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Efficient Simulation of a Random Knockout Tournament

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ABSTRACT

We consider the problem of using simulation to efficiently estimate the win probabilities for participants in a general random knockout tournament. Both of our proposed estimators, one based on the notion of "observed survivals" and the other based on conditional expectation and post-stratification, are highly effective in terms of variance reduction when compared to the raw simulation estimator. For the special case of a classical 2^n -player random knockout tournament, where each survivor of the previous round plays in the current round, a second conditional expectation based estimator is introduced. At the end, we compare our proposed simulation estimators based on a numerical example and in terms of both variance reduction and the time to complete the simulation experiment. Based on our empirical study, the method of "observed survivals" is the most efficient method.

1. INTRODUCTION AND SUMMARY

Selecting a single winner from a set of many contestants is the common theme in many sporting events. Tournaments of varying structures are used for choosing a winner in situations where players can be compared pairwise. In a knockout tournament players compete head-to-head in matches; with the losers being eliminated from the tournament. In a random knockout tournament at the outset of each round players are paired at random.

A general knockout tournament involving N players is specified by positive integer parameters n, $m_1,\ldots, m_{n-1}, m_n = 1$, satisfying

$$2m_i \le N - \sum_{j=1}^{i-1} m_j, \qquad i = 1,\ldots,n-1$$

$$\sum_{j=1}^{n} m_j = N - 1$$

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The interpretation of these parameters is that the tournament is to consist of n rounds, with round i involving m_i matches. Let $r_k = N - \sum_{j=1}^{k-1} m_j$ be the number of players remaining at the beginning of round k.

We say that the knockout tournament is a random knockout tournament if the players involved in round i are randomly chosen, and then randomly paired, from the $N - \sum_{j=1}^{i-1} m_j$ players who have not been eliminated in any of the preceding rounds. We assume that the outcome of a match between two players in any round depends only on the pair and not on the round or previous rounds, and we let $P_{ij} = 1-P_{ji}$ denotes the probability that player i defeats player j if they are paired.

The player who wins the match in round n is called the winner of the tournament. Letting P_i be the probability that player i is the winner of the tournament $i = 1, \dots, N$, we are interested in using simulation to efficiently estimate these P_i's

In section 2 we propose an efficient simulation procedure based on the notion of "observed survivals". In section 3 we combine conditional expectation and post-stratified sampling for another simulation approach. In section 4, we consider the "classical knockout tournament" where $N = 2^n$ and round k consists of 2^{n-k} matches, and present a second conditional expectation estimator. Two numerical examples are considered in section 5. In these examples, a considerable variance reduction over the raw simulation estimator is gained when using any of our suggested methods. The time to complete the simulation experiment is also considered for each method. Considering the variance reduction criterion in parallel with the time to obtain the simulation results, we conclude that the observed survival method is best.

2. METHOD 1: OBSERVED SURVIVALS AND THE PRODUCT ESTIMATOR

We can model the evolution of the random knockout tournament from each round to the next one by the transitions of a Markov chain whose states are the sets of players alive at the outset of each round. Let N_i be the total number of rounds that player i survives (that is, for $0 \le j < n$, $N_i = j$ if player i loses in round $j+1$). Our problem of estimating $P_i = P(N_i > n-1)$ is in fact a special case of a more general problem of using simulation to estimate the distribution of the number of transitions it takes a Markov chain to enter a particular set of states. Specifically, N_i+1 is the number of transitions that it takes the Markov chain whose state is the set of surviving players at the beginning of each round to enter a state that does not contain i. In Ross and Schechner (1985) simulation estimators based on observed hazards were developed to efficiently estimate first passage distributions to a given set of states of a Markov chain. Our method 1 uses the estimator developed in Ross and Schechner (1985).

Each simulation run involves simulating the tournament through $n-1$ rounds. (There is no need to simulate round n).

Observe that

$$P_i = P(N_i > n-1) = \prod_{k=1}^{n} P(N_i > k-1 \mid N_i \ge k-1)$$

Note that $s_k(i) = P(N_i > k-1 | N_i \geq k-1)$ is the discrete survival rate value representing the probability that player i survives round k given that he has survived up to round k. Using simulation, we illustrate how to estimate $P(N_i > n-1) = \prod_{k=1}^{n} s_k(i)$.

