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Fuzzy Hierarchical Location-Allocation Models for Congested Systems

Hassan Shavandi[1*], Hashem Mahlooji[2]
[1, 2] Department of Industrial Engineering , Sharif University of Technology
Tehran , Iran
(shavandi@sharif.edu)
(mahlooji@sharif.edu)

ABSTRACT

There exist various service systems that have hierarchical structure. In hierarchical service networks, facilities at different levels provide different types of services. For example, in health care systems, general centers provide low-level services such as primary health care services, while the specialized hospitals provide high-level services. Because of demand congestion in service networks, location of servers and allocation of demand nodes have a strong impact on the length of queues at servers as well as on the response times to service calls. The thrust of this article is the development of hierarchical location-allocation models for congested systems by employing queueing theory in a fuzzy framework. The new models allow partial coverage of demand nodes and approximate determination of parameters. Using queueing theory and fuzzy conditions, both referral and nested hierarchical models are developed for the maximal covering location problem (MCLP). An example is solved for both an existing probabilistic model and the new fuzzy models and the results are compared.

Keywords: Location, Hierarchical, Queueing, Fuzzy Sets, Congested Systems, Referral systems, Nested systems

1. INTRODUCTION AND LITERATURE REVIEW

There exist many examples of hierarchical structures in service networks both in the public and private sectors. In health care systems, general centers provide low-level services such as primary health care services, while the specialized hospitals provide high-level facilities. Schools have hierarchical structure in nature, because there are primary schools, middle schools, and high schools. Numerous other examples of hierarchical structures can be found at: airports, computer service centers, day-care centers, emergency medical centers, regional health facilities, social service centers, police centers, warehouses, and distribution.

In hierarchical systems, facilities at different levels provide different types of services. However, there is often a linkage between the different levels, which makes it impossible to solve the location problem for each level separately. For example, in the area of health care services, when deemed appropriate, the customers of a particular primary health care center can be referred to a hospital which is designated to provide high-level services. So the location problems with a hierarchical structure should be modeled and solved simultaneously for both low- and high-levels as a unified problem. Due to the nature of relations among the various levels both on the demand as well as the service side, analysis of hierarchical service systems is a challenge to be met.

As of now, numerous models have been developed to provide the highest level of service and hence, achieve the lowest level of congestion possible. For a short review of discrete location

*Corresponding Author

models, it should be noted that the location set covering problem (LSCP) that is a version of set covering problem, was developed by Toregas et al. (1971). This model was an attempt to locate the least number of servers to cover all the demand nodes with at least one server within the time or distance standard. This model's main drawback was its unrealistic assumption of unlimited budget that leads to the full coverage of all nodes. Efforts to remedy this flaw, led to the emergence of the Maximal Covering Location Problem (MCLP) by Church and ReVelle (1974). This model sought to maximize the population of calls which have a server within the time or distance standard at the presence of a limit on the number of servers. As such, not all nodes receive coverage. This model did not consider the idea of server congestion and assumed that free servers were available at the time of any call. Besides, parameters and structure of constraints were assumed to be deterministic.

The idea of server congestion was first considered by Larson (1974, 1975). He developed models in which the steady state busy fraction of servers was calculated and used to solve the location problem. In spite of using the queueing theory notions in location models, such models lacked a probabilistic structure until Daskin (1983) presented a probabilistic version of MCLP which he called Maximal Expected Covering Location Problem (MEXCLP). This model attempts to maximize the expected population coverage. Later, Berman, Larson et al. (1985 , 1987) employed queueing theory to develop models for congested networks.

Marianov and ReVelle (1994, 1996) and Marianov and Serra (1998) later proposed several models in which the number of requests for service was not constant in time, but instead, a stochastic process.

Hierarchical service systems can be classified according to their structures as nested and non-nested systems (Narula, 1985). In a nested system, the high-level servers provide low-level services as well, while in non-nested systems each level offers one unique service. A hierarchical system is labeled as coherent if all customers of a particular low-level server are the customers of a particular high-level server as well. In a referral system the users can go to a higher-level server only when they are referred by a low-level server. A non-referral system lacks such restriction.

