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Bertrand's Paradox Revisited: More Lessons about that Ambiguous Word, *Random*

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ABSTRACT

The Bertrand paradox question is: *"Consider a unit-radius circle for which the length of a side of an inscribed equilateral triangle equals* $\sqrt{3}$*. Determine the probability that the length of a 'random' chord of a unit-radius circle has length greater than* $\sqrt{3}$*."* Bertrand derived three different 'correct' answers, the correctness depending on interpretation of the word, *random*. Here we employ geometric and probability arguments to extend Bertrand's analysis in two ways: (1) for his three classic examples, we derive the probability distributions of the chord lengths; and (2) we also derive the distribution of chord lengths for five new plausible interpretations of randomness. This includes connecting (and extending) two random points within the circle to form a random chord, perhaps being a most natural interpretation of *random*.

***Keywords*:** Bertrand paradox, geometrical probability, randomness, mathematical modeling.

1. INTRODUCTION

Over 120 years ago Joseph Louis Bertrand (1888) studied an applied probability problem and published his findings as a 'paradox'. Bertrand's paradox is instructional in that it offers three different 'answers' to a probability question, each answer 'correct' for a given interpretation of the word 'random'. The Bertrand question is as follows: Consider a unit-radius circle for which the length of a side of an inscribed equilateral triangle equals $\sqrt{3}$. Determine the probability that the length of a 'random' chord of the unit-radius circle has length greater than $\sqrt{3}$. The three 'correct answers' are 1/3, 1/4 and 1/2, each representing respectively a different but plausible interpretation of the word *random*.

As shown by many authors including Kendall and Moran (1963), Larson and Odoni (1981) and Aristoff *et al* (2009), Bertrand's Paradox is useful pedagogically because it can be presented to students and practitioners alike to demonstrate the extreme care that must be taken when

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interpreting randomness. Bertrand's Paradox has been the subject of much discussion, debate and new insights over the years, as shown by Holbrook (2000), Jaynes (1973), Marinoff (1994), Rosenberg (2004), Streit (1978) and Tissler (1984). In addition to deriving the distributions of random chord length for the original Bertrand scenarios, Chiu (2008) also addresses the importance of specifying/identifying the source(s) of randomness in several examples so that modelers can agree on what to disagree.

In this paper we extend Bertrand's Paradox to several other realizations, including perhaps the most natural interpretation of a random chord, one that Bertrand chose not in include in his classic 1888 paper. In addition to deriving the probabilities that the associated *random* chord will exceed $\sqrt{3}$ in length, we derive the probability distribution for the length of each type of random chord over a circle of radius *r*. As a review, here are the three classic Bertrand "random chords".

Bertrand 1. Chord's two end points are *random* on the circumference of the circle.

Consider two points uniformly and independently distributed over the circumference of the circle. Connecting these two points defines a random chord. Without loss of generality (*wlg*) we can position ourselves at one (say, point ***A***) of the two points and consider the second as uniformly distributed over the circumference. By referring to Figure 1, we see that the chord will exceed $\sqrt{3}$ in length only when the second point is in one third of the circumference, that is, over arc (***B***, ***C***). Thus the desired probability P_1 is 1/3.

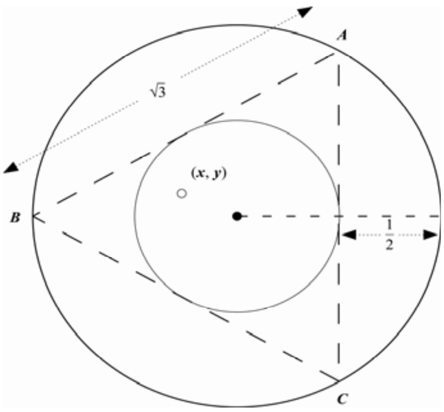

Figure 1

Bertrand 2. The mid-point of the chord is random inside the circle.

Select a random point (*x, y*) whose location is selected from a uniform distribution over the entire circle. This point (*x, y*) becomes the center of the random chord. If this random point (denoted as a hollow dot) is inside a (smaller) concentric circle of radius 1/2, its associated chord length will be longer than $\sqrt{3}$ in length – with the probability being the ratio of the area of the inner circle (of radius 1/2) to that of the unit circle, which equals P_2 =1/4.

Bertrand 3. The *radial distance* of the chord mid-point is random in (0, 1).

The mid-point of a chord uniquely determines its length, being dependent on its distance from the circle's center – or the *radial* distance of the chord mid-point. We may therefore assume that a random chord has its *radial distance* being uniformly distributed in (0, 1). Again referring to Figure 1, since one half of the radius lies inside the pictured inscribed equilateral triangle, the desired probability is $P_3 = 1/2$.

Interestingly, the classic Bertrand paradox does not consider perhaps the most natural interpretation of the word *random* as one of its three candidate solutions: *Two points are located randomly and independently in the interior of the circle. An extended straight-line connection of the two random points determines a random chord.*

Landscape Charts to Illustrate Various Sources of Randomness

We extend Bertrand's classic consideration of randomness to eight scenarios, including his original three. The *CDF* (cumulative distribution function) or *PDF* (probability density function) of the random chord length will be derived for each case, from which the Bertrand probability P can be readily evaluated. We will express our results in terms of the circle radius r instead of simply a circle with unit radius.

To provide a graphical impression, we create *landscape* charts to graphically illustrate the random chord selection process, whether *randomness* occurs on the circle circumference or inside of the circle. In these charts we shall use (1) *hollow dots* to denote the random location of a point; (2) a *dashed line* to represent the random chord generated; and (3) *solid line(s) or dot(s)* to represent a fixed geometric object. We also implicitly assume the rotational symmetry inherent in a circle.

Circumference Randomness: *angular* randomness: The Many Faces of Bertrand 1

The original Bertrand formulation is to choose two independent and uniformly distributed points over the circle circumference to define a random chord, which is equivalent to defining a random chord from a random arc with its arc length being uniformly distributed in $(0, 2\pi r)$ (Bertrand 1a, Figure 2). Since a circle's central angle is linearly related to its associated arc length, generating a random central angle θ uniformly distributed in $(0, \pi)$ will produce the same arc length, and thus chord length, uncertainty (Bertrand 1b, Figure 2). Similarly, generating a random inscribed angle ϕ uniformly in $(0, \pi/2)$ relative to a reference radius will result in the same arc length randomness, and thus chord randomness (Bertrand 1c, Figure 2). Finally, Bertrand 1d creates a random chord by generating a random tangential angle γ, uniform in $(0, \pi/2)$. The Bertrand 1 randomness can be characterized as being generated from *angular randomness*.

Case 8 is the Evan's Robot (see its originating story in a subsequent section). Starting from its home base on the circumference, a robot travels clockwise around the circle at a constant angular speed ω until the first occurrence of a robotic Poisson failure with failure rate λ. The straight-line distance between the failure location and its home base defines the length of a random chord. It turns out that the chord length randomness approaches that of (and converges to) Bertrand 1 when the failure rate is (very) small relative to its angular speed, a not too surprising result considering the Poisson *randomness* of robot failure.

Bertrand 2: Chord Center Uniformly Distributed Inside the Circle

Bertrand 2 chooses a random point uniformly distributed in the interior of a circle to become the center of a random chord.

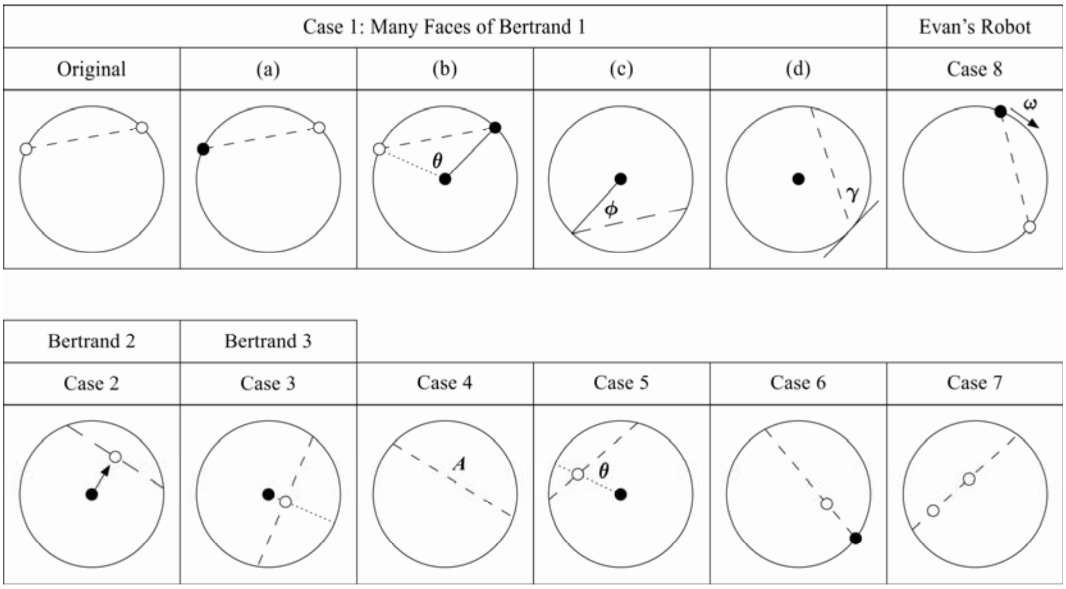

Figure 2. Eight scenarios

Bertrand 3: Chord Center Uniformly Distributed over a Reference Radius

Bertrand 3 chooses a random point on a reference radius, uniformly distributed in (0, r), which becomes the center of the random chord.