Remark. Using that i will play in round 1 with probability $2m_1/N$ and, if he does play, that his opponent is equally likely to be any of the other $N-1$ players, we have that

$$s_1(i) = 1 - \frac{2m_1}{N} \sum_{l \neq i} P_{li} / (N-1)$$

Let N_i^j be the number of rounds that i survives in the jth simulation run. We define "observed survivals" at the jth simulation run as point estimates of $s_k(i)$, $k = 1,\ldots,n$, as follows. Let

$$\lambda_k^j(i) = P(N_i^j > k-1 | \beta_{k-1}^j)$$

where β_k^j is the simulated set of players present at the beginning of round k. Thus, if N_i^j is the number of rounds that i survives in the jth simulation run, then

$$\lambda_k^j(i) = \begin{cases} 1 - \dfrac{2m_k}{r_k(r_k-1)} \sum_{l \in \beta_k^j, l \neq i} P_{li} & \text{if} \quad N_i^j \geq k-1 \\ 0 & \text{if} \quad N_i^j < k-1 \end{cases}$$

Note that $\lambda_k^j(i)$ is an unbiased estimator of $s_k(i)$ provided that $N_i^j \geq k-1$.

Suppose that we make a total of r simulation runs. Let

$$\overline{\lambda}_k(i) = \sum_{j=1}^{r} \lambda_k^j(i) / \{\# j : N_i^j \geq k-1\}$$

Then $\overline{\lambda}_k(i)$ is an estimate of the discrete survival rate value $s_k(i) = P(N_i > k-1 | N_i \geq k-1)$ and $\prod_{k=1}^{n} \overline{\lambda}_k(i)$ is a consistent estimate of $P(N_i > n-1) = \prod_{k=1}^{n} s_k$ as r, the number of simulation runs, goes to infinity.

Remark. Suppose performing r simulation runs, we have not been able to obtain point estimates of $s_m(i),\ldots, s_n(i)$. In other words, consider the situation where $N_i^j < m-1$ for all $j = 1, .., r$. To estimate $P(N_i > n-1)$ based on the aforementioned estimator it is useful to derive analytic bounds for $s_k(i)$.

Consider the set of probabilities for player i, $\{P_{ij}: j \neq i, j = 1, 2, \ldots, N\}$, and order them so that $P_{iI_1} \leq P_{iI_2} \leq \ldots \leq P_{iI_{N-1}}$. Then, lower and upper bounds for $s_k(i)$ are as follows

$$1-\frac{2m_k}{r_k(r_k-1)}\sum_{j=1}^{r_k-1}P_{I_j i}\le s_k(i)\le 1-\frac{2m_k}{r_k(r_k-1)}\sum_{j=N-r_k+1}^{N-1}P_{I_j i}$$

Approximation of *Var* ($\prod_{k=1}^{n}\overline{\lambda}_k$)

In Ross and Schechner (1985) a two-step method to approximate *Var* ($\prod_{k=1}^{n}\overline{\lambda}_k$) is developed. First *Var* ($\overline{\lambda}_k$), $k=1, \ldots, n$ is approximated, and then using the Delta method and combining these approximations, an approximation of *Var* ($\prod_{k=1}^{n}\overline{\lambda}_k$) is obtained. (See Ross and Schechner, 1985 for details.)

3. METHOD 2: CONDITIONAL EXPECTATION AND POST STRATIFICATION

Until player i loses a match, player i is designated as the surrogate for i. If a player beats i in some round, then at the beginning of the next round that player becomes the surrogate for i. Any player beating a surrogate for i during a round takes over as the surrogate for i at the beginning of the next round. In this manner, at the beginning of each round exactly one of the remaining players is the surrogate for i. Let T_i be the number of matches played by players while they are the surrogate of i.

Suppose that $T_i=t$. Let $F_{i,1}$ denote the first player to play i in a match; let $W_{i,1}$ be the winner of that match. Let $F_{i,2}$ denote the next player to play $W_{i,1}$ in a match; let $W_{i,2}$ be the winner of that match. Let $F_{i,3}$ denote the next player to play $W_{i,2}$ in a match; let $W_{i,3}$ be the winner of that match, and so on up to $F_{i,t}$. Consequently, in order to win the tournament, player i would have to successively beat all the players $F_{i,j}$, $j=1,\ldots,t$.