Church and Eaton (1987) and Gerrard and Church (1994) provide reviews of early hierarchical models. Serra and ReVelle (1993, 1994) combined hierarchical location and coherent districting in a later effort. Serra, Marianov and ReVelle (1992) developed a hierarchical maximum capture model for location in a competitive environment. Later, Serra (1996) presented his model for the coherent covering location problem.

The assumption of demand congestion at servers has not been considered in any of the above models. Once the demand rate (for service) exceeds the service rate, congestion occurs and waiting lines emerge. To enhance the quality of rendering service in congested systems, it is naturally obvious that resorting to queueing theory can be quite helpful. Marianov and Serra (2001) published an article on hierarchical location-allocation models for congested systems. In this article they developed a number of hierarchical location models for LSCP and MCLP based on queueing theory. The probabilistic nature of their approach makes their models more realistic even though they adopt the often used crisp conditions. In fact, to make the models more realistic one can use the fuzzy conditions. As for application of fuzzy theory toward location models, most efforts can be categorized in the class of the qualitative models. For the quantitative fuzzy location models a short review is presented here. Canos et al. (1999), treated the classical *p*_median problem as a fuzzy model and came up with an exact method of solution. Combining set covering and fuzzy sets, Woodyat, et al. (1993) presented an application to optimally assign metallurgical grades to customer orders. A comprehensive review of newly developed hierarchical location models can be found in Sahin and Sural (2007). The very first fuzzy model using queueing theory in the area of location-allocation in congested systems was developed by Shavandi and Mahlooji (2004). They incorporated fuzzy parameters and variables in their work. In their model, no customer is required to receive service from a single server, rather he can select the appropriate server with priorities from a list of servers according to degrees of membership. Shavandi and Mahlooji (2006) also developed a fuzzy queueing maximal covering location-allocation model with a genetic algorithm.

The first fuzzy model for location-allocation in hierarchical systems was developed by Shavandi, et al. (2006). They introduced a fuzzy hierarchical queueing location-allocation model for MCLP in coherent systems.

The main aim of this article is developing fuzzy hierarchical queueing models for MCLP, in both nested and referral systems.

2. A REVIEW OF PROBABILISTIC HIERARCHICAL MAXIMAL COVERING PROBLEM (HIQ_MCLP)

To lay the foundation for presenting the fuzzy hierarchical queueing maximal covering location problem (FHQ_MCLP), it is appropriate to review the hierarchical queueing maximal covering location problem (HIQ_MCLP) for referral systems presented by Marianov and Serra (2001):

$$\max Z = \sum_i \sum_j \sum_k a_i X_{ijk} \tag{1}$$

s.t.

$$\sum_{j,k} X_{ijk} \le 1 \qquad \forall i \tag{2}$$

$$X_{ijk} \le W_j \qquad \forall i, j, k \tag{3}$$

$$X_{ijk} \le Z_k \qquad \forall i, j, k \tag{4}$$

$$\sum_j W_j = P_l \tag{5}$$

$$\sum_k Z_k = P_h \tag{6}$$

$$\text{P[low - level server } j \text{ has} \le b \text{ people in queue]} \ge \alpha \qquad \forall j \tag{7}$$

$$\text{P[high - level server } k \text{ has} \le b \text{ people in queue]} \ge \alpha \qquad \forall k \tag{8}$$

$$X_{ijk}, W_j, Z_k = 0,1 \qquad \forall i, j, k$$

where,

X_{ijk} : allocation variable that takes value 1 if population at demand node *i* is allocated to the low-level server *j* and to the high-level server *k*; otherwise it is zero.

W_j : location variable which takes value 1 if a low-level server is located at node *j*, and zero otherwise.

Z_k : location variable that takes value 1 if a high-level server is located at node *k*, and zero otherwise.

a_i : the population at demand node *i*.

P_l : the number of low-level servers to be located.

P_h : the number of high-level servers to be located.

