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The Effect of Gauge Measurement Capability and Dependency Measure of Process Variables on the MC_p

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ABSTRACT

It has been proved that process capability indices provide very efficient measures of the capability of processes from many different perspectives. These indices have been widely used in the manufacturing industry for measuring process reproduction capability according to manufacturing specifications. In the past few years, univariate capability indices have been introduced and used to characterize process performance, but are comparatively neglected for multivariate processes where multiple dependent characteristics are involved in quality measurement. Also, most of researches related to process capability indices have assumed no gauge measurement errors. Unfortunately, such an assumption does not reflect real situations accurately even with highly sophisticated advanced measuring instruments. Conclusions drawn from process capability analysis are hence unreliable. In this paper, we consider the effect of process variables correlation coefficient on the multivariate process capability index (MC_p) for different gauge measurement capabilities. Also, with respect to correlation coefficient and measurement capability we investigate the statistical properties of the estimated MC_p. The results indicate that gauge measurement capability has an important role in determining process capability. This factor would increase the effect of correlation coefficient on estimating the process capability, such that for different gauge measurement capabilities, correlation coefficients will change the results of estimating and testing the process capability.

***Keywords*:** Capability analysis, Correlation coefficient, Critical value, Hypothesis testing, Multivariate process, Gauge measurement errors.

1. INTRODUCTION

In manufacturing industry, there is growing interest in quantitative measures of industrial processes variation. One of the measuring tools most frequently used to measure the capability of a manufacturing process is process capability indices, designed to quantify the relation between the actual performance of the process and its specified requirements. These indices have received much interest in statistical literature during recent years (Vannman and Hubele , 2003). It has been proved that process capability indices provide very efficient measures of the capability of processes from many different perspectives (Chang and Wu, 2008). These capability indices, quantifying process potential and performance, are important for any successful quality improvement activity and quality program implementation**.**

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Capability indices, C_p, C_{pk}, $C_{pm,}$ and C_{pmk}, have been proposed in the manufacturing and service industries providing numerical measures on whether a process is capable of reproducing items within the specification limits preset in the factory. A large number of papers have dealt with the statistical properties and the estimation of these univariate indices. Kotz and Johnson (2002) provided a compact survey and commented on some 170 publications on process capability indices during the years 1992 to 2000. Also Pearn and Kotz (2006) provided a comprehensive survey on process capability indices during the beginning of introducing these indices up to late 2005.

One interesting fact about the characteristic measuring in a process is that the inevitable variations in process measurements come from two sources: the manufacturing process and the gauge. Gauge capability reflects the gauge's precision, or lack of variation, but is not the same as calibration which assures the gauge's accuracy. As it has been emphasized in numerous occasions, process capability measures the ability of a process to meet reassigned specifications.

Most of research papers related to capability measure have assumed no gauge measurement errors (Pearn and Liao, 2005). Unfortunately, such an assumption does not reflect real situations accurately even with highly sophisticated advanced measuring instruments. Montgomery and Runger (1993) pointed out that the quality of data on the process characteristics relies very much on the gauge. Pearn et al. (2007) mentioned that any variation in the measurement process has a direct impact on the ability to make sound judgment about the manufacturing process. An inaccurate measurement system can thwart all the benefits of improvement endeavors resulting in poor quality. On the other hand, improving the gauge measurements and employing properly trained operators can reduce the measurement errors. However, the reality is that no measurement is free from error or uncertainty even if it is carried out with the aid of highly sophisticated and precise measuring instruments. Some research has been done for the case of univariate process capability while considering the measurement error in particular states. Pearn and Kotz (2006) provided a compact survey on pervious researches capability indices with gauge measurement errors during the beginning of introducing these indices up to late 2005.

Another point that is crucial in process capability indices is the bulk of the studies associated with analyzing the quality and efficiency of a process due to a single quality specification; but in modern manufacturing environments where complex processes require monitoring, the possibility of simultaneously monitoring and controlling two or more quality features is rapidly gaining importance. So, our studies on capability indices can not be restricted to the univariate domain. For this reason, multivariate methods for assessing process capability are proposed.

Chan et al. (1991), Taam et al. (1993), Pearn et al. (1992), Chen (1994), Karl et al. (1994), Shahriari et al. (1995), Boyles (1996), Wang and Du (2000), Wang et al. (2000), and others have developed and presented multivariate capability indices for assessing capability. Wang and Chen (1998) and Wang and Du (2000) proposed multivariate extensions for C_p, C_{pk}, $C_{pm,}$ and C_{pmk} based on the principal component analysis, which transforms numbers of original related measurement variables into a set of uncorrected linear functions. A comparison of three novel multivariate methodologies for assessing capability is illustrated in Wang et al. (2000). Although some multivariate capability indices have been studied, and an extensive study has been done for the case of univariate process capability while considering the measurement error, there is a real need for considering this effect on the multivariate quality characteristics as there no study conducted in this regard.

In this paper we focus on the common capability index, MC_p in multivariate state and consider the effect of correlation coefficient on the MC_p and its statistical properties for different gauge measurement capabilities; in other words, we try to answer the question if there is a sound effect of correlation coefficient on the estimation of MC_p, when the measurement error increases.

2. MULTIVARITAL PROCESS CAPABILITY IN PRESENCE OF GAUGE MEASUREMENT ERRORS

The multivariate capability index MC_p is defined as (Taam et al. (1993)):

$$MC_p = \frac{Vol(\text{modified tolerance region})}{Vol[(X-\mu)'\Sigma^{-1}(X-\mu) \le k(q)]} = \frac{Vol(\text{modified tolerance region})}{(\pi\chi^2_{v,0.9973})^{v/2}\,|\Sigma|^{1/2}\,[\Gamma(v/2+1)]^{-1}} \tag{1}$$

Where *k(q)* is the 99.73th percentile of the χ^2 distribution with v degrees of freedom or the dimension of variables; μ is the mean vector and Σ represents the variance–covariance matrix of *X*; $|\Sigma|$ is the determinant of Σ and $\Gamma(.)$ is the gamma function; Vol (modified tolerance region) is the largest ellipsoid centered at the target completely within the original tolerance region; and $\mathrm{vol}[(X-\mu)'\Sigma^{-1}(X-\mu) \le k(q)]$ indicates a scaled 99.73% process elliptical region.

Gauge repeatability and reproducibility (GR&R) studies focus on quantifying the measurement errors. Suppose that in the multivariate case, the measurement errors are described by a random variable M ~ Normal (μ_{Me}, Σ_{Me}), where $\mu_{Me} = 0$, is the mean vector, and Σ_{Me} is the Variance covariance matrix of the measurement error. So, based on the definition of Montgomery and Runger (1993), the gauge capability for the multivariate case is defined by:

$$\lambda^M = \frac{Vol[(X-\mu_{Me})'\Sigma^{-1}_{Me}(X-\mu_{Me}) \le k(q)]}{\text{Vol(modified tolerance region)}} = \frac{(\pi\chi^2_{v,0.9973})^{v/2}\,|\Sigma_{Me}|^{1/2}\,[\Gamma(v/2+1)]^{-1}}{Vol(\text{modified tolerance region})} \tag{2}$$

Where $|\Sigma_{Me}|$ is the determinant of Σ_{Me} and and λ^M is the gauge capability index for the multivariate case. For the measurement system to be deemed acceptable, the variability in the measurements due to the measurement system must be less than a predetermined percentage of the engineering tolerance. So based on the recommendations, some guidelines for gauge acceptance are offered (Montgomery, 1996).