Now, let

$$p_j=\frac{2m_j}{r_j},\quad j=1,\ldots,n$$

denote the proportion of the players that have survived the first $j-1$ rounds who play in round j. Also, let I_j, $j=1,\ldots,n$, be independent Bernoulli random variables such that

$$P(I_j=1)=p_j,\quad j=1,\ldots,n$$

It is easy to see that

$$P(T_i=k)=P(\sum_{j=1}^{n}I_j=k)$$

To compute the preceding probabilities, let

$$P_s(k)=P(\sum_{j=1}^{s}I_j=k),\quad s\le n,\ k\le s$$

Conditioning on I_s gives that

$$P_s(k) = p_s P_{s-1}(k-1) + (1-p_s)P_{s-1}(k)$$

Starting with $P_1(1) = p_1$, $P_1(0)=1-p_1$, we can recursively solve the preceding until we have the quantities $P_n(k)$, $k=0,...,n$.

Let I_i be the indicator of the event that i wins the tournament. Because

$$E[I_i | T_i = t, F_{i,1},...,F_{i,t}] = \prod_{j=1}^{t} P_{i,F_{i,j}}$$

we could use $\prod_{j=1}^{t} P_{i,F_{i,j}}$ as a conditional expectation estimator of P_i. However, because there would appear to be a strong negative correlation between I_i and T_i, a post stratification estimator should have an even smaller variance. That is, write

$$P_i = \sum_{k=1}^{n} E[I_i | T_i = k] P_n(k)$$

and then, after completing r simulation runs, estimate $E[I_i | T_i = k]$ by the average of the quantities $\prod_{j=1}^{T_i} P_{i,F_{i,j}}$ obtained in those runs having $T_i=k$. It can be shown (see Glasserman, 2004, p. 235) that as the number of simulation runs goes to infinity the variance of the post-stratified estimator is of the same order of magnitude as that of the stratified estimator that does the proportion $P_n(k)$ of its runs conditional on $T_i=k$. We can also estimate the variance of the estimator of $E[I_i | T_i = k]$ with the sample variance of the values $\prod_{j=1}^{T_i} P_{i,F_{i,j}}$ having $T_i=k$ divided by the generated number of such values.

Remark. For a fixed value of i, there may be values of k for which there are no runs for which $T_i=k$. If the value of $P_n(k)$ is sufficiently small, we can use 0 as the estimate of $E[I_i | T_i = k] P_n(k)$. However, if there are enough such values of k, the preceding post stratification will not work for estimating P_i. In this case, we suggest using the simulated data to estimate P_i by using the conditional expectation estimator $\prod_{j=1}^{T_i} P_{i,F_{i,j}}$ in conjunction with using T_i as a control variable with known mean $E[T_i] = \sum_{j=1}^{n} p_j$. (Of course, we would still use the post stratification procedure to estimate the values P_j for which the preceding difficulty did not arise.)

Example. To get an idea of the amount of variance reduction the preceding methods yield, suppose we want to estimate P_i for a specified i when $T_{ij}=\alpha, j \neq i$. Now, with I_j, $j=1,...,n$, as previously defined

$$P_i = E[\alpha^{T_i}]$$

$$= E[\alpha^{\sum_{j=1}^{n} I_j}]$$

$$= \prod_{j=1}^{n} E[\alpha^{I_j}]$$

$$= \prod_{j=1}^{n} (\alpha p_j + 1 - p_j)$$

$$= \prod_{j=1}^{n} (1 - (1-\alpha) 2 m_j / r_j)$$

The straight conditional expectation estimator reduces, in this case, to the estimator α^{T_i}. Its variance is

$$Var(\alpha^{T_i}) = E[(\alpha^{T_i})^2] - P_i^2$$

$$= E[(\alpha^2)^{T_i}] - P_i^2$$

$$= \prod_{j=1}^{n} (1 - (1-\alpha^2) 2 m_j / r_j) - P_i^2$$

For instance, when N=5, $m_j \equiv 1$, α=1/2, the preceding gives that

$$Var((1/2)^{T_i}) = \prod_{j=1}^{4} (1 - \frac{3}{2(6-j)}) - (1/5)^2 \approx .015$$

whereas the variance of the raw simulation estimator is $P_i(1-P_i) = 0.16$.