The objective function (1) attempts to maximize the covered population. The first constraint, (2), means that each demand node can be covered by just one server. The constraints (3) and (4) assume that allocation variables can take the value 1 only when a low-level server and a high-level server have already been located at nodes *j* and *k*. The constraints (5) and (6) are related to the specific and bounded number of servers. The constraints (7) and (8) are related to the demand congestion at servers, or the quality of service to make sure that the queue length at each server does not exceed *b* with probability at least α .

To write the constraints (7) and (8) in non-probabilistic form, Marianov and Serra (2001) borrow notions from queueing theory to arrive at the final form of these constraints as:

$$\sum_{i,k} f_i X_{ijk} \leq \mu_j^l \sqrt[b+2]{1-\alpha} \qquad \forall j \qquad (7')$$

or

$$\sum_{i,j} \beta_j f_i X_{ijk} \leq \mu_k^h \sqrt[b+2]{1-\alpha} \qquad \forall k \qquad (8')$$

where the following definitions are in order:

f_i : rate of arrival of requests for service at node *i*.
μ_j^l : service rate at low-level server *j*.
μ_k^h : service rate at high-level server *k*.
β_j : percentage of the requests referred by the low-level server *j* to high-level service.

Since it is assumed that at each service center there exists just one server, and because each node can be served by just one server, the servers operate independently, and the queueing model at each server functions as an M/M/1 model.
So by substituting the constraints (7'), and (8') for constraints (7), and (8) we will arrive at the final form of HIQ_MCLP. Basically, the above formulation presented for referral systems can be applied directly to the nested systems, when modeled as co-located high-level and low-level servers. In the nested systems the low-level services, are provided by high-level servers as well. This is achieved by modifying constraint (3) as

$$X_{ijk} \leq W_j + Z_k \qquad \forall i, j, k \qquad (3')$$

Furthermore, in the nested systems there is often the additional requirement that customers attending a high-level site must be able to receive both types of services there. The following additional constraint is intended to consider such a requirement:

$$X_{ijj} = Z_j , \qquad \forall i, j \qquad (9)$$

3. THE FUZZY HIERARCHICAL QUEUEING MAXIMAL COVERING LOCATION PROBLEM

This section is devoted to the development of two types of FHQ-MCLP models. We first present a fuzzy hierarchical queueing maximal covering location formulation for referral systems, which can easily be applied to non-referral systems as well. Then a similar model is developed for nested

systems. In the case of nested systems a server providing both high-level and low-level services is modeled as a low-level server co-located with a high-level server.
At first we define the parameters, variables, and fuzzy sets that are used in developing such models. The parameters are as follows:

a_i : the population of demand node i; a crisp number.
$\tilde{b}_l$: (b_l^p, b_l^m, b_l^o) : a triangular fuzzy number which stands for the maximum allowable number of customers at each low-level server.
$\tilde{b}_h$: (b_h^p, b_h^m, b_h^o) : a triangular fuzzy number which stands for the maximum allowable number of customers at each high-level server.
$\tilde{f}_i$: (f_i^p, f_i^m, f_i^o) : a triangular fuzzy number which stands for the low-level demand rate at node i.
β_i : the high-level demand percentage at node i, a crisp number.
$\tilde{\mu}_j^l$: $(\mu_j^{lp}, \mu_j^{lm}, \mu_j^{lo})$: service rate at low-level server j ; a triangular fuzzy number.
$\tilde{\mu}_k^h$: $(\mu_k^{hp}, \mu_k^{hm}, \mu_k^{ho})$: service rate at high-level server k ; a triangular fuzzy number.
s_{ij}^{dl} : degree of membership for the distance between node i and the low-level server j being almost less than or equal to the distance standard.
s_{ik}^{dh} : degree of membership for the distance between node i and the high-level server k being almost less than or equal to the distance standard.
s_{jk}^{lh} : degree of membership for the distance between the low-level server j and the high-level server k being less than or equal to the distance standard.
$\tilde{N}_j^{Sl}$: average number of customers at the low-level server j during the steady state; a triangular fuzzy number.
$\tilde{N}_k^{Sh}$: average number of customers at the high-level server k during the steady state; a triangular fuzzy number.
$\tilde{\lambda}_j^l$: $(\lambda_j^{lp}, \lambda_j^{lm}, \lambda_j^{lo})$: arrival rate of demand at the low-level server j; a triangular fuzzy number.
$\tilde{\lambda}_k^h$:$(\lambda_k^{hp}, \lambda_k^{hm}, \lambda_k^{ho})$: arrival rate of demand at the high-level server k; a triangular fuzzy number.
P_l : the number of low-level servers to be located; a crisp number.
P_h : the number of high-level servers to be located; a crisp number.
α : the predefined truth value for the service quality constraint at low-level servers.
β : the predefined truth value for the service quality constraint at high-level servers.