Considering the process capability in the measurement error system, we assume that the observations *X* have a multivariate normal distribution $N_v(\mu, \Sigma)$ and show the relevant quality characteristic of a manufacturing process. Because of measurement errors, the observed variable Y ~ $N_v(\mu_Y = \mu, \Sigma_Y = \Sigma + \Sigma_{Me})$ is measured by the assumption that *X* and *Me* are stochastically independent, instead of measuring the true variable *X*. The empirical process capability index (MC_p^Y) is obtained after substituting Σ_Y for Σ, so the multivariate capability index MC_p^Y is defined as:

$$MC_p^Y = \frac{Vol(\text{modified tolerance region})}{Vol[(X-\mu)'\Sigma_Y^{-1}(X-\mu) \le k(q)]} = \frac{Vol(\text{modified tolerance region})}{(\pi\chi^2_{v,0.9973})^{v/2}\,|\Sigma_Y|^{1/2}\,[\Gamma(v/2+1)]^{-1}} \tag{3}$$

It is easy to show that the relationship between the true process MC_p and the empirical process capability MC_p^Y is given as (Shishebori and Hamadani, 2008):

$$MC_p^Y = \frac{MC_p}{\sqrt{(\lambda^M \ MC_p)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}}} \tag{4}$$

Since the variation of data we observe is larger than that of the original data, the denominator of the index MC_p becomes larger and we will underestimate the true capability of the process.

3. ESTIMATION OF MC_p IN PRESENCE OF GAUGE MEASUREMENT ERRORS

An estimator of MC_p can be expressed as

$$\hat{M}C_p = \frac{Vol(\text{mod}\,ified\ tolerance\ region)}{Vol.(estimated\ \ 99.73\%\ \ process\ \ region)} = \frac{Vol(\text{mod}\,ified\ tolerance\ region)}{(\pi\chi^2_{v,0.9973})^{v/2}\ |\,S\,|^{1/2}\ [\Gamma(\tfrac{v}{2}+1)]^{-1}}$$

Where S is the sample variance-covariance matrix from process and |S| is the determinant of S. $\hat{M}C_p$ is a biased estimator of MC_p multiplied by b_v given as:

$$b_v = (\frac{2}{n-1})^{v/2}\ \frac{\Gamma[1/2(n-1)]}{\Gamma[1/2(n-v)-1/2]}$$

We get an unbiased estimation of MC_p as $\tilde{M}C_p = b_v\ \hat{M}C_p$. Pearn et al. (2007) showed that $\tilde{M}C_p$ is the UMVUE (Uniformly Minimum Variance Unbiased Estimator) of MC_p.

With respect to gauge measurement capability and using the estimators, S_Y, S_{Me} and $\hat{M}C_p$ for the parameters $\sum_Y$, $\sum_{Me}$ and MC_p, the biased estimator of MC_p is given as:

$$\hat{M}C_p^Y = \frac{Vol(\text{mod}\,ified\ tolerance\ region)}{(\pi\chi^2_{v,0.9973})^{v/2}\ |\,S_Y\,|^{1/2}\ [\Gamma(\tfrac{v}{2}+1)]^{-1}} = \frac{Vol(\text{mod}\,ified\ tolerance\ region)}{(\pi\chi^2_{v,0.9973})^{v/2}\ |\,S + S_{Me}\,|^{1/2}\ [\Gamma(\tfrac{v}{2}+1)]^{-1}} \tag{5}$$

Where S_Y is the sample variance-covariance matrix and $|S_Y|$ is the determinant of S_Y.
So the relationship between the estimators of the true process MC_p and the empirical process capability MC_p^Y is given as:

$$\hat{M}C_p^Y = \frac{\hat{M}C_p}{\sqrt{(\hat{\lambda}^M\ \hat{M}C_p)^2 + \frac{|S_Y|-|S_{Me}|}{|S_Y - S_{Me}|}}} \quad or \quad \tilde{M}C_p^Y = \frac{\tilde{M}C_p}{\sqrt{(\tilde{\lambda}^M\ \tilde{M}C_p)^2 + \frac{|S_Y|-|S_{Me}|}{|S_Y - S_{Me}|}}} \tag{6}$$

Illustration Example

It is assumed a bivariate quality control involving joint control of the length (L) and width (W) of a plastic product from a multivariate normality (both quality characteristics/ dimensions have the same unit of measure). Twenty five observations were collected from a plastic production line using the same gauging device. The specification limits for L and W were set at (112.7, 241.3) and (32.7, 73.3), respectively. The center of the specifications was μ_0^T = [177, 53]. The sample mean vector and sample covariance matrix were

$$\overline{X}^T = [177.2, 52.32] \quad , \quad S_Y = \begin{bmatrix} 348.8347 & 85.3308 \\ 85.3308 & 44.6594 \end{bmatrix}$$

Using $\chi^2_{2,0.9973}$ = 11.829 and $|S_Y|$ = 8297.4, then we obtain the practical estimated value of process capability index as:

$$\hat{M}C_p^Y = \frac{\pi \times [(241.3 - 112.7)/2] \times [(73.3 - 32.7)/2]}{(\pi \times \chi^2_{2,0.9973}) \, | S_Y |^{1/2}} = 1.2114$$

and $\tilde{M}C_p^Y = (11/12) \times (1.2114) = 1.1104$.

This value is calculated by ignoring the gauge measurement capability. Now by considering the gauge measurement error for the data, and with respect to the independence of measuring instruments for two variables, we assume that the variance-covariance matrix of gauge measurement is:

$$S_{Me} = \begin{bmatrix} 11.0347 & 0 \\ 0 & 11.0347 \end{bmatrix}$$

Thus we get $\left|S_{Me}\right| = 121.7637$ as an estimation of $\left|\Sigma_{Me}\right|$. Using (2), one can get the gauge measurement capability as 0.1 $(\lambda^M = 0.1)$; therefore, from (6) we obtain $\hat{M}C_p = 1.7282$ and $\tilde{M}C_p = 1.5842$.

Comparing $\hat{M}C_p$ ($\tilde{M}C_p$) with $\hat{M}C_p^Y$ ($\tilde{M}C_p$), it is obvious that the effect of λ^M on the MC_p^Y will increase; in other words, with increasing λ^M , the MC_p^Y will decrease.

