Journal of Industrial and Systems Engineering
Vol. 2, No. 1, pp 41-50
Spring 2008

Deriving the Exact Cost Function for a Two-Level Inventory System with Information Sharing

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ABSTRACT

In this paper we consider a two-level inventory system with one warehouse and one retailer with information exchange. Transportation times are constant and retailer faces independent Poisson demand. The retailer applies continuous review (*R*,*Q*)-policy. The supplier starts with *m* initial batches (of size *Q*), and places an order to an outside source immediately after the retailer's inventory position reaches *R+s*. In this system the lead time of the retailer is determined not only by the constant transportation time but also by the random delay incurred due to the availability of stock at the supplier. A recent paper has obtained the approximate value of the expected cost for this system by using the expected value of the retailer's lead time and hence has pointed out that the optimal supplier policy is an open question. In this paper we tackle this open question and obtain the exact value of the expected system cost by using the idea of the one-for-one ordering policy and implicitly incorporating the distribution function of the random delay.

Keywords: Multi-echelon inventory, Information sharing, Continuous review, Poisson demand

1. INTRODUCTON

This paper deals with an inventory system with one warehouse (supplier) and one retailer with information exchange. Transportation times from an outside source to the supplier and from the supplier to the retailer are constant.

The literature of incorporating information on multi-echelon inventory systems is rather limited and relatively recent. (Milgrom & Roberts, 1990) identified information as a substitute for inventory on economic terms. (Lee & Whang, 1998) discuss the use of information sharing in supply chains in practice, relate it to academic research and outline the challenges facing the area. (Cheung & Lee, 1998) examine the impact of information availability in order coordination and allocation in a Vendor Managed Inventory (VMI) environment. (Cachon & Fisher, 2000) consider an inventory system with one supplier and N identical retailers. Inventories are monitored periodically and the supplier has information about the inventory position of all retailers. All locations follow an (R, nQ) ordering policy with the supplier's batch size being an integer multiple of that of the

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retailers. Cachon and Fisher show how the supplier can use such information to better allocate stock to retailers.

(Gavirneni, 2002) illustrates how information flows in supply chains can be better utilized by appropriately changing the operating policies in the supply chain. The author considers a supply chain containing a capacitated supplier and a retailer facing independent and identically distributed demands. In his setting two models were considered. (1) the retailer is using the optimal (s, S) policy and providing the supplier information about her inventory levels; and (2) the retailer, still sharing information on her inventory levels, orders in a period only if by the previous period the cumulative end-customer demand since she last ordered was greater than a specified value . In model 1, information sharing is used to supplement existing policies, while in model 2; operating policies were redefined to make better use of the information flows.

(Hsiao and Shieh, 2006) consider a two-echelon supply chain, which contains one supplier and one retailer. They study the quantification of the bullwhip effect and the value of information sharing between the supplier and the retailer under an autoregressive integrated moving average (*ARIMA*) demand of (0,1,*q*). Their results show that with an increasing value of *q*, bullwhip effects will be more obvious, no matter whether there is information sharing or not. They show when there exists information sharing, the value of the bullwhip effect is greater than it is without information sharing. With an increasing value of *q*, the gap between the values of the bullwhip effect in the two cases will be larger.

Poisson models with one-for-one ordering policies can be solved very efficiently. (Sherbrooke, 1968) and (Graves, 1985) present different approximate methods. (Seifbarghi & Akbari, 2006) investigate the total cost for a two-echelon inventory system where the unfilled demands are lost and hence the demand is approximately a Poisson process. (Axsäter, 1990a) provides exact solutions for the Poisson models with one-for-one ordering policies. For special cases of (R,Q) policies, various approximate and exact methods have been presented in the literature. Examples of such methods are (Deuermeyer & Schwarz, 1981), (Moinzadeh & Lee, 1986), (Lee & Moinzadeh, 1987a and b), (Svoronos & Zipkin, 1988), (Axsäter, Forsberg, & Zhang, 1994), (Axsäter,1990b and 1993) and (Forsberg, 1996). As a first step (Axsäter,1993) expressed costs as a weighted mean of costs for one-for-one ordering polices. He (Forsberg, 1996) exactly evaluated holding and shortage costs for a two-level inventory system with one warehouse and N different retailers. He also expressed the policy costs as a weighted mean of costs for one-for-one ordering policies. (Forsberg, 1995) considers a two-level inventory system with one warehouse and N retailers. In (Forsberg, 1995) [9], the retailers face different compound Poisson demand processes. To calculate the compound Poisson cost, he uses Poisson costs from (Axsäter,1990a).