Cases 4, 5, and 6: Mixed Randomness

These three cases, seemingly different at first glance, result in the same distribution of random chord length. Case 4 starts with the selection of a random area $\boldsymbol{A}$, uniform in (0, πr^2), defining a random chord that separates the circle into two regions of areas $\boldsymbol{A}$ and $\pi r^2 - \boldsymbol{A}$.

Case 5 starts by choosing a random point uniformly distributed over the interior of a circle. Spin a random angle, uniform in (0, $\pi/2$), pivoted at this random point (and relative to the reference radius defined by this random point), thereby creating a random chord.

Case 6 picks a reference point on the circle circumference and connects it to a randomly chosen point inside the circle to create a random chord.

Cases 5 and 6 generate randomness in two sources: angular and area. It is not entirely surprising that they result in the same chord length randomness. It is, however, more thought challenging that they imply the same chord length randomness as that of case 4.

Case 7: Double Area Randomness

This is perhaps the most natural way to define a random chord: choose two random points independently inside a circle to define a random chord. This is a four-dimensional problem to choose the coordinates of two points over an area. The resultant solution is the most complex involving a combination of (*double*) angular randomness and (*double*) area randomness.

Analysis

We shall use the following notations in our exposition and analysis:

- Circle center $\boldsymbol{O}$, with radius r

- A straight line is defined by its two end points as $\boldsymbol{AB}$. For example, $\boldsymbol{OA}$ represents a radius if $\boldsymbol{A}$ is on the circle circumference. Conversely, $\boldsymbol{A}$ is on the circle circumference if $\boldsymbol{OA}$ is a radius.

- An angle $\angle(\boldsymbol{ABC})$ is the angle at point $\boldsymbol{B}$ formed by the two lines $\boldsymbol{AB}$ and $\boldsymbol{BC}$. For example, $\angle(\boldsymbol{AOC})$ is the central angle, when $\boldsymbol{OA}$ and $\boldsymbol{OC}$ are two radii.

- $Arc(\boldsymbol{AB})$ represents the arc of a circle spanned by two points $\boldsymbol{A}$ and $\boldsymbol{B}$ on the circle circumference. When there is ambiguity, we use the notation $Arc(\boldsymbol{ACB})$ to define precisely the portion of the circumference representing $Arc(\boldsymbol{AB})$: whether it is clockwise or counter-clockwise.

- Area of a polygon is identified by its extreme points; for example: $Area(\boldsymbol{ABCD})$.

- The length of a random chord is represented by the random variable $\boldsymbol{L}$, which has a range of $(0, 2r)$.

- A random variable $\boldsymbol{X}$ has cumulative distribution function (*CDF*) $F_{\boldsymbol{X}}(x) \equiv P\{\boldsymbol{X} \le x\}$ and probability density function (*PDF*) $f_{\boldsymbol{X}}(x) = \dfrac{d}{dx} F_{\boldsymbol{X}}(x)$.

2. TWO RANDOM POINTS ARE UNIFORMLY AND INDEPENDENTLY DISTRIBUTED ON THE CIRCUMFERENCE OF THE CIRCLE, FROM WHICH A RANDOM CHORD IS FORMED (Bertrand 1).

This is the first of the original Bertrand *randomness* characterizations. There are several equivalent randomness formulations/assumptions leading to the same resultant uncertainty of a random chord length. In addition to the Bertrand formulation of two random points on the circle circumference, they include the following:

(a) Fix a point on the circumference; then uniformly select a second point on the circumference. Join the two points to define a random chord: *randomness* is defined by the uncertain location of the second point, being uniform in $(0, 2\pi r)$; or uniform in $(0, \pi r)$ if we limit, by symmetry and without loss of generality (*wlg*), the location of the random point on half of the circumference.

(b) Select a random angle θ in $(0, \pi)$. Form a random chord $\boldsymbol{AB}$ spanned by two radii with a central angle θ: $\angle(\boldsymbol{AOB}) = \theta$. The *randomness* of the central angle θ induces the *randomness* of its associated random chord.

(c) Fix a reference radius $\boldsymbol{OA}$. Form an inscribed angle $\angle(\boldsymbol{OAB}) = \phi$, with $\boldsymbol{B}$ on the circumference and ϕ being uniform in $(0, \pi/2)$. $\boldsymbol{AB}$, thus created, is the random chord. The *randomness* of the inscribed angle ϕ, being uniform in $(0, \pi/2)$, induces the *randomness* of its associated random chord. See Figure 3.

(d) Select a reference tangent line ***TD*** with its associated radius ***OT***. Form a random chord ***TC*** defined by the angle $\angle(\boldsymbol{DTC}) = \gamma$, which is uniform in (0, $\pi/2$) and with point ***C*** on the circumference. The *randomness* of the tangential angle γ, being uniform in (0, $\pi/2$), induces the *randomness* of its associated chord. See Figure 4.

We now derive the distribution of a random chord for case (b). Identical distributions can be similarly derived in the other cases. The random angle $\boldsymbol{\theta}$ is uniformly distributed in (0, π), which is related to the random chord length of ***L*** as:

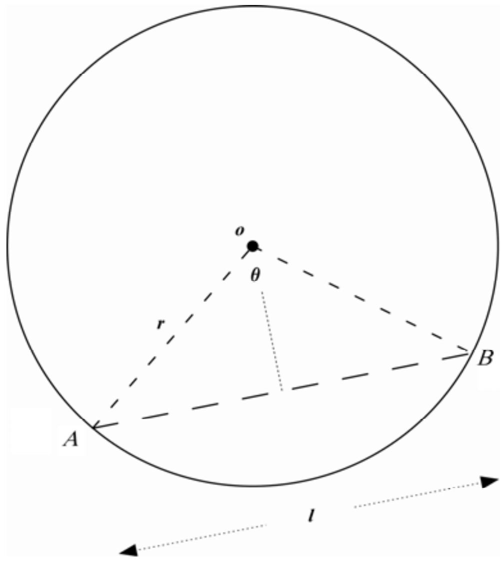

Figure 3

$$\sin\frac{\boldsymbol{\theta}}{2} = \frac{\boldsymbol{L}}{2r}, \text{ with } PDF\ f_{\boldsymbol{\theta}}(\theta) = \frac{1}{\pi}, \quad 0 \le \theta \le \pi.$$

Thus, $$f_{\boldsymbol{L}}(l) = f_{\boldsymbol{\theta}}(\theta)\left|\frac{d\theta}{dl}\right| = \frac{2}{\pi}\frac{1}{\sqrt{4r^2 - l^2}}, \quad 0 \le l \le 2r.$$

The *CDF* of ***L*** is given by:

$$F_{\boldsymbol{L}}(l) = P\{\boldsymbol{L} \le l\} = \frac{2}{\pi}\sin^{-1}\left(\frac{l}{2r}\right), \quad for\ \ 0 \le l \le 2r.$$

Setting $l = \sqrt{3}$ and $r = 1$, we obtain $1 - F_{\boldsymbol{L}}(l) = P(\boldsymbol{L} > \sqrt{3}) = 1 - \frac{2}{\pi}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = 1 - \frac{2}{\pi}\frac{\pi}{3} = \frac{1}{3}$, as expected for Bertrand 1.

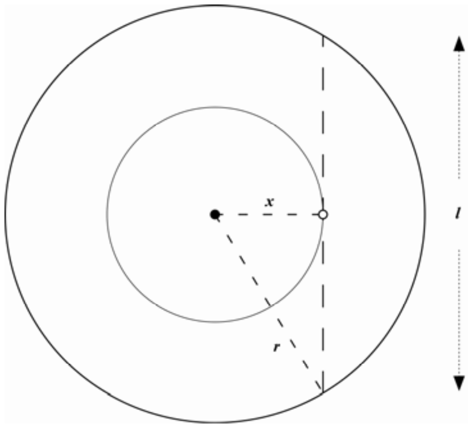

Figure 4

3. CHORD CENTER IS RANDOMLY SELECTED INSIDE THE CIRCLE (Bertrand 2).

Select a random point inside the circle as the center of the chord, identified as a hollow dot, which is equally likely to be anywhere inside the circle of radius *r*. This scenario corresponds to Bertrand's second solution. We use $\boldsymbol{X}$ to denote the random radial distance of the randomly selected point. By radial geometry, it can easily be shown that the *PDF* of $\boldsymbol{X}$ is given as,

$$f_{\boldsymbol{X}}(x) = \frac{2x}{r^2}, \quad 0 \le x \le r.$$

The random chord length $\boldsymbol{L}$ is similarly related to $\boldsymbol{X}$ by,

$$\boldsymbol{X}^2 = r^2 - (0.5\boldsymbol{L})^2.$$

We can, thus, derive

$$f_{\boldsymbol{L}}(l) = \left[f_{\boldsymbol{X}}(x) \left| \frac{dx}{dl} \right| \right]_{x=\sqrt{r^2-(0.5l)^2}} = \frac{2\sqrt{r^2-(0.5l)^2}}{r^2} \frac{l}{2\sqrt{4r^2-l^2}} = \frac{l}{2r^2}, \quad 0 \le l \le 2r.$$

The *CDF* of $\boldsymbol{L}$ can be obtained by integration:

$$F_{\boldsymbol{L}}(l) = \frac{l^2}{4r^2}, \quad \text{for } \ 0 \le l \le 2r.$$

Setting $l = \sqrt{3}$ and $r = 1$, we obtain

$$1 - F_{\boldsymbol{L}}(\sqrt{3}) = P\{L > \sqrt{3}\} = 1 - \frac{3}{4} = \frac{1}{4},$$

as expected for Bertrand 2.