It is assumed that, the correlation matrix between process variables, considering the measurement error is given by:

$$\rho = \begin{bmatrix} 1 & 0.8007 \\ 0.8007 & 1 \end{bmatrix}$$

4. EXPECTED VALUE, VARIANCE AND MSE OF $\hat{M}C_p^Y$

According to Pearn et al. (2007), the probability density function of $\hat{M}C_p^Y$ is expressed as:

$$f(x) = f_Y[g^{-1}(x)] \times \left|\frac{d}{dx} g^{-1}(x)\right| = f_Y\left[\frac{(MC_p^Y)^2 (n-1)^v}{x^2}\right] \times \frac{2(MC_p^Y)^2 (n-1)^v}{x^3} \quad for \;\; x > 0 \tag{7}$$

and the *r*th moment of $\hat{M}C_p^Y$, according to the equation (7), is given by:

$$E((\hat{M}C_p^Y)^r) = (MC_p^Y)^r \times \frac{2^{-v/2} \prod_{i=1}^{v} \Gamma[1/2(n-i) - 1/2]}{(n-1)^{-v/2} \prod_{i=1}^{v} \Gamma[1/2(n-i)]} \tag{8}$$

So, the expected value of $\hat{M}C_p^Y$:

$$E(\hat{M}C_p^Y) = MC_p^Y \times \left(\frac{n-1}{2}\right)^{v/2} \frac{\Gamma[1/2(n-v) - 1/2]}{\Gamma[1/2(n-1)]} = \frac{1}{b_v} \times MC_p^Y \tag{9}$$

Where b_v is a correction factor so that $\tilde{M}C_p^Y = b_v \times \hat{M}C_p^Y$ is an unbiased estimator of MC_p^Y. From equation (8) and the definition of variance, we have the variance of $\hat{M}C_p^Y$ as:

$$Var\,(\hat{M}C_p^Y) = \left[\frac{2^{-v} \prod_{i=1}^{v} \Gamma[1/2(n-i) - 1]}{(n-1)^{-v} \prod_{i=1}^{v} \Gamma[1/2(n-i)]} - \left(\frac{1}{b_v}\right)^2\right] \frac{(MC_p)^2}{(\lambda^M \;\; MC_p)^2 + \frac{|\Sigma_Y| - |\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}} \tag{10}$$

For $\lambda^M > 0$, it is clear that $\tilde{M}C_p^Y$ is a biased estimator of MC_p and the bias is given as:

$$bias(\tilde{M}C_p^Y) = E(\tilde{M}C_p^Y) - MC_p = \left(\frac{1}{\sqrt{(\lambda^M \;\; MC_p)^2 + \frac{|\Sigma_Y| - |\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}}} - 1\right) MC_p \tag{11}$$

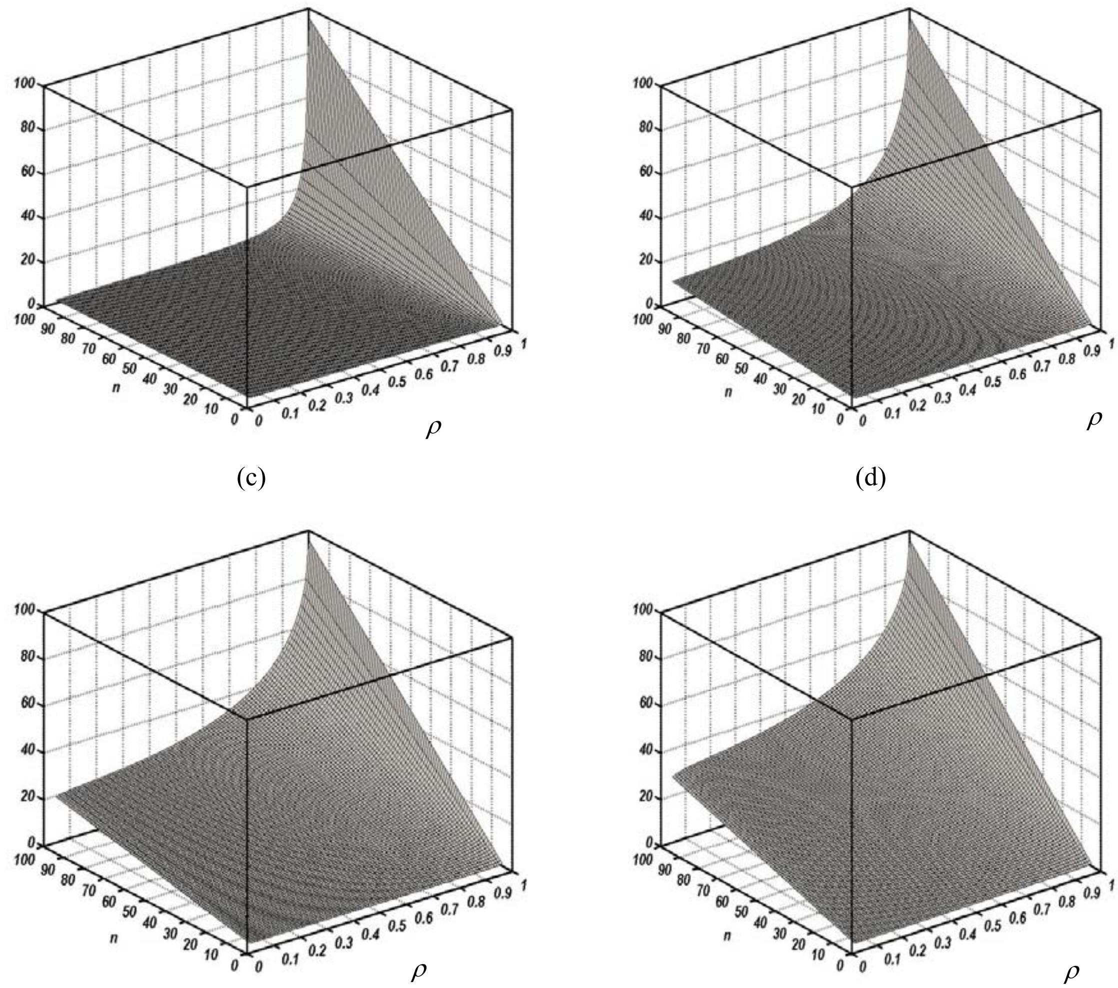
Which is a decreasing function of λ^M.

Taking into account both the bias and the variance of the estimators $\tilde{M}C_p$ and $\tilde{M}C_p^Y$, and using the fact that MSE= $(\text{bias})^2$ +variance, the MSEs of $\tilde{M}C_p$ and $\tilde{M}C_p^Y$, denoted by MSE($\tilde{M}C_p$) and MSE($\tilde{M}C_p^Y$) are given as:

$$MSE\;(\tilde{M}C_p) = MC_p^2 \times \left[b_v^{\,2} \frac{2^{-v} \prod_{i=1}^{v} \Gamma[1/2(n-i) - 1]}{(n-1)^{-v} \prod_{i=1}^{v} \Gamma[1/2(n-i)]} - 1\right] \tag{12}$$

$$MSE(\widetilde{M}C_p^Y) = MC_p^2 \times \left[1 + b_v^{\,2}\, \frac{2^{-v}\prod_{i=1}^{v}\Gamma[1/2(n-i)-1]}{(n-1)^{-v}\prod_{i=1}^{v}\Gamma[1/2(n-i)]} \times \frac{1}{(\lambda^M\ MC_p)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y-\Sigma_{Me}|}} + \left(\frac{2}{\sqrt{(\lambda^M\ MC_p)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y-\Sigma_{Me}|}}} \right) \right] \tag{13}$$

(a) (b)



(c) (d)

(a) $\lambda^M = 0.05$, (b) $\lambda^M = 0.30$ (c) $\lambda^M = 0.60$, (d) $\lambda^M = 0.95$.

Figure 1 Surface plot of γ for n = 5(1)100 and $\rho \in [0, 1]$

For comparing MSE ($\widetilde{M}C_p^Y$) with MSE ($\widetilde{M}C_p$), we consider the function:

$$\gamma = G(MC_p, n, \lambda^M) = \frac{MSE(\widetilde{M}C_p^Y)}{MSE(\widetilde{M}C_p)} = \frac{(bias(\widetilde{M}C_p^Y))^2 + Variance(\widetilde{M}C_p^Y)}{(bias(\widetilde{M}C_p))^2 + Variance(\widetilde{M}C_p)} \tag{14}$$

Equation (14) was written in MATLAB 7.0 software and the three dimensional surface of γ was obtained. Figure 1 shows the three dimensional surface of γ for MC_p = 1.5842 with $\rho \in [0, 1]$ for different gauge measurement errors (λ^M).