(Moinzadeh, 2002), considered an inventory system with one supplier and M identical retailers. All the assumptions that we use in this paper are the same as the one he used in his paper. That is the retailer faces independent Poisson demand and applies continuous review (R,Q)-policy. Excess demand is backordered in the retailer. No partial shipment of the order from the supplier to the retailer is allowed. Delayed retailer orders are satisfied on a first-come, first-served basis. The supplier has online information on the inventory status and demand activities of the retailer. He starts with m initial batches (of size Q), and places an order to an outside source immediately after the retailer's inventory position reaches $R+s$, ($0 \le s \le Q-1$). It is also assumed that outside source has ample capacity.

For evaluating the total system cost, using the results in (Hadley and Whitin, 1963) for one level-one retailer inventory system, Moinzadeh found the holding and backorder costs at each retailer and the holding cost at the supplier. The holding cost at each retailer is computed by the expected on hand inventory at any time (Hadley & Whitin, 1963). In the above system the lead time of the retailer is a random variable. This lead time is determined not only by the constant transportation time but also by the random delay incurred due to the availability of stock at the supplier. In his derivation Moinzadeh used the expected value of the retailer's lead time to approximate the lead time demand and pointed out that "*the form of the optimal supplier policy in the context of our model is an open question and is possibly a complex function of the different combinations of inventory positions at all the retailers in the system*" (Moinzadeh, 2002). As Hadley and Whitin noted, treating the lead time as a constant equal to the mean lead time, when in actuality the lead time is a random variable, can lead to carrying a safety stock which is much too low. The amount of the error increases as the variance of the lead time distribution increases (Hadley and Whitin, 1963).

In this paper, we implicitly derive the exact probability distribution of this random variable and obtain the exact system costs as a weighted mean of costs for one-for-one ordering policies, using the Axsäter's exact solutions for Poisson models with one-for-one ordering policies (Axsäter, 1990a).

In what follows we provide a detailed formulation of the problem and we show how to derive the total cost expression of this inventory system.

2. PROBLEM FORMULATION

In this paper we use the following notation:

S_0	Supplier inventory position in an inventory system with a one- for-one ordering policy
S_1	Retailer inventory position in an inventory system with a one-for-one ordering policy
L	Transportation time from the supplier to the retailer
L_0	Transportation time from the outside source to the supplier (Lead time of the supplier)
λ	Demand intensity at the retailer
h	Holding cost per unit per unit time at the retailer
h_0	Holding cost per unit per unit time at the supplier
β	Shortage cost per unit per unit time at the retailer
t_i	Arrival time of the ith customer after time zero
$c(S_0, S_1)$	Expected total holding and shortage costs for a unit demand in an inventory system with a one-for-one ordering policy
R	The retailer's reorder point

Q Order quantity at both the retailer and the supplier

m Number of batches (of size Q) initially allocated to the supplier

K Expected total holding and shortage costs for a unit demand

$TC(R,m,s)$ Expected total holding and shortage costs of the system per time unit, when the supplier starts with m initial batches (of size Q), and places an order to an outside source immediately after the retailer's inventory position reaches $R+s$

In this paper we use the following assumptions;

1) Transportation time from the outside source to the supplier is constant.

2) Transportation time from the supplier to the retailer is constant.

3) Arrival process of customer demand at the retailer is a Poisson process with a known and constant rate.

4) Each customer demands only one unit of product.

5) Supplier has online information on the inventory position and demand activities of the retailer.

To find K , the expected total holding and shortage costs for a unit demand, we express it as a weighted mean of costs for the one-for-one ordering policies. As we shall see, with this approach we do not need to consider the parameters L, L_0, h, h_0 and β explicitly, but these parameters will, of course, affect the costs implicitly through the one-for-one ordering policy costs. To derive the one-for-one carrying and shortage costs, we suggest the recursive method in (Axsäter,1990a and 1993).