Both the conditional expectation stratification estimator and the previous section's estimator based on survival probabilities have 0 variance when $P_{ij} \equiv \alpha$.

4. CLASSICAL KNOCKOUT TOURNAMENT

In a classical random knockout tournament $N=2^n$, and there are 2^{n-k} matches in the kth round of the tournament. In performing a simulation run we can determine pairings in each round by generating a single random permutation $R_1, ..., R_{2^n}$ of 1,…, 2^n that would determine how to assign players to starting positions that are numbered from 1 to 2^n. Matches in the first round are specified by pairing players assigned to adjacent starting positions. In the kth round, adjacent winners of the previous round are paired, $k = 2, ..., n$. Consider, for instance, an eight-player random knockout tournament with, [1, 2, 3, 4, 5, 6, 7, 8], as the generated random permutation. In the first round, player 1 plays against player 2, player 3 plays against player 4, player 5 plays against player 6, and player 7 plays against player 8. In the second round, the winner of player 1 and 2 plays against the winner of player 3 and 4, the winner of player 5 and 6 plays against the winner of player 7 and 8. In the last round the survivors of the second round play against each other. A recursive procedure has been introduced in Edwards (1996) to compute winning probabilities for a classical random knockout tournament when players' starting positions are known and from that point on pairings are determined as described above. Thus, given the random permutation R, we can compute the conditional win probabilities, P_i, $i=1,\ldots,2^n$. Thus, in this case, we have a second conditional expectation estimator.

Remark. When compared to method 2, the second conditional expectation estimator is derived by conditioning on less information and so it is preferable to method 2 in terms of the variance reduction criterion.

5. NUMERICAL EXAMPLES

Consider a knockout tournament with 8 players. It is common to represent players' probabilities, P_{ij}, i=1,2,…,8; j=1,2,…,8, in a matrix which is usually called the "preference matrix". Consider the following preference matrix for an 8-player random knockout tournament:

$$\begin{pmatrix} - & .2 & .3 & .5 & .4 & .7 & .8 & .6 \\ .8 & - & .4 & .6 & .7 & .8 & .5 & .5 \\ .7 & .6 & - & .2 & .3 & .9 & .5 & .3 \\ .5 & .4 & .8 & - & .7 & .6 & .1 & .6 \\ .6 & .3 & .7 & .3 & - & .8 & .4 & .5 \\ .3 & .2 & .1 & .4 & .2 & - & .6 & .6 \\ .2 & .5 & .5 & .9 & .6 & .4 & - & .3 \\ .4 & .5 & .7 & .4 & .5 & .4 & .7 & - \end{pmatrix}$$

We consider both a one-match-per-round random knockout tournament (N=8, n=7, $m_i \equiv 1$), and a classical 8-player random knockout tournament. Using 10^5 simulation runs, we estimate P_i's for both types of tournaments.

The result. The following first three tables refer to the simulation result of the one-match-per-round random knockout tournament, and the last three tables refer to the classical random knockout tournament. (Note that in tables given below, numbers below each P_i refer to simulation estimates related to player i). Recall that method 1 is the method of observed survivals, and method 2 refers to the conditional expectation combined with post-stratification. In the simulation of the classical knockout tournament method 3 refers to the second conditional expectation estimator introduced in section 4. Based on our numerical example, method 2 in the one-match-per-round tournament and both method 2 and 3 in the classical knockout tournament are preferable to the method of observed survivals in terms of variance reduction. However, in addition to the variance reduction, the factor of time has also been taken into account by measuring the amount of time MATLAB takes to complete our simulation experiments. It is interesting to observe that our estimator based on the notion of observed survivals takes considerably less time compared to the conditional expectation based estimators in both types of tournaments. This can be explained by noting that when computing simulation estimators based on the notion of conditional expectation introduced in this paper, there are many multiplication operations involved in a single simulation run. In contrast, in method 1, point estimates of $s_k(i)$ are first collected for $k=1,2,\ldots,n$ over 10^5 simulation runs, and then averaged. P_i is then estimated by the product of these averages at the end of the simulation experiment. Based on the values obtained in our numerical example, method 1 is the most efficient simulation procedure for estimating P_i's in random knockout tournaments.