According to figure 1, it is obvious that for a known value of measurement capability, the increase in mean square error of MC_p^Y is more pronounced than that in mean square error of MC_p. Therefore one can say that for a known value of λ^M , increasing the correlation coefficient between process variables and also increasing the sample size will increase the mean square error of MC_p. Of course, one can see in figure 1 that for a known value of measurement capability and correlation coefficient, γ increases with the growth of sample size, because the effect of correlation coefficient and also of measurement errors on process capability estimation is more observable with growing of sample size.

5. CONFIDENCE INTERVAL FOR MC_p

Since $\widetilde{M}C_p$ is a statistical estimator like other statistics, it is subject to the sampling variation, therefore one needs to compute an interval to provide a range that includes the true MC_p with high probability. Based on the definition, a 100(1−*α)*% confidence interval for MC_p can be established (Pearn et al. (2007)). 100(1− α)% confidence interval bound can be written as (15):

$$\left[(\hat{M}C_p)^2 \frac{F_Y^{-1}(1-\frac{\alpha}{2})}{(n-1)^v},\ (\hat{M}C_p)^2 \frac{F_Y^{-1}(\frac{\alpha}{2})}{(n-1)^v} \right] \quad or \quad \left[(\frac{\widetilde{M}C_p}{b_v})^2 \frac{F_Y^{-1}(1-\frac{\alpha}{2})}{(n-1)^v},\ (\frac{\widetilde{M}C_p}{b_v})^2 \frac{F_Y^{-1}(\frac{\alpha}{2})}{(n-1)^v} \right] \tag{15}$$

Furthermore, a 100(1−α)% lower confidence bound for MC_p can be obtained as:

$$\left[(\hat{M}C_p)^2 \frac{F_Y^{-1}(\alpha)}{(n-1)^v} \right] \quad or \quad \left[(\frac{\widetilde{M}C_p}{b_v})^2 \frac{F_Y^{-1}(\alpha)}{(n-1)^v} \right] \tag{16}$$

However, as a result of the measurement errors, we take $\widetilde{M}C_p^Y$ as an estimator of MC_p. Thus the confidence bounds are:

$$\left[(\hat{M}C_p^Y)^2 \frac{F_Y^{-1}(1-\frac{\alpha}{2})}{(n-1)^v},\ (\hat{M}C_p^Y)^2 \frac{F_Y^{-1}(\frac{\alpha}{2})}{(n-1)^v} \right] \quad or \quad \left[(\frac{\widetilde{M}C_p^Y}{b_v})^2 \frac{F_Y^{-1}(1-\frac{\alpha}{2})}{(n-1)^v},\ (\frac{\widetilde{M}C_p^Y}{b_v})^2 \frac{F_Y^{-1}(\frac{\alpha}{2})}{(n-1)^v} \right] \tag{17}$$

$$\left[(\hat{M}C_p^Y)^2 \frac{F_Y^{-1}(1-\alpha)}{(n-1)^v},\ \infty \right) \quad or \quad \left[(\frac{\widetilde{M}C_p^Y}{b_v})^2 \frac{F_Y^{-1}(1-\alpha)}{(n-1)^v},\ \infty \right) \tag{18}$$

In the discussed example, a 95% confidence interval and lower bound for MC_p are given as $[0.5416\ ,\ 2.8265]$ and $[0.6295\ ,\ \infty)$ respectively.

It is interesting to find out the confidence coefficient θ (the probability that the confidence interval contains the actual MC_p value) for the confidence bound given in (17). One can calculate this coefficient using the following definition:

$$\theta = P\left((\hat{M}C_p^Y)^2 \frac{F_Y^{-1}(1-\frac{\alpha}{2})}{(n-1)^v} \le MC_p \le (\hat{M}C_p^Y)^2 \frac{F_Y^{-1}(\frac{\alpha}{2})}{(n-1)^v} \right) = P\left(\frac{F_Y^{-1}(1-\frac{\alpha}{2})}{(\hat{\lambda}^M\, M\hat{C}_p)^2 + \frac{|S_Y|-|S_{Me}|}{|S_Y - S_{Me}|}} \le y \le \frac{F_Y^{-1}(\frac{\alpha}{2})}{(\hat{\lambda}^M\, M\hat{C}_p)^2 + \frac{|S_Y|-|S_{Me}|}{|S_Y - S_{Me}|}} \right) \tag{19}$$

By substituting y in the above equation we get:

$$\left(\frac{MC_p^Y}{\hat{M}C_p^Y} \right)^2 \sim \frac{y}{(n-1)^v} \Rightarrow y \sim \frac{M\hat{C}_p^2\,(n-1)^v}{(\hat{M}C_p^Y)^2 \left((\hat{\lambda}^M\; M\hat{C}_p)^2 + \frac{|S_Y|-|S_{Me}|}{|S_Y - S_{Me}|} \right)}$$

If we are interested in evaluating θ for the discussed example, then $\theta = 0.5560$, in other words, the probability that the calculated confidence interval contains the real value of MC_p is equal to 0.5560, which is small compared to 0.95. Accordingly, producers will be damaged if they ignore the effect of measurement error on the calculation of confidence interval which will result in rejecting many of their conformed products and making a lot of losses for their process.

In order to improve the confidence interval for the given confidence coefficient (α=1-θ), one can recalculate the confidence bounds such that it contains the actual value of MC_p with the probability of θ. Hence, if we consider the proposed confidence interval to be L^* and U^*, then with respect to the gauge measurement capability, the adjusted 100(1 − α)% confidence interval bound can be written as (20):