4. CHORD CENTER IS UNIFORMLY DISTRIBUTED ON A REFERENCE RADIUS (Bertrand 3).

The distance between the center of the random chord and the center of circle is uniform in (0, r), formulation 3 in Bertrand's classical treatment of the subject. This is equivalent to selecting a random point on a (reference) radius to become the center of a random chord, identified by the hollow dot. We define the random variable $\boldsymbol{X}$, with realization $\boldsymbol{X} = x$, as uniform over (0, r) being the radial distance on the reference radius (See Figure 5). As in Case 2, $\boldsymbol{X}$ and its associated chord length $\boldsymbol{L}$ are related as,

$$\boldsymbol{X}^2 = r^2 - (0.5\boldsymbol{L})^2,$$

with $f_{\boldsymbol{X}}(x) = \frac{1}{r}, \quad 0 \le x \le r.$

Thus,

$$f_{\boldsymbol{L}}(l) = f_{\boldsymbol{X}}(x)\left|\frac{dx}{dl}\right| = \frac{1}{r}\frac{l}{2\sqrt{4r^2 - l^2}}, \quad 0 \le l \le 2r.$$

The *CDF* of $\boldsymbol{L}$ is given by

$$F_{\boldsymbol{L}}(l) = 1 - \frac{\sqrt{4r^2 - l^2}}{2r}, \quad \mathit{for} \quad 0 \le l \le 2r.$$

Setting $l = \sqrt{3}$ and $r = 1$, we obtain

$$1 - F_{\boldsymbol{L}}(l) = P\{L > \sqrt{3}\} = \frac{\sqrt{4-3}}{2} = \frac{1}{2},$$

as expected for Bertrand 3.

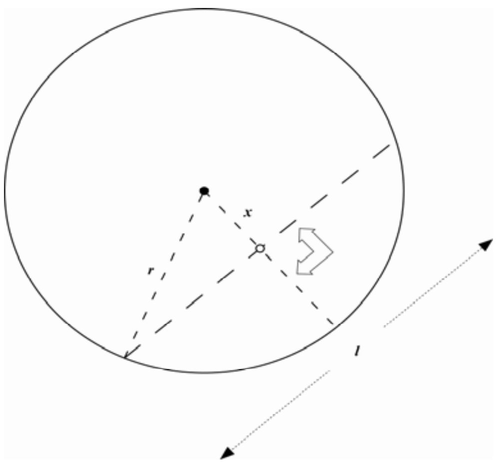

Figure 5

This completes the analysis for the three classic cases of Bertrand *randomness*. We now continue to new formulations.

5. DEFINE A RANDOM CHORD AS ONE THAT SEPARATES THE CIRCLE INTO TWO PARTS, WITH THE SMALLER AREA, DENOTED BY *A*, BEING EQUALLY LIKELY IN (0, $0.5\pi r^2$).

Geometric consideration allows us to relate the area $\boldsymbol{A}$ and the chord length $\boldsymbol{L}$ as,

$$\boldsymbol{A} = \frac{\theta}{2\pi}\pi r^2 - 0.5\boldsymbol{L}\sqrt{r^2 - (0.5\boldsymbol{L})^2} = r^2 \sin^{-1}\frac{\boldsymbol{L}}{2r} - \frac{\boldsymbol{L}}{4}\sqrt{4r^2 - \boldsymbol{L}^2} = h(\boldsymbol{L}).$$

The area $\boldsymbol{A}$ (shaded in Figure 6) equals the difference between the area of the *pie slice* with a central angle of $\boldsymbol{\theta}$ and the second term being the area of the triangle with base $\boldsymbol{L} = l$. We have also used $\sin\frac{\boldsymbol{\theta}}{2} = \frac{\boldsymbol{L}}{2r}$.

Since the function $h(\boldsymbol{L})$ is monotonically increasing, we can derive the *PDF* of $\boldsymbol{L}$ from $f_A(a) = \frac{2}{\pi r^2}, \quad 0 \le a \le \frac{\pi r^2}{2}$.

$$f_{\boldsymbol{L}}(l) = f_{\mathbf{A}}(a)|_{a=h(l)} \left|\frac{dh(l)}{dl}\right| = \frac{2}{\pi r^2}\left[\frac{r^2}{\sqrt{4r^2 - l^2}} - \frac{\sqrt{4r^2 - l^2}}{4} + \frac{l^2}{4\sqrt{4r^2 - l^2}}\right] = \frac{l^2}{\pi r^2\sqrt{4r^2 - l^2}}, \quad 0 \le l \le 2r.$$

The *CDF* is obtained by integration,

$$F_L(l) = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{l}{2r}\right) - \frac{l\sqrt{4r^2 - l^2}}{4r^2}\right], \quad 0 \le l \le 2r.$$

Numerically, the Bertrand probability in this case is found to be 0.609.

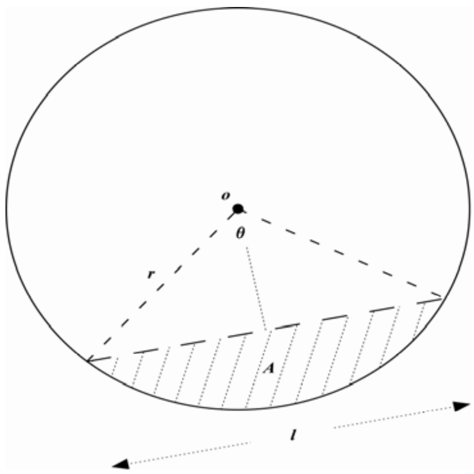

Figure 6

6. SELECTED A RANDOM POINT *R* INSIDE THE CIRCLE, SPIN A RANDOM ANGLE AT *R* RELATIVE TO REFERENCE LINE *OR*

Pick a random point inside the unit circle, denoted by ***R*** (and identified as a hollow dot in Figure 7). We use ***Y*** to express the random radial distance of this point from the circle center (the length of the line ***OR***). Pivoting at this random point, we spin a random angle θ relative to the reference line ***OR*** uniformly distributed in $(0, \pi/2)$ to determine a random chord. The random radial distance from the chord center is denoted by the random variable ***X***, with $\boldsymbol{X} = \boldsymbol{Y}\sin\theta$.

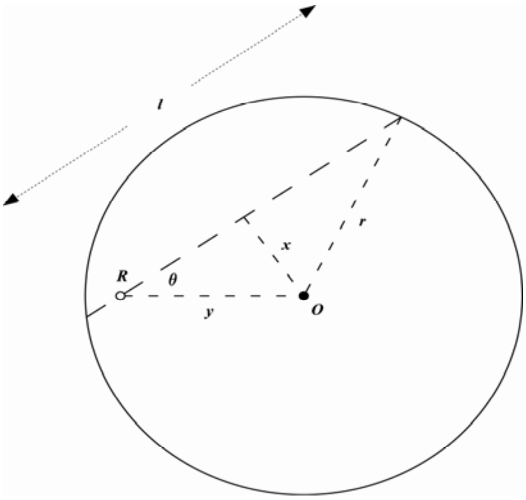

Figure 7

We first derive the distribution for ***X***, from which the *PDF* of the chord length ***L*** can be readily obtained as in cases 2 and 3. First, the *PDF*'s for ***Y*** and θ are independent, expressed as:

$$f_Y(y) = \frac{2y}{r^2}, \quad 0 \le y \le r; \quad f_\theta(\theta) = \frac{2}{\pi}, \quad 0 \le \theta \le \pi/2.$$

To derive the *PDF* of $\boldsymbol{X} = \boldsymbol{Y}\sin\theta$ is a change-of-variable exercise from the distribution of $(\theta, \boldsymbol{Y})$ to that of ***X***. We use Figure 8 to show the limits of integration when we evaluate the *PDF* of ***X***, $f_X(x)$.