3. DERIVING THE MODEL

To find the total cost, first, following the (Axsäter,1990a)'s idea, we consider an inventory system with one warehouse and one retailer with a one-for-one ordering policy. Also, as in (Axsäter,1990a) let S_0 and S_1 indicate the supplier and the retailer inventory positions respectively in this system. When a demand occurs at the retailer, a new unit is immediately ordered from the supplier and the supplier orders a new unit at the same time. If demands occur while the warehouse is empty, shipment to the retailer will be delayed. When units are again available at the warehouse the demands at the retailer are served according to a first come first served policy. In such situation the individual unit is, in fact, already virtually assigned to a demand when it occurs, that is, before it arrives at the warehouse.

For the one-for-one ordering policy as described above, we can say that any unit ordered by the supplier or the retailer is used to fill the S_i th ($i=0,1$) demand following this order. In other words, an arbitrary customer consumes S_1 th (S_0 th) order placed by the retailer (supplier) just before his arrival to the retailer. Axsäter (1990a) obtains the expected total holding and shortage costs for a unit demand, that is, $c(S_0, S_1)$ for the one-for-one ordering policy.

In this paper, based on the one-for-one ordering policy as described above, we first show that the expected holding and shortage costs for a unit demand, say j^{th} customer's order in a retailer's order of size Q, $j=1\ldots,Q$, is exactly equal to the total costs for a unit demand in a base stock system with supplier and retailer's inventory positions $S_0=s+mQ$ and $S_1=R+j$, respectively and so is equal to $c(s+mQ, R+j)$ (see (A.12)). Then, considering Q separate base stock systems in which the inventory positions of the supplier and the retailer for the j^{th} base stock system are $s+mQ$ and $R+j$ respectively, we obtain the exact value of $TC(R,m,s)$, the expected total holding and shortage

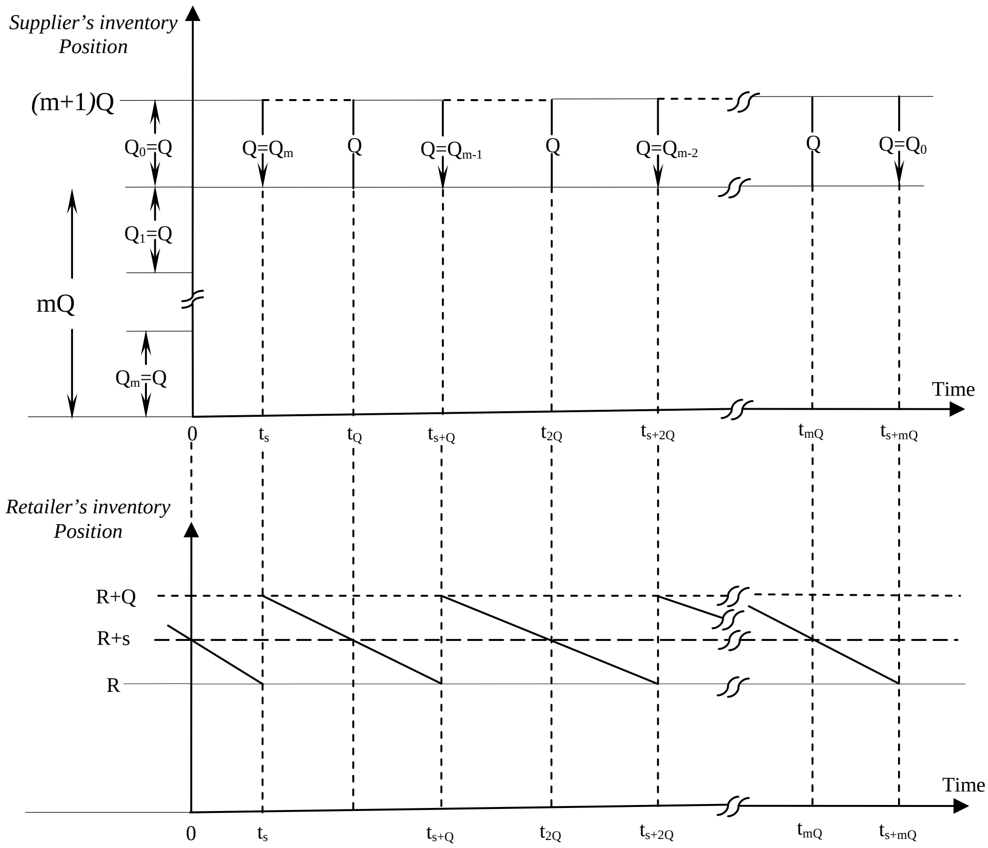

Figure 1. Supplier's inventory and inventory position of the retailer

costs per time unit for an inventory system with the following characteristics:

- The single retailer faces independent Poisson demand and applies continuous review (R,Q)-policy.

- The supplier starts with m initial batches (of size Q) and places an order to an outside source immediately after the retailer's inventory position reaches $R+s$.

- The outside source has ample capacity.

We intend to show that the total cost per unit time is the sum of Q separate base stock systems in which the inventory positions of the supplier and the retailer for the j^{th} base stock system are $s+mQ$ and $R+j$ respectively, $j=1,\ldots,Q$, i.e.,

$$TC(R,m,s)=\frac{\lambda}{Q}.\sum_{j=1}^{Q}c(s+mQ,R+j)$$

To prove this assertion, let us consider a time at which the supplier places an order to the outside source. We designate this time as time zero (see Figure 1). We also denote the batch which the supplier orders at time zero by Q_0. At this time, the retailer's inventory position is exactly $R+s$ and the supplier's inventory position will just reach $(m+1)Q$. Thus the batch Q_0 will fill the $(m+1)^{\text{th}}$ retailer's order to the warehouse. Let us denote the arrival times of customers who arrive after time zero by $t_1, t_2, \ldots$. Then at time t_s when the s^{th} customer arrives, the retailer will order one batch of size Q, and the supplier's inventory position will drop to mQ (see Figure 1). We note that after time zero, at the arrival time of $(s+mQ)^{\text{th}}$ customer, i.e., at time t_{s+mQ}, the retailer will order a batch of size Q. This retailer's order will be fulfilled by the (same) batch Q_0 that was ordered by the supplier at time zero. This means that the batch Q_0 is released from the warehouse when $(s+mQ)^{\text{th}}$ system demand has occurred after this order, i.e. after time zero.

The first unit in the batch Q_0 will be used in the same way to fill the $(R+1)^{\text{th}}$ retailer demand after the retailer's order for batch Q_0 (that is, after time t_{s+mQ}).Thus the first unit in the batch Q_0 will have the same expected retailer and warehouse costs as a unit in a base stock system with $S_0=s+mQ$ and $S_1=R+1$ (call it the first base stock system). Therefore the corresponding expected holding and shortage costs will be equal to $c(s+mQ\ , R+1)$ (see (A.12)).

By the same reasoning, it can be seen that the j^{th} unit in the batch Q_0 will be used to fill the $(R+j)^{\text{th}}$ retailer demand after the retailer's order for batch Q_0. Then the j^{th} unit in the batch Q_0 will have the same expected retailer and warehouse costs as a unit in a base stock system with $S_0=s+mQ$ and $S_1=R+j$ (call it the j^{th} base stock system). Therefore the expected holding and shortage costs for the j^{th} unit in the batch Q_0 will be equal to $c(s+mQ\ , R+j)$, $j=1,\ldots,Q$ (see (A.12)).

It should be noted that each customer, demands only one unit of a batch of size Q. If we number the customers who use all Q units of this batch from 1 to Q, then the demand of any customer will be filled randomly by one of these Q units. That is, each unit of a batch of size Q will be consumed by the j^{th} ($j=1,2,\ldots,Q$) customer according to a discrete uniform distribution on $1,2,\ldots,Q$. In other words, the probability that the i^{th} ($i=1,2,\ldots,Q$) unit of a batch of size Q is used by the j^{th} ($j=1,2,\ldots,Q$) customer is equal to $\frac{1}{Q}$. Therefore we can now express the expected total cost for a unit demand as:

$$K=\frac{1}{Q}.\sum_{j=1}^{Q}c(s+mQ,R+j) \qquad (1)$$

Since the average Poisson demand per unit of time is equal to λ, the total cost of the system per unit time can then be written as:

$$TC(R,m,s) = \lambda.K$$
$$= \frac{\lambda}{Q}.\sum_{j=1}^{Q} c(s+mQ, R+j) \qquad (2)$$

which proves our assertion.