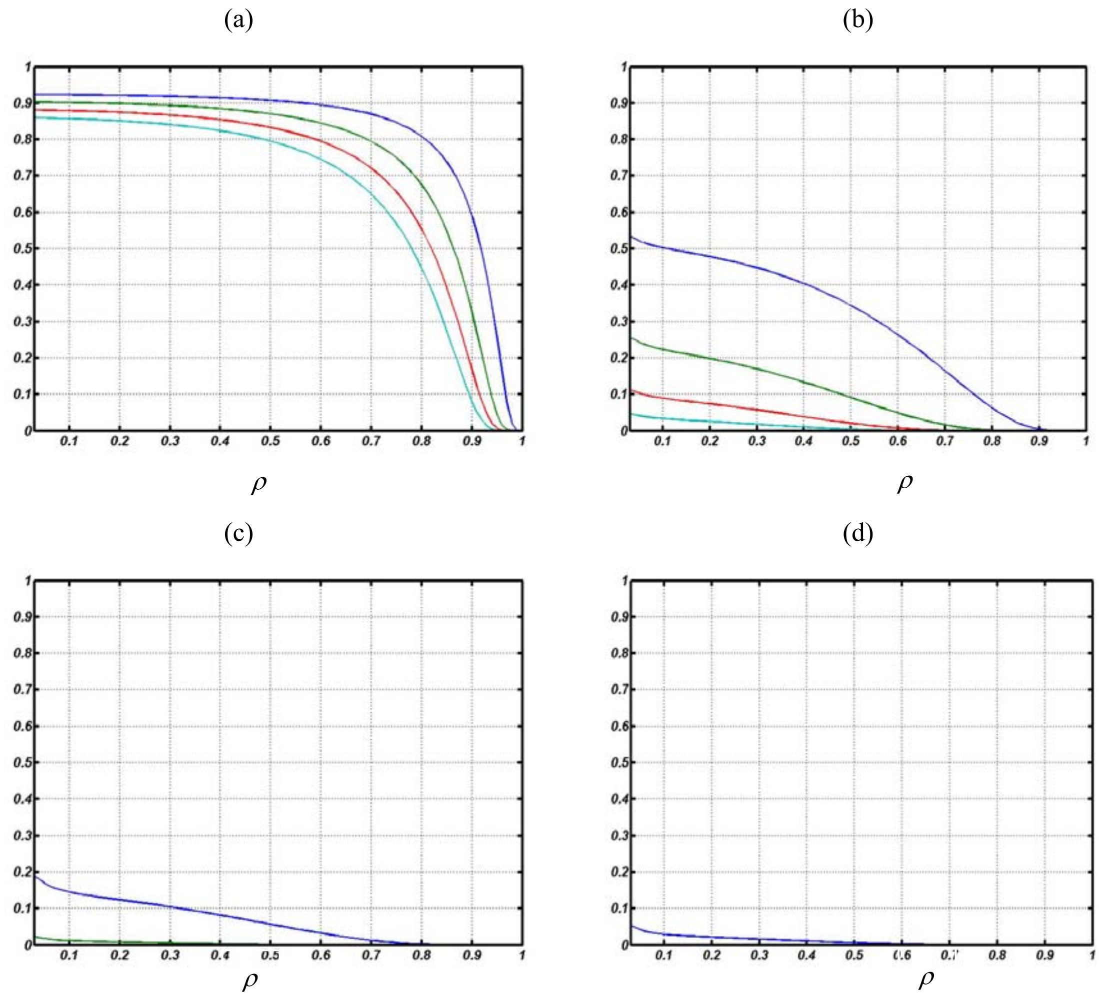
$$L^* = \sqrt{\frac{F_Y^{-1}(1-\frac{\alpha}{2}) \times \left(\hat{M}C_p^Y\right)^2 \times \left(\frac{|S_Y|-|S_{Me}|}{|S_Y - S_{Me}|} \right)}{(n-1)^v - (\hat{\lambda}^M\;\hat{M}C_p^Y)^2\, F_Y^{-1}(1-\frac{\alpha}{2})}} \quad , \quad U^* = \sqrt{\frac{F_Y^{-1}(\frac{\alpha}{2}) \times \left(\hat{M}C_p^Y\right)^2 \times \left(\frac{|S_Y|-|S_{Me}|}{|S_Y - S_{Me}|} \right)}{(n-1)^v - (\hat{\lambda}^M\;\hat{M}C_p^Y)^2\, F_Y^{-1}(\frac{\alpha}{2})}} \tag{20}$$

For the discussed example, the new confidence interval is given as $L^* = 1.03073$, $U^* = 2.4673$; therefore, the 0.95 confidence interval for the actual value of MC_p is [1.0307 , 2.4673].

Figure 2 shows the changing pattern of θ for different sample sizes, different λ^M and also different correlation coefficients at 95% confidence interval.

According to figure 2, one can see that for a known sample size, by growing the amount of correlation coefficient, the decreasing pattern of confidence coefficient (θ) is affected considerably; in other words, by increasing the value of correlation coefficient, the effect of measurement capability on the value of θ will increase and the probability that the 95% calculated confidence interval contains the true value of MC_p will decrease considerably. In addition, one can conclude that, with growing the value of gauge measurement errors (λ^M), the effect of correlation coefficient on reducing the value of confidence coefficient (θ) will increase considerably (figure 2).

Figure 3 shows the curve of the unadjusted lower confidence bound as a function of the correlation coefficients for different gauge measurement capabilities and α = 0.05. In figure 4, one can see the behavior of the adjusted lower confidence bound as a function of the correlation coefficients. In both figures, the upper continuous straight line shows the lower confidence bound for the case of no measurement error, and the striped lines show the lower confidence bounds for different gauge measurement capabilities.



(a) λ^M = 0.05, **(b)** λ^M =0.30, **(c)** r λ^M =0.60, **(d)** λ^M =0.95

Figure 2 Changing procedure of θ with n = 25(25)100 (from bottom to top) and ρ in [0, 1]

According to Figure 3, it is obvious that by increasing the measurement capability index, the effect of correlation coefficient on the lower confidence bound for MC_p will decrease, such that for large ρ and small λ the change is not considerable, but for large λ the effect of correlation on the lower confidence bound is noticeable. In other words, the lower confidence bound will be underestimated and it will reduce the precision of estimated process parameters.

The lower bound estimation improves considerably with correcting the lower bound of MC_p^Y (Figure 4), and the effect of this improvement is more observable on the small values of correlation coefficients.

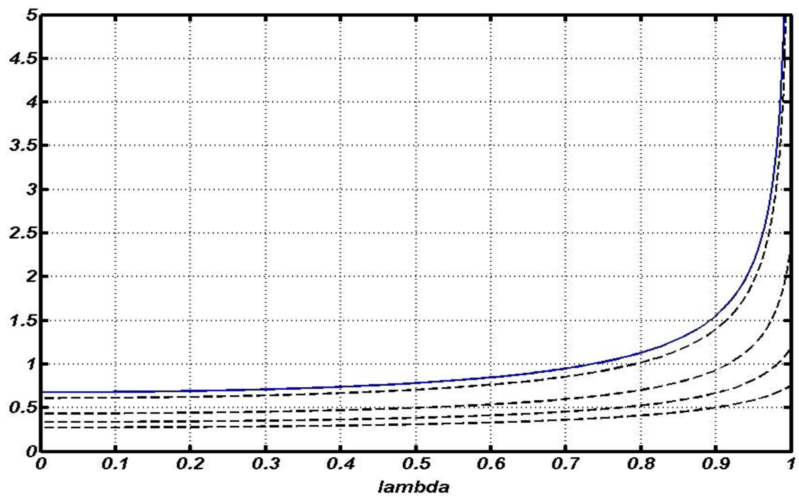


Figure 3 $\rho \in [0, 1]$ for (a) $\lambda^M = 0.05$, (b) $\lambda^M = 0.30$, (c) $\lambda^M = 0.60$, (d) $\lambda^M = 0.95$

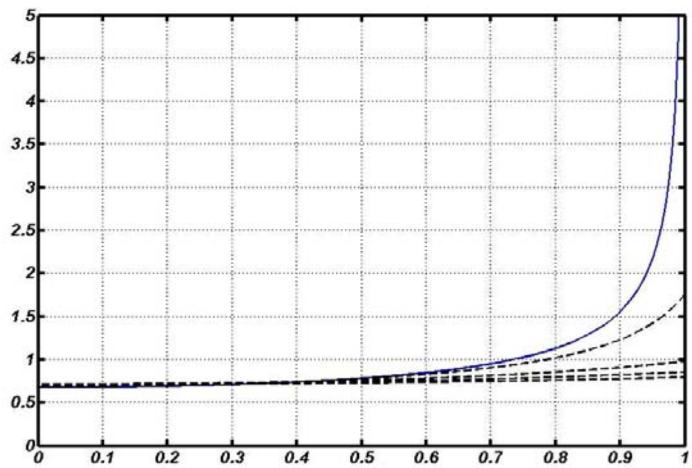


Figure 4 $\rho \in [0, 1]$ for (a) $\lambda^M = 0.05$, (b) $\lambda^M = 0.30$, (c) $\lambda^M = 0.60$, (d) $\lambda^M = 0.95$