To derive an expression for $f_X(x)$, $f_X(x)dx$ is evaluated by integrating $f_{\theta Y}(\theta, y)dyd\theta$ over the *strip* of space (with *width dx*) in the $(\theta, \boldsymbol{Y})$ sample space. We express $f_{\theta Y}(\theta, y)$ as the product of two marginal density functions since θ and ***Y*** independent. Note the limits of integration for θ in $\left(\sin^{-1}\frac{x}{r}, \pi/2\right)$ where the joint density $f_{\theta Y}(\theta, y)$ is nonzero.

$$f_X(x) = \int_{\sin^{-1}\frac{x}{r}}^{0.5\pi} f_{\theta Y}(\theta, y)\left|\frac{\delta y}{\delta x}\right| d\theta = \int_{\sin^{-1}\frac{x}{r}}^{0.5\pi} \left(\frac{2y}{r^2}\frac{2}{\pi}\right)\frac{1}{\sin\theta}d\theta = \frac{4x}{\pi r^2}\int_{\sin^{-1}\frac{x}{r}}^{0.5\pi} \frac{d\theta}{\sin^2\theta}$$

$$= \frac{4x}{\pi r^2}\left[-\cot\theta\right]_{\sin^{-1}\frac{x}{r}}^{0.5\pi} = \frac{4\sqrt{r^2 - x^2}}{\pi r^2}, \quad 0 \le x \le r.$$

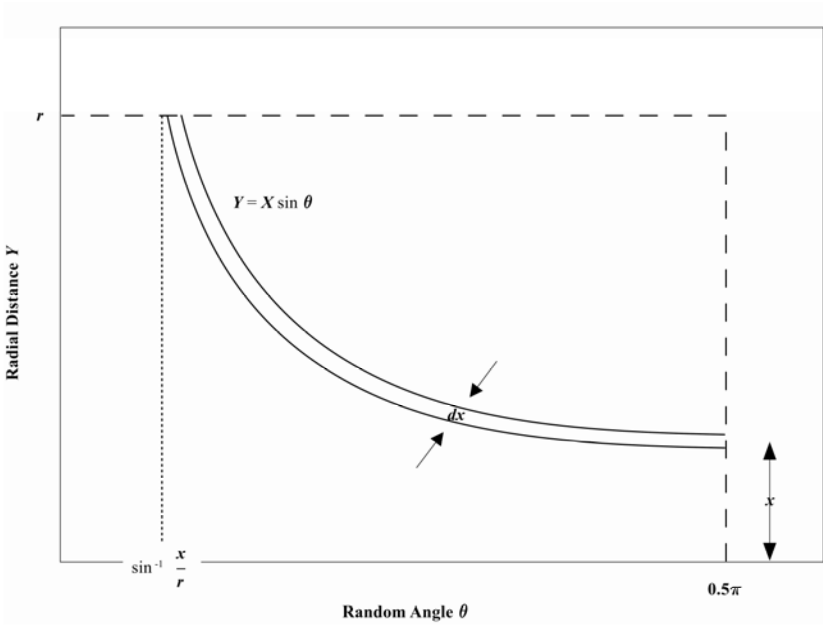

Figure 8. Sample Space for (θ, Y)

The random chord length $\boldsymbol{L}$ is similarly related to $\boldsymbol{X}$ as in case 2 by

$\boldsymbol{X}^2 = r^2 - (0.5\boldsymbol{L})^2$. We can, thus, write

$$f_{\boldsymbol{L}}(l) = f_{\boldsymbol{X}}(x)\left|\frac{dx}{dl}\right| = \frac{4\sqrt{r^2 - x^2}}{\pi r^2}\frac{l}{2\sqrt{4r^2 - l^2}} = \frac{4(0.5l)}{\pi r^2}\frac{l}{2\sqrt{4r^2 - l^2}}$$
$$= \frac{1}{\pi r^2}\frac{l^2}{\sqrt{4r^2 - l^2}}, \quad 0 \le l \le 2r$$

The *CDF* is obtained via integration,

$$F_{\boldsymbol{L}}(l) = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{l}{2r}\right) - \frac{l\sqrt{4r^2 - l^2}}{4r^2}\right], 0 \le l \le 2r\,.$$

This *randomness* characterization leads to the same random chord length distribution as that of Case 4, a fact not entirely obvious at first glance! And we have the same Bertrand probability, 0.609.

7. ONE RANDOM POINT IS ON THE CIRCUMFERENCE AND ONE IS INSIDE THE CIRCLE.

Connecting and extending these two random points creates a random chord. We seek to obtain the *CDF* for the length of this random chord, $F_{\boldsymbol{L}}(l) \equiv P\{\boldsymbol{L} \le l\}$.

We first identify a random point on the circle circumference as K in Figure 9. If the random point inside the circle is within the shaded area $KEFK$ or $KABK$, the chord length will be smaller than l. The evaluation of $P\{\boldsymbol{L} \le l\}$ becomes a ratio of areas, since the random point inside the circle is uniformly distributed inside the circle of radius r.

$$P\{\boldsymbol{L} \le l\} = \frac{Area(KEFK) + Area(KABK)}{\pi r^2} = \frac{2 * Area(KABK)}{\pi r^2}.$$

$$Area(OKABO) = \frac{2\beta}{2\pi}\pi r^2 = \beta r^2 = r^2 \sin^{-1}\frac{l}{2r},$$

$$Area(OKBO) = 0.5lx = 0.5l\sqrt{r^2 - (0.5l)^2} = \frac{l\sqrt{4r^2 - l^2}}{4},$$

$$\begin{aligned} Area(KABK) &= Area(OKABO) - Area(OKBO) \\ &= r^2 \sin^{-1}\frac{l}{2r} - \frac{l\sqrt{4r^2 - l^2}}{4}, \end{aligned}$$

$$\begin{aligned} P\{\boldsymbol{L} \le l\} &= \frac{2}{\pi r^2}\left[r^2 \sin^{-1}\frac{l}{2r} - \frac{l\sqrt{4r^2 - l^2}}{4}\right] \\ &= \frac{2}{\pi}\left[\sin^{-1}\frac{l}{2r} - \frac{l\sqrt{4r^2 - l^2}}{4r^2}\right], 0 \le l \le 2r. \end{aligned}$$

This *randomness* is identical to that of case 5 (and case 4): pick a random point $\boldsymbol{R}$ inside the circle, spin a random angle at $\boldsymbol{R}$ relative to line $\boldsymbol{OR}$ to establish a random chord --- a result not readily obvious. Again, the Bertrand probability is 0.609.

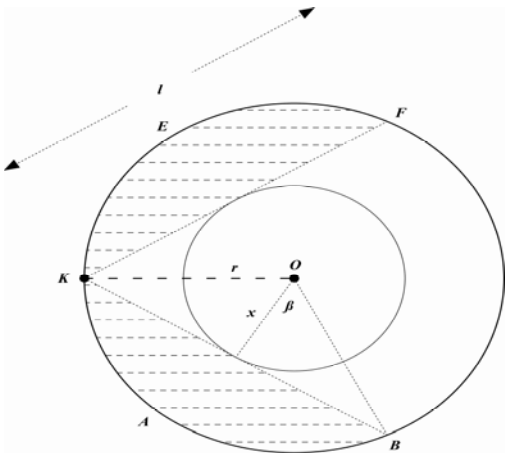

Figure 9

8. Select two random points inside the circle, connecting and extending to form a random chord

Identify two points independently and uniformly distributed inside a circle of radius *r*, joining these two points and extending that straight line to create a random chord. This is perhaps the most natural idea of a random chord, one that was not included in Bertrand's classic paper. We first condition on the location of the first random point. This random point has a radial distance denoted by the random variable $\boldsymbol{Y}$ with its *PDF* identified in Bertrand Case 2,

$$f_{\boldsymbol{Y}}(y) = \frac{2y}{r^2}, \quad 0 \le y \le r.$$

The conditional *CDF* of a random chord length, condition on $\{\boldsymbol{Y} = y\}$, is then derived using a geometric argument based on a ratio of areas. The *CDF* of the random chord length, $F_{\boldsymbol{L}}(l)$, will be obtained using the law of total probability.

We use Figure 10 to relate various geometrical and area relationships.

The point $\boldsymbol{R}$, identified as a hollow dot, is the conditional location of the first random point, whose radial distance is expressed as a random variable $\boldsymbol{Y}$, now conditional as $\boldsymbol{Y} = y$. The resultant random chord length is less than l (the length of the straight line $\boldsymbol{AG}$) if its random location is inside the two (equal-area) shaded areas: $\boldsymbol{ABCRA}$ and $\boldsymbol{EFGRE}$. The lines $\boldsymbol{OT}$ and $\boldsymbol{ATG}$ are perpendicular to each other. We now seek to find the area of the shaded region to compute:

$P\{\boldsymbol{L} \le l \mid \boldsymbol{Y} = y\} = \dfrac{2 * Area(\boldsymbol{ABCRA})}{\pi r^2}$, from which we can write

$$F_{\boldsymbol{L}}(l) = P\{\boldsymbol{L} \le l\} = \int_0^r P(\boldsymbol{L} \le l \mid \boldsymbol{Y} = y) f_{\boldsymbol{Y}}(y)dy$$
$$= \int_0^r P(\boldsymbol{L} \le l \mid \boldsymbol{Y} = y) \frac{2y}{r^2} dy$$

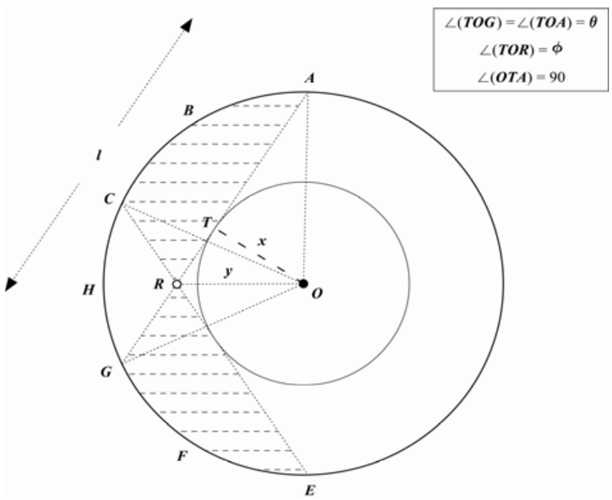

Figure 10

We will use ordered sequences of points (either extreme points, or interior points on an arc) to represent areas.

