,is the smallest integer value of *R* which satisfies the relation

$$h + \frac{(h+\widehat{\pi})}{\rho_2-\rho_1}\left[\rho_1^{R+2}(1-\rho_2)-(1-\rho_1)\rho_2^{R+2}\right] \geq 0$$

or equivalently, $R^*$is the smallest integer value of *R* which satisfies the relation

$$1 + \frac{(1+\widehat{\pi}/h)}{\rho_2-\rho_1}\left[\rho_1^{R+2}(1-\rho_2)-(1-\rho_1)\rho_2^{R+2}\right] \geq 0 \tag{10}$$

3.2. Case 2:$\rho_1 = \rho_2$

When $\rho_1 = \rho_2 = \rho$ from relation (4) we can write:

$$P(O=j) = (1-\rho)^2 j\rho^j$$

Now for finding $\overline{b}(R)$ using (6) and the above relation we have:

$$\overline{b}(R) = \sum_{j=R+1}^{\infty}(j-R)(1-\rho)^2 j\rho^j = (1-\rho)^2\left[\sum_{j=R+1}^{\infty} j^2\rho^j - R\sum_{j=R+1}^{\infty} j\rho^j\right]$$

Using calculus the above equation can be simplified as follows

$$\overline{b}(R) = \rho^{R+2}\left(R + (2/(1-\rho))\right) \tag{11}$$

Now with substituting (11) in (2) we can find the sum of holding and shortage costs as

$$K(R) = h(R - \nu) + (h + \hat{\pi})\rho^{R+2}\left(R + (2/(1-\rho))\right) \tag{12}$$

To find the optimal solution we first prove the convexity of the cost function in (12). To do this, we obtain $\Delta K(R)$ and $\Delta^2 K(R)$ which are as follow

$$\Delta K(R) = h + (h + \hat{\pi})\rho^{R+2}\left[(R+1)(\rho - 1) - 1\right] \tag{13}$$

and

$$\Delta^2 K(R) = (h + \hat{\pi})\left[\rho^{R+3}\left[(R+2)(\rho-1)-1\right] - \rho^{R+2}\left[(R+1)(\rho-1)-1\right]\right]$$

or equivalently,

$$\Delta^2 K(R) = (h + \hat{\pi})\left[\rho^{R+2}(\rho-1)^2(R+2)\right]$$

Clearly $\Delta^2 K$ is greater than zero for all feasible values of ρ. Thus the cost function is convex and the optimal solution, R^*,is the smallest integer value of R which satisfies the relation

$$\Delta K(R) \geq 0$$

Or from (13) the optimal solution, R^*,is the smallest integer value of R which satisfies the relation

$$h + (h + \hat{\pi})\rho^{R+2}\left[(R+1)(\rho-1)-1\right] \geq 0$$

or equivalently, $R^*$is the smallest integer value of R which satisfies the relation

$$1 + (1 + \hat{\pi}/h)\rho^{R+2}\left[(R+1)(\rho-1)-1\right] \geq 0 \tag{14}$$

4. NUMERICAL EXAMPLE

In this section we present a numerical example for our model. To obtain the optimal value of the inventory position of the retailer we apply relation (10) for Case 1 and relation (14) for Case 2. To use these relations we need the values of unit holding cost per unit time at the retailer, h, and unit backorder cost per unit time at the retailer. We also need the values of traffic intensities at the repair center, and at the supplier which are $\rho_1 = \alpha\lambda/\mu_1$ and $\rho_2 = \lambda/\mu_2$ respectively. In this numerical example we consider the following values for these parameters

h=1,

$\hat{\pi}$ = either 1, 1.5, 2, 2.5, 3, 3.5, or 4

ρ_1 =0.7 or 0.75

For Case 1 ($\rho_1 \neq \rho_2$) we consider ρ_2 = 0.8 ($\rho_1 < \rho_2$) and ρ_2 = 0.4 ($\rho_1 > \rho_2$).

Applying relation (10) for Case 1 and relation (14) for Case 2 we have obtained the optimal values of R, R^*. These values are given in Table 1.

As it can be seen from this Table the value of R^* increases as $\hat{\pi}/h$ increases.

Table 1The optimal value of *R* for ρ_1=0.7 or 0.75 and ρ_2=0.4, 0.7, 0.75 and 0.8

h	$\hat{\pi}$	$\hat{\pi}$ / ***h***	ρ_1	$\rho_1 > \rho_2$ **=0.4** R^*	$\rho_1 = \rho_2$ R^*	$\rho_1 < \rho_2$ **=0.8** R^*
1	0.5	0.5	0.7	1	0	3
1	1.0	1.0	0.7	2	2	5
1	1.5	1.5	0.7	3	3	6
1	2.0	2.0	0.7	3	4	7
1	2.5	2.5	0.7	4	5	8
1	3.0	3.0	0.7	4	5	9
1	3.5	3,5	0.7	5	6	9
1	4.0	4.0	0.7	5	6	10
1	0.5	0.5	0.75	2	1	4
1	1.0	1.0	0.75	3	3	6
1	1.5	1.5	0.75	3	5	7
1	2.o	2.0	0.75	4	5	8
1	2.5	2.5	0.75	5	6	9
1	3.0	3.0	0.75	5	7	10
1	3.5	3,5	0.75	6	8	10
1	4.0	4.0	0.75	6	8	11

5. CONCLUSION

In this research we studied a system consisting of one supplier with a single processing unit, a retailer, a repair center with a single server, and Poisson demand. The retailer applies one for one ordering policy with backordering case.Supplier processing time and repair center service time are both exponentially distributed with given means. A certain fraction,α, of the items produced by the supplier are defective. Upon arrival of a defective item to the retailer, he sends it to the repair center for repair. Each defective item after repair will go immediately to the retailer. For this system we considered two cases for the ratio of traffic intensities of the supplier and the repair center. We derived the long run unit total cost of the system. Further, for each case we proved that the total system cost function is convex and obtained the relation to determine the optimal value of the inventory position of the retailer which minimizes the long run unit total cost of the system. Finally, we provided a numerical example.

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