6. HYPOTHESIS TESTING FOR CAPABILITY INDEX UNDER GAUGE MEASUREMENT ERRORS

In hypothesis testing, we determine whether or not a hypothesized value of a parameter is true based on the sample taken and the parameter estimate derived from it. That is, we are trying to find out where the estimated capability is relative to either true capability, hypothesized capability, or how different the estimated and true capabilities are. To do this, we estimate an index value, compare it to a lower bound c_0, and compute the so-called p-value. The quantity p refers to the actual risk of incorrectly concluding that the process is capable of a particular test. In general, we want p-value to be no greater than 0.05. To test whether a given process is capable, we may consider the following statistical hypothesis testing:

$$H_0: MC_p \le c \ (process\ is\ not\ capable)$$

$$H_1: MC_p > c \ (process\ is\ capable)$$

Where c is the standard minimal criteria for MC_p. The critical value, c, can be determined as:

$$P\left(\widetilde{M}C_p \geq c_0 \middle| MC_p = c\right) = P\left(\frac{b_v MC_p}{\sqrt{\chi^2_{n-1}\times\chi^2_{n-2}\times\ldots\times\chi^2_{n-v}/(n-1)^v}} \geq c_0 \middle| MC_p = c\right) = \alpha \quad (21)$$

With respect to $y = (\chi^2_{n-1}\times\chi^2_{n-2}\times\ldots\times\chi^2_{n-v})$, then we have:

$$P\left(\frac{b_v c}{\sqrt{y/(n-1)^v}} \geq c_0\right) = \alpha \rightarrow P\left(y \leq \frac{(n-1)^v (b_v c)^2}{c_0^2}\right) = \alpha \rightarrow F_Y^{-1}(\alpha) = \frac{(n-1)^v (b_v c)^2}{c_0^2}$$

Thus, the critical value can be expressed as:

$$c_0 = b_v\, c \sqrt{\frac{(n-1)^v}{F_Y^{-1}(1-\alpha)}} \quad (22)$$

And the power of the test (the chance of correctly judging a capable process as capable) can be computed as:

$$\pi(MC_p) = P\left\{\hat{M}C_p > c_0 \mid MC_p\right\} = P\left(y < \frac{F_y^{-1}(1-\alpha)\times\left(b_v\ MC_p\right)^2}{c^2} \middle| MC_p\right) \quad (23)$$

In the presence of measurement errors, the critical value (denoted by c_0^Y) α-risk (denoted by α^Y) and the power of the test (denoted by π^Y) are:

$$c_0^Y = b_v\, c \sqrt{\frac{(n-1)^v}{F_Y^{-1}(1-\alpha)}} \times \frac{1}{\sqrt{(\lambda^M\ c)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}}} \quad (24)$$

$$\alpha^Y = P\left(y < \frac{(n-1)^v \left(b_v\, c\right)^2}{\left(c_0^Y\right)^2} \times \frac{1}{(\lambda^M\ c)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}}\right) \quad (25)$$

$$\pi^Y(MC_p) = P\left(y < \frac{F_y^{-1}(1-\alpha)\times\left(b_v\ MC_p\right)^2}{c^2} \times \frac{1}{(\lambda^M\ MC_p)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}} \middle| MC_p\right) \quad (26)$$

With respect to (25), it can be seen that the right side of this probability equation is multiplied by $\left[(\lambda^M c)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}\right]^{-1}$. So, we underestimate the true capability of the process when we calculate process capability index using $\widetilde{M}C_p^Y$ instead of $\widetilde{M}C_p$, and the probability that $\widetilde{M}C_p^Y$ is greater than c_0 will be less than the probability of that using $\widetilde{M}C_p$. Thus, the α-risk using $\widetilde{M}C_p^Y$ to estimate MC_p is less than the α-risk using $\widetilde{M}C_p$ to estimate MC_p ($\alpha^Y \leq \alpha$). Also in comparison of (23) with (26); one can see that equation (26) is multiplied by $\left[(\lambda^M \, MC_p)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}\right]^{-1}$; so the power using $\widetilde{M}C_p^Y$ to estimate MC_p is also less than the power using $\widetilde{M}C_p$ to estimate MC_p ($\pi^Y \leq \pi$).

To improve the method of testing hypothesis for the MC_p, one can reconsider the testing procedure such that in the case of gauge measurement errors, a better estimation of critical region and power of the test is obtained.

If we define $\widetilde{M}C_p^Y$, using the mentioned definitions in the previous sections then:

$$\frac{MC_p^Y}{\hat{M}C_p^Y} = \frac{b_v \, MC_p^Y}{\widetilde{M}C_p^Y} \sim \sqrt{\frac{\chi_{n-1}^2 \times \chi_{n-2}^2 \times \ldots \times \chi_{n-v}^2}{(n-1)^v}} \Rightarrow \widetilde{M}C_p^Y = \frac{b_v \, MC_p^Y}{\sqrt{\chi_{n-1}^2 \times \chi_{n-2}^2 \times \ldots \times \chi_{n-v}^2 / (n-1)^v}} \tag{27}$$

Therefore, the new α value is given by:

$$\alpha^* = P\left(\widetilde{M}C_p^Y \geq c_0^* \middle| \widetilde{M}C_p^Y = c\right) = P\left(y < \frac{MC_p^Y \, (n-1)^v \, b_v^2}{c_0^* \left((\lambda^M \; MC_p)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}\right)} \middle| \widetilde{M}C_p^Y = c\right)$$

Based on the above probability phrase, the new critical value is obtained as (28):

$$c_0^* = \frac{b_v \times c \times (n-1)^{v/2}}{\left[F_Y^{-1}(1-\alpha)\right]^{1/2} \sqrt{(\lambda^M \; c)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}}} \tag{28}$$

Also, to improve the power function of the mentioned testing hypothesis, one can use $\widetilde{M}C_p^Y$ based on (27):

$$\pi^*(MC_p) = P\left(\widetilde{M}C_p^Y \geq c_0^* \middle| \widetilde{M}C_p^Y = c\right) =$$
$$P\left(y < \frac{F_y^{-1}(1-\alpha) \times (MC_p)^2}{c^2} \times \frac{(\lambda^M \; c)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}}{(\lambda^M \; MC_p)^2 + \frac{|\Sigma_Y|-|\Sigma_{Me}|}{|\Sigma_Y - \Sigma_{Me}|}} \middle| MC_p\right) \tag{29}$$

For the discussed example, assume that we want to test the following hypothesis at $(\alpha = 0.05)$:

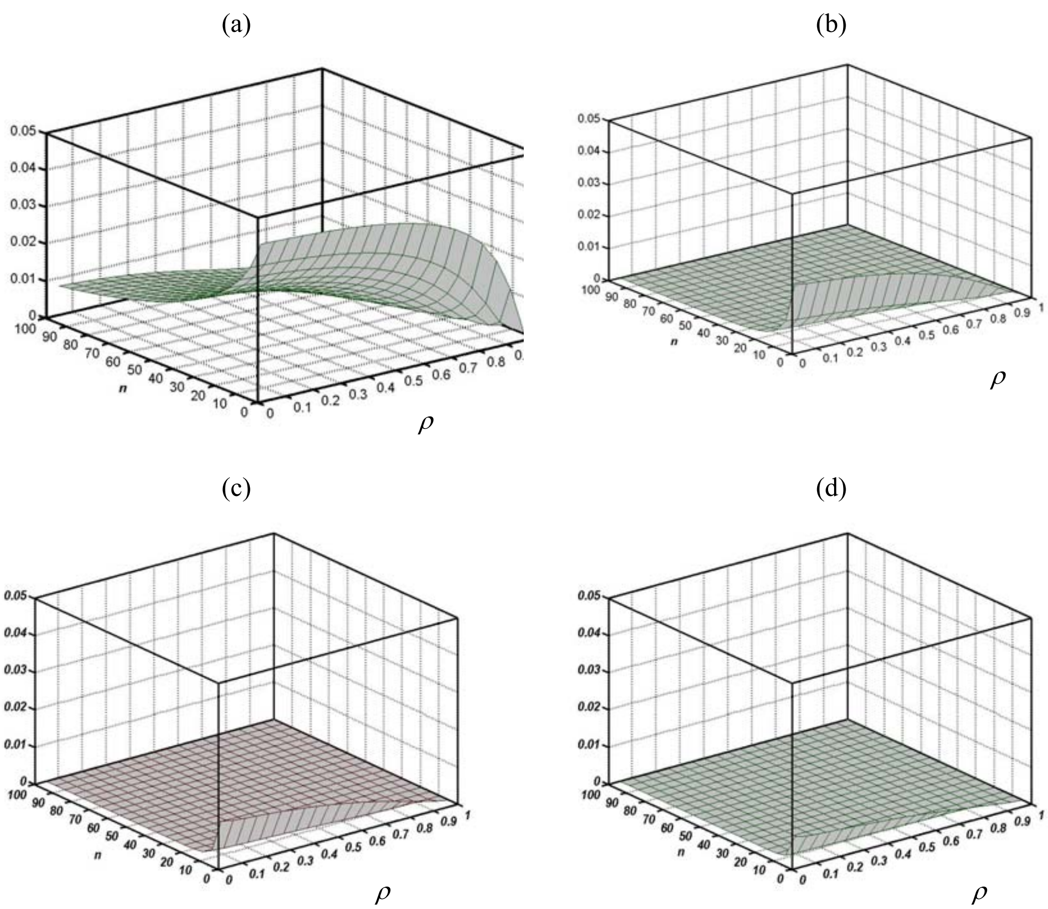
$H_0: MC_p \leq 2$

$H_1: MC_p > 2$

Using (24) and (25), the critical value (c_0^Y), and alpha value (α^Y) are calculated 3.0179 and 0.001 respectively; so the calculated α^Y is considerably smaller than the significant level of the test ($(\alpha^Y = 0.001) < (\alpha = 0.05)$), and it will lead to accept the null hypothesis in many cases.

Accepting the null hypothesis means rejecting the actual capability of the process with respect to consumer view; therefore, it is essential to calculate the critical value by using (28) for testing hypothesis in the presence of measurement errors in order to avoid the false decision.

For the discussed example, the critical value using (28) is $c_0^* = 2.1102$. Using this value for testing hypothesis we get the desired α value ($\alpha = 0.05$). Now if the capability index is increased to $MC_p = 3$ then the power of the test without considering the measurement error (26) is given as $\pi^Y(MC_p) = 0.0677$. If the measurement error is taken into account for evaluating the power of the test (29), then $\pi^*(MC_p) = 0.5521$. Comparing these two values shows that taking into account the gauge measurement errors will cause a great deal of improvement in testing hypothesis for process capability index.

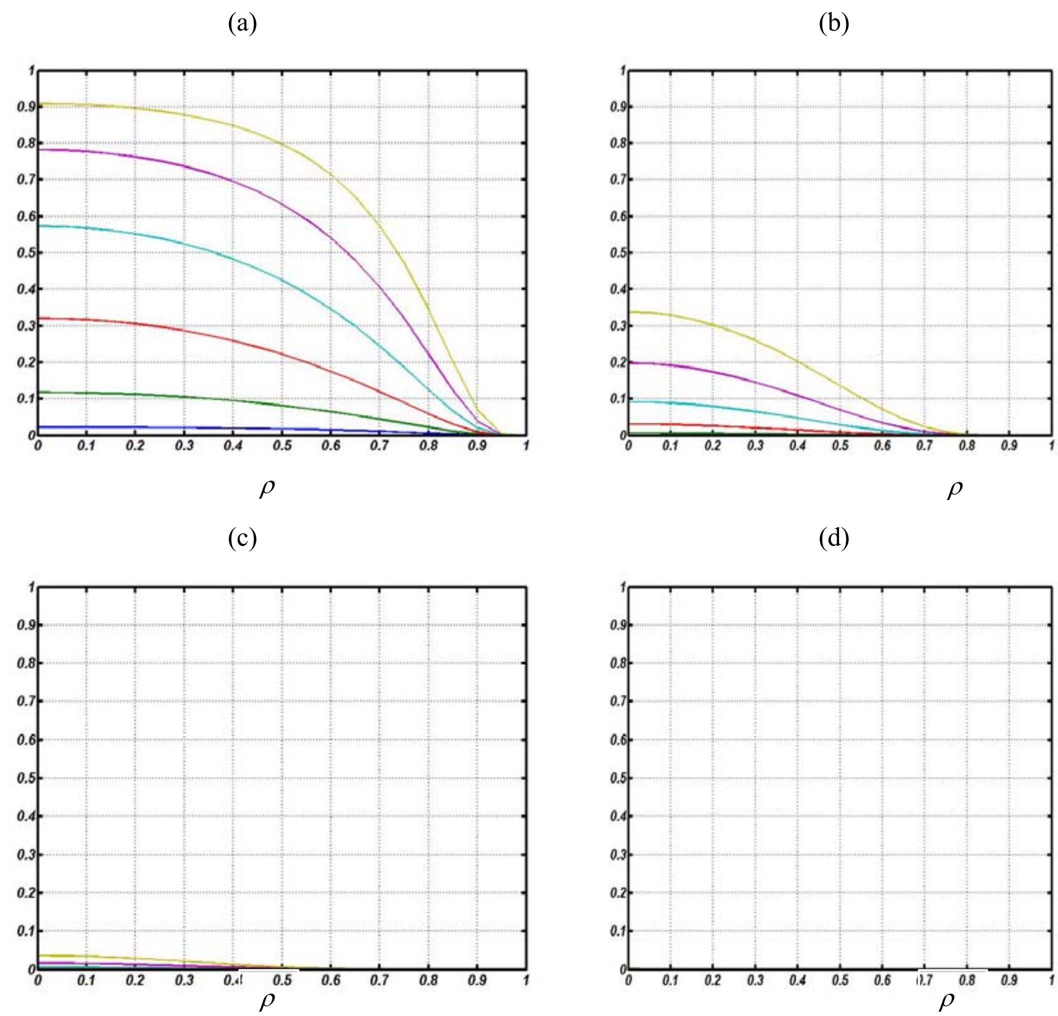


(a) $\lambda^M = 0.05$, **(b)** $\lambda^M = 0.30$, (c) $\lambda^M = 0.60$, (d) $\lambda^M = 0.95$.