ABCRA = ***OABCHGO*** – ***OGAO*** – ***CHGRC***,

Where

OABCHGO = area of the cone spanned by a central angle of 2θ,

OGAO = area of a triangle, spanned by a central angle equaling 2θ,

CHGRC = a *sliver* of a *pie* near the edge of the circle
= ***OCHGO*** – ***OCRO*** – ***OGRO*** = ***OCHGO*** –2****OGRO***

OCHGO = area of the cone spanned by the central angle $\angle(\boldsymbol{COG})$, which equals $2(\theta - \phi)$:

$\angle(\boldsymbol{COG}) = \angle(\boldsymbol{COR}) + \angle(\boldsymbol{GOR}) = 2\angle(\boldsymbol{GOR}) = 2*\{\angle(\boldsymbol{TOG}) - \angle(\boldsymbol{TOR})\} = 2(\theta - \phi)$,

OCRO = ***OGRO*** = area of a triangle.

The following quantities are essential in our algebra, thus highlighted below:

l: length of the random chord

y: radial distance from the first random point to the circle center

x: radial distance from the center of the random chord to the circle center

The algebra simplifies when we use the following identities:

$$x^2 = r^2 - (0.5l)^2, \quad x = \frac{\sqrt{4r^2 - l^2}}{2}.$$

We now seek to find expressions for the underlined quantities, using geometric arguments inside a circle of radius r.

$$\boldsymbol{OABCHGO} = \pi r^2 \frac{2\theta}{2\pi} = r^2\theta.$$

It turns out that θ will be cancelled out, and will not appear in the analysis.

$$\boldsymbol{OGAO} = \frac{1}{2}lx = \frac{1}{2}l\sqrt{r^2 - (0.5l)^2} = \frac{l\sqrt{4r^2 - l^2}}{4},$$

$$\boldsymbol{OCHGO} = \pi r^2 \frac{2(\theta - \phi)}{2\pi} = r^2(\theta - \phi), \text{ with}$$

$$\phi = \cos^{-1}\left(\frac{x}{y}\right) = \cos^{-1}\left(\frac{\sqrt{4r^2 - l^2}}{2y}\right),$$

$$\begin{aligned}\boldsymbol{OGRO} &= \frac{1}{2}\text{base}*\text{height} = \frac{1}{2}(GT-RT)x = \frac{1}{2}\left(\frac{l}{2}-\sqrt{y^2-x^2}\right)x \\ &= \frac{l\sqrt{4r^2-l^2}}{8}-\frac{\sqrt{4r^2-l^2}\sqrt{4y^2-4r^2+l^2}}{8},\end{aligned}$$

where GT and RT are straight-line distances.

We can now express the area $\boldsymbol{ABCRA}$:

$$\begin{aligned}\boldsymbol{ABCRA} &= \boldsymbol{OABCHGO}-\boldsymbol{OGAO}-\boldsymbol{CHGRC}, \\ &= \boldsymbol{OABCHGO}-\boldsymbol{OGAO}-(\boldsymbol{OCHGO}-2*\boldsymbol{OGRO}) \\ &= r^2\phi-\frac{\sqrt{4r^2-l^2}\sqrt{4y^2-4r^2+l^2}}{4}.\end{aligned}$$

Given the radial distance of one of the random points $\boldsymbol{R}$ being y, the conditional probability that the random chord length is less than l becomes

$$P\{\boldsymbol{L}\le l\,|\,\boldsymbol{Y}=y\}=\frac{2\boldsymbol{ABCRA}}{\text{area of circle}}=\frac{2\phi}{\pi}-\frac{\sqrt{4r^2-l^2}\sqrt{4y^2-4r^2+l^2}}{2\pi r^2}.$$

As described earlier, we will make the substitution from l to x as it will simplify our algebra.

$$P\{\boldsymbol{L}\le l\,|\,\boldsymbol{Y}=y\}=\frac{2\boldsymbol{ABCRA}}{\text{area of circle}}=\frac{2\phi}{\pi}-\frac{2x\sqrt{y^2-x^2}}{\pi r^2}=\frac{2}{\pi}\cos^{-1}\left(\frac{x}{y}\right)-\frac{2x\sqrt{y^2-x^2}}{\pi r^2}.$$

The CDF of $\boldsymbol{L}$ can now be derived using the Law of Total Probability, by integrating over the random variable $\boldsymbol{Y}$. The lower limit of integration recognizes the fact that the radial distance of the first random point has to be larger than x (the radial distance of the center of the random chord to the circle center) if the random chord length is to be less than l.

$$\begin{aligned}F_{\boldsymbol{L}}(l) &= P\{\boldsymbol{L}\le l\} \\ &= \int_{y=x}^{r} P(\boldsymbol{L}\le l\,|\,\boldsymbol{Y}=y)f_{\boldsymbol{Y}}(y)dy=\int_{y=x}^{r} P(\boldsymbol{L}\le l\,|\,\boldsymbol{Y}=y)\frac{2y}{r^2}dy \\ &= \int_{y=x}^{r}\left\{\frac{4y\phi}{\pi r^2}-\frac{4xy\sqrt{y^2-x^2}}{\pi r^4}\right\}dy, \quad \text{with } \phi=\cos^{-1}\left(\frac{x}{y}\right) \\ &= \frac{2}{\pi}\cos^{-1}\left(\frac{x}{r}\right)-\frac{x\sqrt{r^2-x^2}\,(10r^2-4x^2)}{3\pi r^4} \\ &= \frac{2}{\pi}\cos^{-1}\left(\frac{\sqrt{4r^2-l^2}}{2r}\right)-\frac{l(6r^2+l^2)\sqrt{4r^2-l^2}}{12\pi r^4}, 0\le l\le 2r.\end{aligned}$$

The last equality substitutes x using $x^2=r^2-(0.5l)^2$.

The integration of the inverse cosine term appears in an appendix, derived as,

$$\int_{y=x}^{r} \cos^{-1}\left(\frac{x}{y}\right) y\, dy = \frac{1}{2}\left[r^2 \cos^{-1}\left(\frac{x}{r}\right) - x\sqrt{r^2 - x^2}\right].$$

We have also used the following formula for the integral of the other term:

$$\int_{y=x}^{r} \sqrt{y^2 - x^2}\, y\, dy = \frac{1}{3}\sqrt{\left(r^2 - x^2\right)^3}, \quad \text{for } 0 \le x \le r.$$

Differentiating the *CDF* produces its surprisingly simple *PDF*:

$$f_{\boldsymbol{L}}(l) = \frac{1}{3\pi\sqrt{4r^2 - l^2}}\left(\frac{l}{r}\right)^4, \quad 0 \le l \le 2r.$$

The Bertrand probability in this case is a surprisingly high 0.7468.

Evan's Robot.

Over a cup of coffee on the Stanford campus, Evan Larson (son of the 2nd author) suggested Poisson failures to model randomness.

Starting at its home base and at time $t = 0$, a robot begins traveling clockwise at a constant angular speed of ω (radians) per second around and on the circumference of a circle of radius r. The robot has attached to it a rubber band rooted at its home base. The robot travels around the circle, with the attached rubber band stretched and flexed until the robot stops with a failure episode. Failure occurs as a Poisson process with rate λ per second.

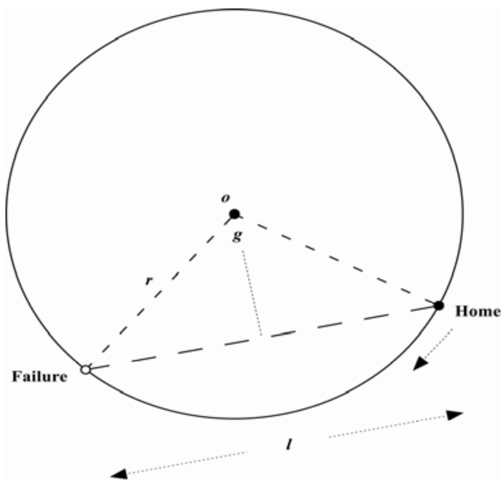

Figure 11

As a function of λ, we shall derive the *CDF* of the length of the rubber band upon the occurrence of the first failure episode. See Figure 11.

Define:

ω = angular speed of the robot, measured in radiant per second.

λ = Poisson failure rate of the robot, with unit as number per second.