4. CONCLUSION

In this paper we showed how to obtain the exact value of the total holding and shortage costs for a two-level inventory system which consists of one warehouse and one retailer with information exchange. The transportation times from the outside source to the supplier and from the supplier to the retailer are both known and constant. In this system it is assumed that the supplier starts with m initial batches of size Q, and places an order to an outside source immediately after the retailer's inventory position reaches $R+s$.

For the above inventory system the delivery time for the retailer is equal to the transportation time plus a random delay due to stock out at the supplier. In a recent paper it was pointed out that the optimal supplier policy in this inventory system is an open question and for evaluating the approximate cost of the system, the expected value of this random delay is used. In this paper we tackled this open question.

In this paper using the idea of the one-for-one ordering policies, we implicitly incorporated the distribution function of the random delay to obtain the exact value of the system costs. To do so we derived the total costs of this two-echelon inventory system as a weighted mean of costs for the one-for-one ordering policies.

APPENDIX

Evaluation of the One-For-One Ordering Policies

This Appendix is a summary of Axsäter (1990). For more details one can see Axsäter's paper.
We define (as in Axsäter (1990) for one retailer) the following notations:

$g^{S_0}(t) =$ Density function of the Erlang (λ, S_0)

and,

$G^{S_0}(t) =$ Cumulative distribution function of $g^{S_0}(t)$.
thus,

$$g^{S_0}(t) = \frac{\lambda^{S_0} t^{S_0-1}}{(S_0-1)!} e^{-\lambda t}, \qquad (A.1)$$

and,

$$G^{S_0}(t) = \sum_{k=S_0}^{\infty} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \tag{A.2}$$

The average warehouse holding costs per unit is:

$$\gamma(S_0) = \frac{h_0 S_0}{\lambda}(1 - G^{S_0+1}(L_0)) - h_0 L_0 (1 - G^{S_0}(L_0)), \quad S_0 > 0 \tag{A.3}$$

and for $S_0 = 0$,

$$\gamma(0) = 0. \tag{A.4}$$

Given that the value of the random delay at the warehouse is equal to t, the conditional expected costs per unit at the retailer is:

$$\pi^{S_1}(t) = e^{-\lambda(L+t)} \frac{h+\beta}{\lambda} \sum_{k=0}^{S_1-1} \frac{(S_1-k)}{k!}(L+t)^k \lambda^k + \beta(L + t - \frac{S_1}{\lambda}) \tag{A.5}$$

($0! = 1$ by definition),

The expected retailer's inventory carrying and shortage cost to fill a unit of demand is:

$$\Pi^{S_1}(S_0) = \int_0^{L_0} g_0^{S_0}(L_0 - t)\pi^{S_1}(t)dt + (1 - G_0^{S_0}(L_0))\pi^{S_1}(0) \tag{A.6}$$

And,

$$\Pi^{S_1}(0) = \pi^{S_1}(L_0) \tag{A.7}$$

Furthermore, for large value of S_0, we have

$$\Pi^{S_1}(S_0) \approx \pi^{S_1}(0) \tag{A.8}$$

The procedure starts by determining $\bar{S}_0$ such that

$$G^{\bar{S}_0}(L_0) < \varepsilon \tag{A.9}$$

where ε is a small positive number.
The recursive computational procedure is:

$$\Pi^{S_1}(S_0 - 1) = \Pi^{S_1-1}(S_0) + (1 - G^{S_0}(L_0)) \times (\pi^{S_1}(0) - \pi^{S_1-1}(0)), \tag{A.10}$$

$$\Pi^0(S_0) = G^{S_0}(L_0)\beta L_0 - G^{S_0+1}(L_0)\beta\frac{S_0}{\lambda} + \beta L \qquad \text{(A.11)}$$

and, the expected total holding and shortage costs for a unit demand in an inventory system with a one-for-one ordering policy is:

$$c(S_0, S_1) = \Pi^{S_1}(S_0) + \gamma(S_0) \qquad \text{(A.12)}$$

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