The decision variables for the models are as follows:

W_j : A zero-one variable which assumes value 1 if a low-level server is located at node j, otherwise it is zero.
Z_k : A zero-one variable which assumes value 1 if a high-level server is located at node k, otherwise it is zero.
X_{ij} : The degree of membership for node i being covered by the low-level server j.
V_{jk} : The degree of membership for referring the high-level services from the low-level server j to the high-level server k.
Y_{ik} : The degree of membership for node i being covered by the high-level server k.

The fuzzy sets that are used in the models are as follows:

$\tilde{N}_j^{dl}$ **:** This discrete fuzzy set represents the distance from any node to the low-level server *j* and is defined as

$$\tilde{N}_j^{dl} = \left\{ \frac{s_{1j}^{dl}}{1}, \frac{s_{2j}^{dl}}{2}, \ldots, \frac{s_{ij}^{dl}}{i} \right\}, \quad \forall\, j$$

where s_{ij}^{dl} stands for the degree of membership for the distance of node *i* from the low-level server *j* to be approximately smaller than or equal to the distance standard. In order to calculate $\tilde{N}_j^{dl}$ **,** first let d_{ij} represent the distance between node *i* and the low-level server *j*, also let S_{dl} denote the distance standard for low-level services. Now it is clear that the statement " the node *i*'s distance from node *j* is approximately less than or equal to the distance standard" can be represented by the following fuzzy notation :

$$d_{ij} \tilde{\leq} S_{dl} \tag{10}$$

Such a definition allows us to put any node *i* in $\tilde{N}_j^{dl}$ for the low-level server *j* according to its degree of membership. The degree of membership, s_{ij}^{dl} , can be calculated as

$$s_{ij}^{dl} = \begin{cases} 0 & , \quad d_{ij} > u_{dl} \\ \dfrac{u_{dl} - d_{ij}}{u_{dl} - s_{dl}} & , \quad s_{dl} \leq d_{ij} < u_{dl} \\ 1 & , \quad d_{ij} \leq s_{dl} \end{cases} \tag{11}$$

where u_{dl} stands for the acceptable upper bound of the distance standard. The relation in (11) is obtained on the basis of Figure (1).

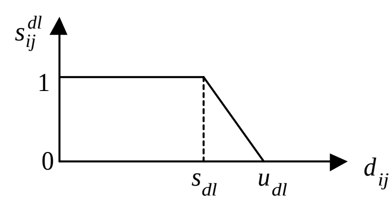

Figure 1. The membership function of the distance standard

Thus the set $\tilde{N}_j^{dl}$ is defined as a fuzzy set like

$$\tilde{N}_j^{dl} = \left\{ \frac{s_{1j}^{dl}}{1}, \frac{s_{2j}^{dl}}{2}, \ldots, \frac{s_{ij}^{dl}}{i} \right\} \quad , \qquad \forall j \tag{12}$$

where each node belongs to set $\tilde{N}_j^{dl}$ according to a membership degree.

$\tilde{N}_k^{dh}$: This discrete fuzzy set stands for the distance from any node to the high-level server *k* and is defined as

$$\tilde{N}_k^{dh} = \left\{ \frac{s_{1k}^{dh}}{1}, \frac{s_{2k}^{dh}}{2}, \ldots, \frac{s_{ik}^{dh}}{i} \right\} \quad , \quad \forall k \tag{13}$$

$\tilde{N}_k^{lh}$: This discrete fuzzy set represents the distance from a low-level server to the high-level server *k* and is defined as

$$\tilde{N}_k^{lh} = \left\{ \frac{s_{1k}^{lh}}{1}, \frac{s_{2k}^{lh}}{2}, \ldots, \frac{s_{jk}^{lh}}{j} \right\} \quad , \quad \forall k \tag{14}$$

Technicalities of evaluating s_{ik}^{dh} and s_{jk}^{lh} are similar to those of evaluating s_{ij}^{dl}.