$\boldsymbol{L}$ = length of the rubber band.

$\boldsymbol{T}$ = time until failure, negative exponential with parameter λ.

$\boldsymbol{\theta}$ = the total angle (measured in radians) traveled by the robot at the moment it failed.

$\boldsymbol{\theta} = \omega \boldsymbol{T}$.

$\beta = \lambda/\omega$, a function of λ and ω.

$\boldsymbol{G}$ = the acute central angle spanning the robot's home to the point of failure

We first obtain the *CDF* of $\boldsymbol{G}$, from which we shall derive that for $\boldsymbol{L}$.

$$F_{\boldsymbol{G}}(g) = P(\boldsymbol{G} \le g)$$

$$= P(0 < \boldsymbol{\theta} \le g) + P(2\pi - g < \boldsymbol{\theta} \le 2\pi + g) + P(4\pi - g < \boldsymbol{\theta} \le 4\pi + g) + P(6\pi - g < \boldsymbol{\theta} \le 6\pi + g) + \ldots$$

$$= P(0 < \boldsymbol{\theta} \le g) + \sum_{k=1}^{\infty} P(2k\pi - g < \boldsymbol{\theta} \le 2k\pi + g)$$

$$= P(0 < \boldsymbol{T} \le \frac{g}{\omega}) + \sum_{k=1}^{\infty} P\left(\frac{2k\pi - g}{\omega} < \boldsymbol{T} \le \frac{2k\pi + g}{\omega}\right), \quad \text{with } \boldsymbol{\theta} = \omega\boldsymbol{T}$$

$$= 1 - e^{-\lambda\frac{g}{\omega}} + \sum_{k=1}^{\infty}\left[e^{-\lambda\frac{2k\pi-g}{\omega}} - e^{-\lambda\frac{2k\pi+g}{\omega}}\right], \quad \boldsymbol{T} \text{ is an exponential } RV \text{ with parameter } \lambda$$

$$= 1 - e^{-\beta g} + \frac{e^{-2\pi\beta}\left(e^{\beta g} - e^{-\beta g}\right)}{1 - e^{-2\pi\beta}}, \quad \text{substituting } \beta = \frac{\lambda}{\omega}$$

$$= 1 + \frac{e^{\beta g}\left(e^{-2\pi\beta} - e^{-2\beta g}\right)}{1 - e^{-2\pi\beta}}.$$

The *CDF* for the length of the rubber band $\boldsymbol{L}$ when the robot stops can be derived by observing the relationship between $\boldsymbol{G}$ and $\boldsymbol{L}$ as

$$\boldsymbol{L} = 2r\sin\left(\frac{\boldsymbol{G}}{2}\right), \qquad or \quad \boldsymbol{G} = 2\sin^{-1}\left(\frac{\boldsymbol{L}}{2r}\right).$$

The *CDF* of $\boldsymbol{L}$ can be derived from that of $\boldsymbol{G}$ with direct substitution,

$$F_L(l) = 1 + \frac{e^{2\beta\sin^{-1}\left(\frac{l}{2r}\right)}}{\left(1-e^{-2\pi\beta}\right)}\left[e^{-2\pi\beta} - e^{-4\beta\sin^{-1}\left(\frac{l}{2r}\right)}\right], \qquad 0 \le l \le 2r, \quad \text{where } \beta = \frac{\lambda}{\omega}.$$

The *PDF* is derived through differentiation,

$$f_L(l) = \frac{2\beta e^{2\beta\sin^{-1}\left(\frac{l}{2r}\right)}}{\left(1-e^{-2\pi\beta}\right)\sqrt{4r^2-l^2}}\left[e^{-2\pi\beta} + e^{-4\beta\sin^{-1}\left(\frac{l}{2r}\right)}\right], \qquad 0 \le l \le 2r.$$

When $\beta = \lambda/\omega$ is large, the robot will fail almost immediately – the linear distance between the base and the point of failure will be very small – resulting in a virtual zero Bertrand probability. On the other hand, when $\beta = \lambda/\omega$ is small, the robot will likely fail after many revolutions around the circle – resulting in failure location (approaching) equally likely anywhere on the circle circumference, which is the Bertrand 1 uncertainty.

In fact, the chord length *CDF* (and *PDF*) becomes that of Bertrand 1 when $\beta = 0$ (l'Hospital's Rule). This is intuitively plausible considering the *randomness* of a Poisson event: Evan's robot is traveling at a constant speed around the circumference and its first failure is in the distant future (β being small), neutralizing the initial potential early failure favoring short chord length.

SUMMARY

We first provide the various distributions of random chord length below, grouped according to their similarities:

Case 1, Bertrand 1: Angular Randomness

The random chord length distribution is a function of a trigonometry function of chord length:

$$F_{\boldsymbol{L}}(l) = P(\boldsymbol{L} \le l) = \frac{2}{\pi}\sin^{-1}\left(\frac{l}{2r}\right), \quad for \;\; 0 \le l \le 2r.$$

Case 8, Evan's Robot: Becoming Bertrand 1 when $\beta \to 0$

$$F_L(l) = 1 + \frac{e^{2\beta \sin^{-1}\left(\frac{l}{2r}\right)}}{\left(1 - e^{-2\pi\beta}\right)}\left[e^{-2\pi\beta} - e^{-4\beta \sin^{-1}\left(\frac{l}{2r}\right)}\right], \quad 0 \le l \le 2r, \quad \text{where } \beta = \frac{\lambda}{\omega}.$$

Case 2, Bertrand 2: Chord Center is Uniformly Distributed inside the Circle

The random chord length *CDF* is a function of the square of chord length.

$$F_{\boldsymbol{L}}(l) = \frac{l^2}{4r^2}, \quad \textit{for} \;\; 0 \le l \le 2r.$$

Case 3, Bertrand 3: Chord Center is Uniformly Distributed over a Reference Radius

The random chord length distribution equals the square root of a quadratic function of chord length.

$$F_{\boldsymbol{L}}(l) = 1 - \frac{\sqrt{4r^2 - l^2}}{2r}, \quad \textit{for} \;\; 0 \le l \le 2r.$$

Cases 4, 5 and 6: Mixed Randomness

The random chord length distribution combines angular (trigonometric function) randomness and area (square) randomness of chord length.

$$F_{\boldsymbol{L}}(l) = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{l}{2r}\right) - \frac{l\sqrt{4r^2 - l^2}}{4r^2}\right], 0 \le l \le 2r$$

Case 7: Double Area Randomness

The random chord length distribution combines angular (trigonometric function) randomness and (double) area (fourth power) randomness of chord length.

$$F_{\boldsymbol{L}}(l) = \frac{2}{\pi}\cos^{-1}\left(\frac{\sqrt{4r^2 - l^2}}{2r}\right) - \frac{l\left(6r^2 + l^2\right)\sqrt{4r^2 - l^2}}{12\pi r^4}, \quad 0 \le l \le 2r.$$

Table 1 shows various Bertrand probabilities (the probability that a random chord length will exceed $\sqrt{3}r$), median, mean and standard deviation of random chord length for each of the cases. The statistics reported here are in the unit of the circle radius r. The distribution and associated statistics in the case of Evan's Robot (See Table 2) depend on $\beta = \lambda/\omega$, and it approaches that of Bertrand 1 when β is small. We also provide a comparative graph of mean and standard deviation together with the coefficient of variation (the ratio of standard deviation to the mean in Figure 12). It should be noted that the coefficient of variation of Evan's Robot approaches 1 when β goes to infinity, not a surprising result as the exponential failure event dominates the uncertainty.

Table 1. Probabilities and statistics of random chord length of cases 1-7

	Bertrand 1: Two Points on Circumference	Bertrand 2: Chord Center inside Circle	Bertrand 3: Chord Center on Radius	One Point on Circumference, One inside	Both Points inside Circle	Evan's Robot
	Case 1	Case 2	Case 3	Cases 4, 5, 6	Case 7	Case 8
Bertrand Probability	33.33%	25.00%	50.00%	60.90%	74.68%	**
Median	1.4142	1.4142	1.7321	1.8295	1.9021	**
Mean	1.2570	1.3331	1.5453	1.6652	1.7676	**
SD	0.6225	0.4715	0.4772	0.4027	0.3500	**

Table 2. Distribution and statistics for the case of Evan's Robot

$\beta = \lambda / \omega$	Large	2	1	0.5	0.25	0.1	Small
Bertrand Probability	~ 0	1.49%	10.82%	23.81%	30.48%	32.85%	33.33%
Median	~ 0	0.3448	0.6741	1.1074	1.3298	1.4005	1.4142
Mean	~ 0	0.4702	0.7986	1.0793	1.2052	1.2483	1.2570
SD	~ 0	0.4221	0.5945	0.6427	0.6328	0.6245	0.6225