One Match Per Round Random Knockout Tournament

Table 1. Estimates of P_i's

	P1	P2	P3	P4	P5	P6	P7	P8
Raw Simulation	.1097	.2129	.1132	.1388	.1195	.0481	.1208	.1369
Method 1	.1112	.2112	.1141	.1394	.1184	.0483	.1207	.1365
Method 2	.1113	.2098	.1131	.1394	.1183	.0483	.1194	.1370

Table 2. Variance of the estimators based on 10^5 simulation runs (10^{-9} scale)

	P1	P2	P3	P4	P5	P6	P7	P8
Raw Simulation	977	1676	1004	1195	1052	458	1062	1182
Method 1	448.45	122.2	182.1	208.43	104	65.67	130.08	52.31
Method 2	59	111	63	84	55	21	68	62

Table 3. Elapsed time (in second)

Raw Simulation	Method 1	Method 2
101.926	184.9	7362

Classical Random Knockout Tournament

Table 4. Estimates of P_i's

	P1	P2	P3	P4	P5	P6	P7	P8
Raw Simulation	.1083	.2249	.1107	.1422	.1183	.0367	.1209	.138
Method 1	.1078	.2245	.1123	.1439	.1174	.0363	.1188	.1384
Method 2	.1082	.2244	.1124	.1437	.118	.0366	.1192	.1385
Method 3	.108	.2241	.1122	.1437	.1176	.0365	.1193	.1387

Table 5. Variance of the estimators based on 10^5 simulation runs (10^{-9} scale)

	P1	P2	P3	P4	P5	P6	P7	P8
Raw Simulation	966	1743	985	1220	1043	353	1063	1189
Method 1	107	81	114	130	78	38	117	42
Method 2	51	60	68	92	40	10	51	21
Method 3	14	23	22	31	13	3	16	9

Table 6. Elapsed time (in second)

Raw Simulation	Method 1	Method 2	Method 3
9.1791	13.2288	2580.4	3475.3

Remark. Recall in method 1, we compute $\overline{\lambda}_k$'s from r simulation runs to obtain a single point estimate of P_i. We used the method of Ross and Schechner (1985) to estimate the variance of the estimator. Calling this the internal variance estimator, we studied its accuracy by comparing it with an external variance estimator $Var(ext)$ based on a meta-experiment of 1000 independent experiments.

From the aforementioned numerical example, we obtained for the classical tournament that $Var(int)=120\times10^{-9}$ and $Var(ext)=118\times10^{-9}$, leading us to conclude that the internal variance estimator is accurate.

REFERENCES

[1] Chung F.R.D., Hwang F.K. (1978), Do stronger players win more knockout tournaments?; *J. Am. Statist. Ass.* 73; 593-596.

[2] David H.A. (1959), Tournaments and Paired comparisons; *Biometrika* 46; 139-149.

[3] Edwards C.T. (1996), Double-elimination tournaments: Counting and calculating; *Amer. Statistician* 50; 27-33.

[4] Glasserman P. (2004), Monte Carlo Methods in Financial Engineering; *Springer*.

[5] Glenn W.A. (1960), A comparison of the effectiveness of tournaments; *Biometrika* 47; 253-262.

[6] Horen J., Riezman R. (1985), Comparing draws for single elimination tournaments; *Operations Research* 33; 249-262.

[7] Hwang F.K. (1982), New concepts in seeding knockout tournaments; *Am. Math. Monthly* 107; 140-150.

[8] Marchand E. (2002), On the comparison between standard and knockout tournaments; *The Statistician* 51(2); 169-178.

[9] Ross and Schechner (1985) Ross S.M., Schechner Z. (1985), Using Simulation to Estimate First Passage Distribution; *Management Science* 31(2).

[10] Ross S.M. (2006), SIMULATION; fourth ed.; Academic Press.

[11] Searls D.T. (1963), On the probability of winning with different tournament procedures; *J. Amer. Stat. Assoc.*58; 1064-1081.