$\tilde{C}_j^{dl}$: This fuzzy set includes the nodes which are approximately covered by the low-level server *j*, i.e.

$$\tilde{C}_j^{dl} = \left\{ \frac{X_{1j}}{1}, \frac{X_{2j}}{2}, \ldots, \frac{X_{ij}}{i} \right\} \quad , \quad \forall j \tag{15}$$

$\tilde{C}_k^{lh}$: This fuzzy set includes the low-level servers which are approximately covered by the high-level server *k* for referring the high-level services, i.e.

$$\tilde{C}_k^{lh} = \left\{ \frac{V_{1k}}{1}, \frac{V_{2k}}{2}, \ldots, \frac{V_{jk}}{j} \right\} \quad , \quad \forall k \tag{16}$$

$\tilde{C}_k^{dh}$: This fuzzy set includes the nodes which are approximately covered by the high-level server *k*, i.e.

$$\tilde{C}_k^{dh} = \left\{ \frac{Y_{1k}}{1}, \frac{Y_{2k}}{2}, \ldots, \frac{Y_{ik}}{i} \right\} \quad , \quad \forall k \tag{17}$$

In this work we intend to develop models which cover the demand nodes that are within the distance standard. Thus, for the case of low-level servers, one has to find the intersection of the fuzzy sets, $\tilde{C}_j^{dl}$ and $\tilde{N}_j^{dl}$ to determine the issue of coverage for the demand nodes with respect to the distance standard. As such, a new fuzzy set is obtained whose elements consist of the common elements of these two sets. The degree of membership for each element in this set is equal to the minimum of the degrees of membership for the same element across the two fuzzy sets. So if we include the condition that X_{ij} always stays less than or equal to s_{ij}^{dl}, then the coverage of low-level services will be always within the distance standard. The same conditions are needed to ensure that the coverage of high-level services be within the distance standard as well. Therefore the following constraints must be added to the model:

$$X_{ij} \leq S_{ij}^{dl} \tag{18}$$

$$Y_{ik} \leq S_{ik}^{dh} \tag{19}$$

$$V_{jk} \leq S_{jk}^{lh} \tag{20}$$

$\tilde{D}_j^l$: This is the set of demands which are approximately covered by the low-level server *j*, i.e.,

$$\tilde{D}_j^l = \left\{ \frac{X_{1j}}{\tilde{f}_1}, \frac{X_{2j}}{\tilde{f}_2}, \ldots, \frac{X_{ij}}{\tilde{f}_i} \right\} \quad , \quad \forall j \tag{21}$$

Each element of this set is a triangular fuzzy number. In the following section we employ this set to determine the arrival rates of service demands to the low-level servers.

$\tilde{D}_k^h$: This is the set of the demand calls which are approximately covered by the high-level server *k*, i.e.,

$$\tilde{D}_k^h = \left\{ \frac{Y_{1k}}{\beta_1 \tilde{f}_1}, \frac{Y_{2k}}{\beta_2 \tilde{f}_2}, \ldots, \frac{Y_{ik}}{\beta_i \tilde{f}_i} \right\} \quad , \quad \forall k \tag{22}$$

$\tilde{P}_j^l$: the fuzzy set of populations which are approximately covered by the low-level server *j*, where,

$$\tilde{P}_j^l = \left\{ \frac{X_{1j}}{a_1}, \frac{X_{2j}}{a_2}, \ldots, \frac{X_{ij}}{a_i} \right\} \tag{23}$$

$\tilde{P}_j^h$: the fuzzy set of populations which are approximately covered by the high-level server *k*, where,

$$\tilde{P}_j^h = \left\{ \frac{Y_{1k}}{a_1}, \frac{Y_{2k}}{a_2}, \ldots, \frac{Y_{ik}}{a_i} \right\} \tag{24}$$