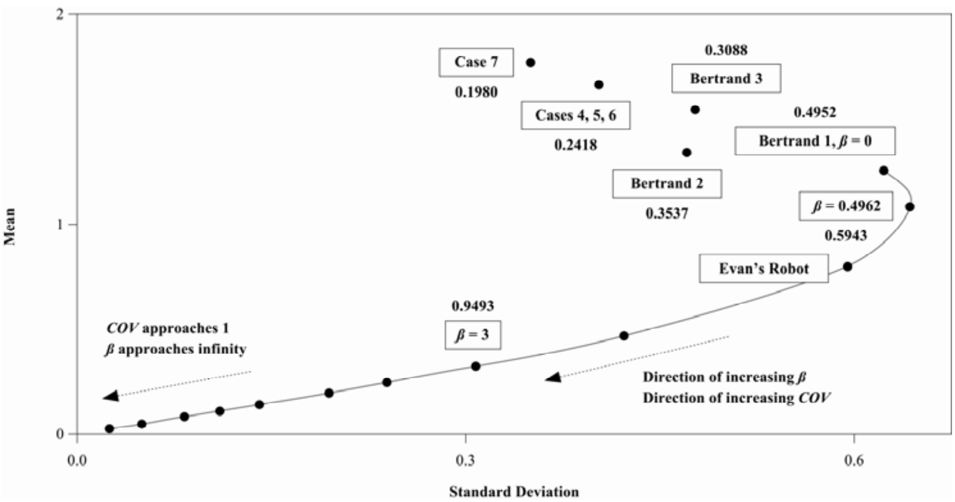

Figure 12. Cofficient of Variation, COV

Figure 14 shows the *CDF*s of a random chord length arising from these various sources of randomness, grouped according to the similarity of their distributions. The *CDF* graph also indicates the location of the Bertrand probability (the vertical dash line when the random chord length equals $\sqrt{3}r$). In fact, the *CDF* value at $\sqrt{3}r$ actually shows one minus the Bertrand probability.

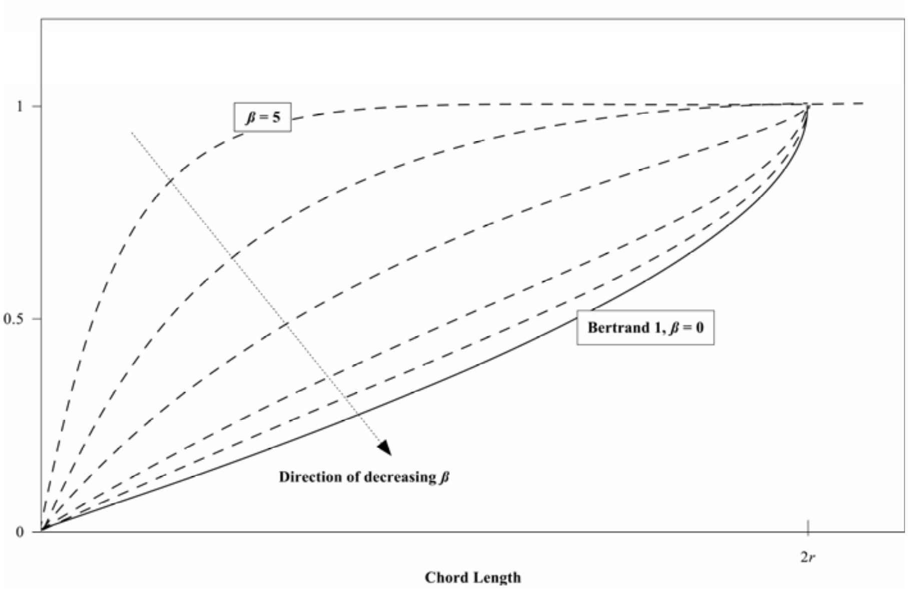

Figure 13. CDF Convergence of Evan's Robot to Bertrand 1

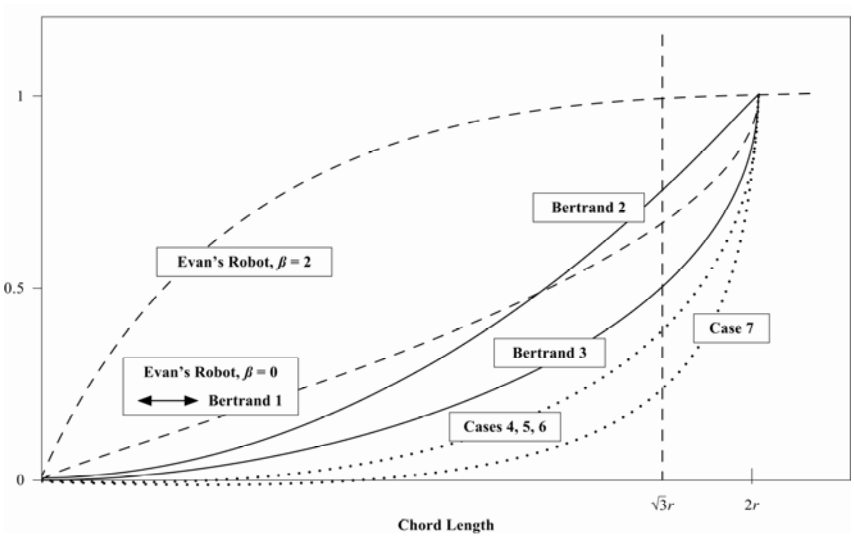

Figure 14. CDF of Random Chord Length

The next graph shows the *PDF*s of the random chord length. We note that, with the exception of Bertrand 2, the *PDF*s go to infinity as the chord length approaches 2*r*.

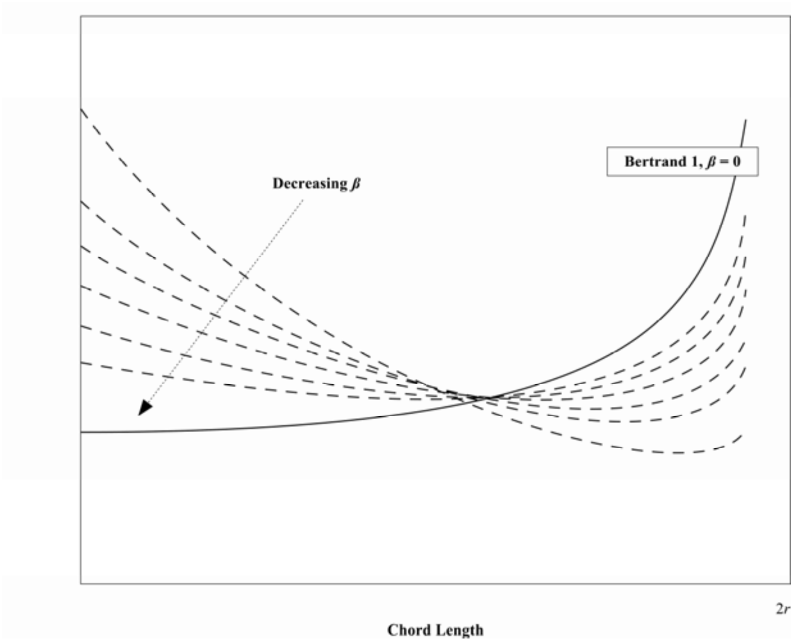

Figure 15. Convergence of Evan's Robot Chord Length PDF

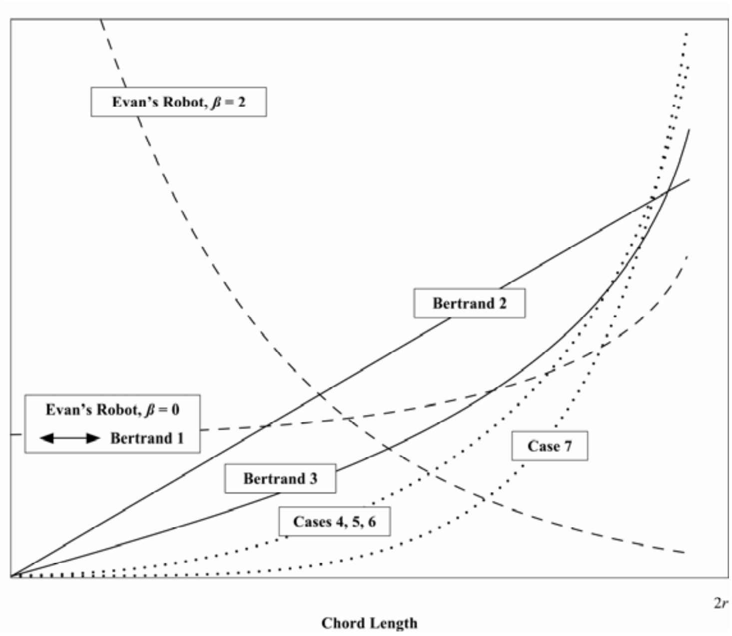

Figure 16. PDF of Random Chord Length

Crofton's Method in Geometric Probability

Students of geometric probability are probably aware of the Crofton's method (Crofton (1885), or Larson and Odoni (1981)) to solve for probabilistic quantities (such as moments and *CDF*) in a geometric setting if certain conditions are satisfied. We provide the Crofton's approach to solve Case 7 in the appendix.

9. CONCLUSION

In this paper we extended Bertrand's Paradox to several other realizations, including perhaps the most natural interpretation of a random chord, one that Bertrand chose not in include in his classic 1888 paper. In addition to deriving the probabilities that the length of an associated *random* chord will exceed $\sqrt{3}$ (on a circle of unit radius), we derived the probability distribution function for the length of each type of random chord. The eight scenarios examined (including Bertrand's original three) result in six distinct distributions for the length of a random chord.