Figure 5 Changing procedure of α (α_M^Y) for ρ in [0, 1] and n= 5(1)100

Figure 5 shows the changing procedure of α (α_M^Y) according to the changing values of correlation coefficient, gauge measurement capability and sample size for a type one error probability (α=0.05). As can be seen, increasing the value of correlation coefficient will reduce the value of α_M^Y with respect to the measurement capability index.

According to figure 5, one can see that with growing the correlation coefficient among process variables, the α_M^Y value decreases depending on the measurement capability. So it can be included that with growing the correlation coefficient and also sample size, the α_M^Y value have a decreasing behavior depending on measurement capability. On the other hand, whenever the measurement error increases, the effect of correlation coefficient on α_M^Y grows and α_M^Y decreases considerably.

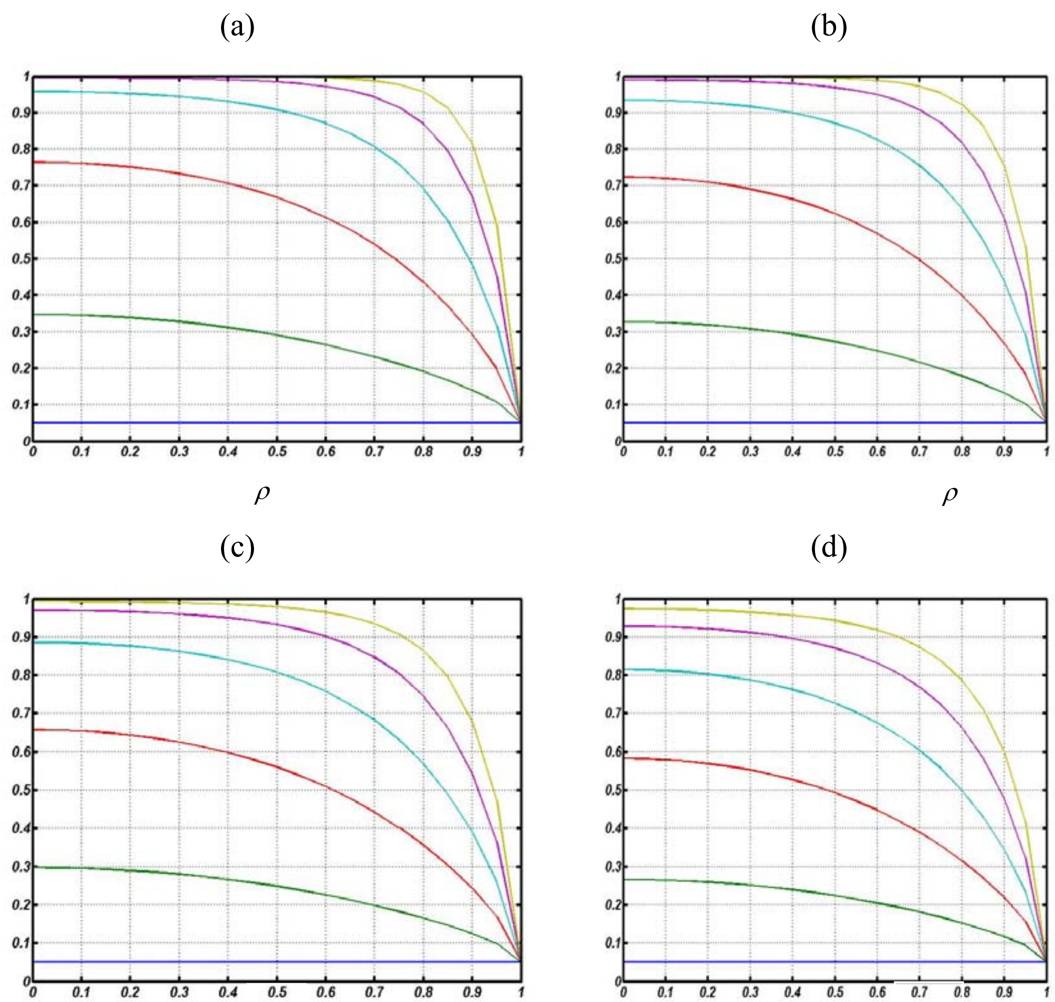


(a) λ^M = 0.05, **(b)** λ^M =0.30, (c) λ^M =0.60, (d) λ^M =0.95.

Figure 6 Changing procedure of $\pi_M^Y(MC_p)$ versus ρ for MC_p =2.00(0.20)3.00 (from bottom to top)

Figure 6 shows the changing procedure of unadjusted power of testing ($\pi_M^Y(MC_p)$) with different values of correlation coefficients, gauge measurement capability for different values of deviation from testing value (c=2). The sample size and α value in this example are n=25 and $\alpha = 0.05$, respectively.

As it is shown in figure 6, for a given value of measurement capability index, $\pi_M^Y(MC_p)$ decreases gradually as the correlation coefficient increases. This result shows that the deviation of process capability from the proposed value of c = 2 is not obvious and the testing hypothesis is not confirmed.



(a) λ^M = 0.05, **(b)** λ^M =0.30, (c) λ^M =0.60, (d) λ^M =0.95.

Figure 7 Changing procedure of $\pi_M^*(MC_p)$ versus ρ for MC_p=2.00(0.20)3.00 (from bottom to top)

The effect of correlation coefficient on the adjusted power of the test given by (27) ($\pi_M^*(MC_p)$) versus ρ for different measurement capabilities, given sample size (n=25), α=0.05 and c=2 is shown in figure 7.

According to figure 7, for a given value of Gauge measurement capability, using the adjusted power of the test ($\pi_M^*(MC_p)$), the effect of correlation coefficient on the power of the test is negligible for $0 < \rho < 0.7$ and the reduced precision is not noticeable.

7. CONCLUSIONS

Most process capability researches in the literatures have been carried out irrespective of gauge measurement errors. Gauge capability has a significant effect on process capability measurement. An inaccurate measurement system can remove the benefits of such endeavors resulting in poor quality. Furthermore, the bulk of the studies associated with analyzing the quality and efficiency of a process are so far limited to discussing one single quality specification, but in real applications, manufactured products often have multiple quality characteristics and multiple characteristics processes are by now so common that our studies on capability indices can't be restricted to the univariate domain. In this paper, we considered the effect of process variables correlation coefficient on the index MC_p for different gauge measurement capabilities. With respect to the results obtained in this paper, it is specified that gauge measurement capability has an important effect on determining the process capability and this effect grows with increasing the correlation coefficients of process variables. On the other hand, the effect of correlation coefficient on incorrect estimation of the index MC_p increases with growing of the gauge measurement errors.

So, conclusions about capability of the process without considering the gauge measurement capability are not reliable especially in processes with high correlation coefficients. Also we showed that the α-risk and the power of the test may decrease with a significant magnitude due to gauge measurement errors, which result in understating capability of the process. Since measurement errors may not be avoided, having proper confidence coefficients and power becomes essential. This necessity will be growing when the correlation coefficient of the process variables increases. Thus, we provided adjusted confidence bounds and critical values for practitioners to use in determining whether their processes meet the capability requirements.

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