APPENDIX

Appendix 1

Integration details involving the inverse cosine function

We shall evaluate the following integral $\int_{y=x}^{r} \cos^{-1}\left(\frac{x}{y}\right) y\, dy$, through a change of variable:

$$z = \cos^{-1}\left(\frac{x}{y}\right), \quad \cos z = \frac{x}{y}, \quad y = \frac{x}{\cos z}, \quad dy = \frac{x \sin z}{\cos^2 z}\, dz,$$

$$\cos^{-1}\left(\frac{x}{y}\right) y\, dy = x^2\, \frac{z \sin z}{\cos^3 z}\, dz.$$

$$z = \cos^{-1}(1) = 0, \text{ when } y = x; \quad z = \cos^{-1}\left(\frac{x}{r}\right), \text{ when } y = r.$$

In the first line below, we use the above transformation to carry out a change of variable exercise to replace the inverse cosine function. In the second line, we up the variables to carry out integration by parts. From third to fourth: integration of the tangent square function.

$$\int_{y=x}^{r} \cos^{-1}\left(\frac{x}{y}\right) y\, dy = \int_{z=0}^{\cos^{-1}\frac{x}{r}} x^2\, \frac{z \sin z}{\cos^3 z}\, dz = x^2 \int_{z=0}^{\cos^{-1}\frac{x}{r}} z \left[\frac{\sin z}{\cos^3 z}\, dz\right]$$

$$= x^2 \int_{z=0}^{\cos^{-1}\frac{x}{r}} z \left\{\frac{1}{2}\, d\left[\tan^2 z\right]\right\}$$

$$= \frac{x^2}{2}\left\{\left[z \tan^2 z\right]_{z=0}^{\cos^{-1}\frac{x}{r}} - \int_{z=0}^{\cos^{-1}\frac{x}{r}} \tan^2 z\, dz\right\}$$

$$= \frac{x^2}{2}\left\{\cos^{-1}\left(\frac{x}{r}\right)\frac{r^2 - x^2}{x^2} - \left[\tan z - z\right]_{z=0}^{\cos^{-1}\frac{x}{r}}\right\}$$

$$= \frac{x^2}{2}\left\{\cos^{-1}\left(\frac{x}{r}\right)\frac{r^2 - x^2}{x^2} - \frac{\sqrt{r^2 - x^2}}{x} - \cos^{-1}\left(\frac{x}{r}\right)\right\}$$

$$= \frac{r^2}{2}\cos^{-1}\left(\frac{x}{r}\right) - \frac{x\sqrt{r^2 - x^2}}{2}.$$

Appendix 2

Crofton's Method to "solve" Case 7:

The Crofton's method appears as an attractive candidate to compute the *CDF* of a random chord length under Case 7: the two points defining a random chord are independently and uniformly distributed inside the interior of the circle. The problem setting satisfies conditions for the application of this method (such as rotational invariance). Interested readers can consult (Larson and Odoni (1981), or Crofton's original paper (1885)). This method involves examination of the infinitesimal changes of the desired quantity (the *CDF* in our case) as the relevant area is infinitesimally enlarged (the radius of our circle). Such consideration results in a differential equation (the desired *CDF* as a function of the circle radius), whose solution leads to what is being sought.

We are seeking $F_{\boldsymbol{L}}(l;\alpha)$, the *CDF* of the random chord length $\boldsymbol{L}$, when the circle radius is α. To simplify notation, we set $\boldsymbol{X}(\alpha) = \boldsymbol{F_L}(\boldsymbol{l};\alpha)$, assuming a fixed value of l. If we extend the circle radius from α to $\alpha + d\alpha$, $\boldsymbol{X}(\alpha)$ will correspondingly change to $X(\alpha) + dX$ – when the change $d\alpha$ is small. $X(\alpha) + dX$ can be evaluated using conditional expectation (since a *CDF* is a special kind of expectation, as that of an indicator random variable): conditional on where these two random points lie. When α increases to $\alpha + d\alpha$, there are three distinct scenarios describing the location of the two random points:

(1) Both points are inside the original circle of radius α. In this case the sought after quantity remains $X(\alpha)$, with probability $p = \dfrac{\pi\alpha^2\,\pi\alpha^2}{\pi(\alpha + d\alpha)^2\,\pi(\alpha + d\alpha)^2}$.

(2) One point is inside the original circle of radius α, while the other point is inside the circular strip of width $d\alpha$. In this case, we denote the sought after quantity as $X_1(\alpha)$, with probability $q = \dfrac{2(2\pi\alpha\, d\alpha)\pi\alpha^2}{\pi(\alpha + d\alpha)^2\,\pi(\alpha + d\alpha)^2}$. The outside factor of 2 represents the two cases as to which of the two points is inside the original circle of radius α.

(3) Both points are inside the expanded circular strip: the probability of this case is of order $o(d\alpha)$, of which we shall ignore (as $d\alpha$ goes to zero).

Using conditional expectation, we can write:

$$X(\alpha)+dX \approx p*X(\alpha)+q*X_1(\alpha)+o(d\alpha).$$

After simplification and ignoring higher order terms, as well as taking the limit $d\alpha \to 0$:
$\frac{dX(\alpha)}{d\alpha}=\frac{4}{\alpha}\left[X_1(\alpha)-X(\alpha)\right]$, with boundary condition $X(0) = 1$ for $l > 0$.

Crofton's method relies on the hope that it is much simpler to find $\boldsymbol{X}_1(\alpha)$ since one of the two points has been restricted to the "boundary" of the circle. This restricted problem is identical to Case 6 of our *tour de Bertrand*. As we observe in the derivation of Case 6, the solution is indeed simpler:

$$\boldsymbol{X}_1(\alpha) \equiv F_{\boldsymbol{L}}(l)=\frac{2}{\pi}\left[\sin^{-1}\left(\frac{l}{2\alpha}\right)-\frac{l\sqrt{4\alpha^2-l^2}}{4\alpha^2}\right], \quad 0<l<2\alpha$$

$$\boldsymbol{X}_1(\alpha)=\frac{2}{\pi}\left\{\cos^{-1}\left[0.5\sqrt{4-\left(\frac{\boldsymbol{l}}{\alpha}\right)^2}\right]-\frac{\boldsymbol{l}}{4\alpha}\sqrt{4-\left(\frac{\boldsymbol{l}}{\alpha}\right)^2}\right\}, \quad 0<\boldsymbol{l}<2\alpha \cdot$$

How will one solve such a difficult differential equation?

$$\frac{4}{\alpha}X(\alpha)+\frac{dX(\alpha)}{d\alpha}=\frac{8}{\pi\alpha}\left[\sin^{-1}\left(\frac{l}{2\alpha}\right)-\frac{l\sqrt{4\alpha^2-l^2}}{4\alpha^2}\right].$$

We have used a geometrical consideration to solve for $X(\alpha)$ as examined in Case 7. The solution to the above differential equation is,

$$\boldsymbol{X}(\alpha)=\frac{2}{\pi}\cos^{-1}\left(\frac{\sqrt{4\alpha^2-l^2}}{2\alpha}\right)-\frac{l\left(6\alpha^2+l^2\right)\sqrt{4\alpha^2-l^2}}{12\pi\alpha^4}, \quad 0\le l\le 2\alpha.$$

REFERENCES

[1] Aristoff J., Delmore S., Hernandez D., Kao S., Leu T., Lin H., Ramiscal J.A., Zackas R., *Bertrand's Paradox, a web-based Animation,* http://web.mit.edu/tee/www/bertrand/problem.html

[2] Bertrand J. (1888), *Calcul des probabilities*; Paris, 1888.

[3] Chiu S. (2008); What is Randomness? in *Probabilistic Analysis: Structuring, Processing and Presenting Probabilistic Information*; Chapter 11, Class manuscript, Stanford University 2002 and revised September.

[4] Crofton M.W.(1885), Probability; *Encyclopedia Britannica* 19; 9*th* edition, 768-788.

[5] Kendall M.G., MORAN P.A.P. (1963), Geometrical Probability; Charles Griffen; London.

[6] Holbrook J., KIM S.S. (2000), Bertrand’s Paradox Revisited; *Math Intelligencer* 22; 16-19.

[7] Jaynes E.T. (1973), The Well-Posed Problem; *Found. Phys* 3; 477-492.

[8] Larson R.C., Odonia A.R. (1981), Urban Operations Research; Prentice Hall, NJ, Sec. 3.3.2.; Book available on the web at http://web.mit.edu/urban_or_book/www/book/.

[9] Marinoff L. (1994), A Resolution of Bertrand’s Paradox; *Philos. Sci.* 61; 1-24.

[10] Rosenberg E. (2004), The Expected Length of a Random Line Segment in a Rectangle; *Operations Research Letters* 32; 99-102.

[11] Streit F. (1978), On a Statistical Approach to Bertrand’s Problem; *Elem. Math.* 33; 134-138.

[12] Tissler P.E. (1984), Bertrand's Paradox; *The Mathematical Gazette (The Mathematical Association)* 68(443); 15-19.