Since the goal of this model is maximizing the covered population around the distance standard, we can write the objective function for the low-level services as

$$\max Z^l = \sum_{i,j} a_i X_{ij} \tag{25}$$

Similarly for the high-level services, the objective function can be represented as

$$\max Z^h = \sum_{i,k} a_i Y_{ik} \tag{26}$$

therefore the FHQ_MCLP's objective function will be

$$\max \; Z = \max Z^l + \max Z^h \tag{27}$$

3.1. The mathematical model for referral FHQ-MCLP

The FHQ-MCLP mathematical model for the referral systems which is a mixed integer programming model is as follows:

$$\max Z = \sum_{i,j} a_i X_{ij} + \sum_{i,k} a_i Y_{ik} \quad (28)$$

s.t.:

$$X_{ij} \le W_j \quad , \forall i, j \quad (29)$$

$$V_{jk} \le Z_k \quad , \forall j, k \quad (30)$$

$$V_{jk} \le W_j \quad , \forall j, k \quad (31)$$

$$Y_{ik} \le X_{ij} \quad , \forall i, j, k \quad (32)$$

$$Y_{ik} \le V_{jk} \quad , \forall i, j, k \quad (33)$$

$$X_{ij} \le S_{ij}^{dl} \quad , \forall i, j \quad (34)$$

$$Y_{ik} \le S_{ik}^{dh} \quad , \forall i, k \quad (35)$$

$$V_{jk} \le S_{jk}^{lh} \quad , \forall j, k \quad (36)$$

$$\sum_{j=1}^{n} W_j = P_l \quad (37)$$

$$\sum_{k=1}^{n} Z_k = P_h \quad (38)$$

$$\tilde{N}_j^{Sl} \le \tilde{b}_l \quad , \forall j \quad (39)$$

$$\tilde{N}_k^{Sh} \le \tilde{b}_h \quad , \forall k \quad (40)$$

$$0 \le X_{ij} \le 1 \,,\; 0 \le V_{jk} \le 1 \,,\; 0 \le Y_{ik} \le 1 \,,\; W_j = 0,1 \;,\; Z_k = 0,1$$

The objective function, (28), attempts to maximize the approximate coverage of populations around the distance standard. The purpose of constraints (29), (30) and (31) is to ensure that unless a server is located at a node, the other nodes can not be covered by that node's server. Since in the referral model the high-level servers are referred to by low-level servers, so the degree of membership for node *i* to be covered by the high-level server *k*, (Y_{ik}), will be the minimum of X_{ij} and V_{jk}. Constraints (32) and (33) reflect this situation. Constraints (34) through (36) have been explained before. Constraints (37) and (38) are intended to enforce the bounded number of servers. Finally, constraints (39) and (40) have to do with the quality of rendering service by the low- and high-level servers. They enforce the condition that the average number of customers for each server be less than or equal to a given value ($\tilde{b}_l$ *or* $\tilde{b}_h$).The average number of customers for each server is obtained as a triangular fuzzy number, an issue which will be elaborated on later. The maximum permissible number in the system is also a triangular fuzzy number. Accordingly, in constraints (39) and (40) a triangular fuzzy number must be less than or equal to another triangular fuzzy number. To include such a constraint we adopt the method proposed by Dubois and Prade (1980). According to this method one must calculate the correctness of the intended inequality holding true. In fact, for any two fuzzy numbers $\tilde{I}$ and $\tilde{J}$, the correctness of $\tilde{I} \le \tilde{J}$ holding true is calculated as

$$T(\tilde{I} \le \tilde{J}) = Sup_{x \le y} \{\min\{\mu_{\tilde{I}}(x), \mu_{\tilde{J}}(y)\}\} \quad (41)$$

where $\mu_{\tilde{I}}(x)$ and $\mu_{\tilde{J}}(y)$ represent the membership functions for *x* belonging to $\tilde{I}$ and *y* belonging to $\tilde{J}$. Using this method, constraints (39) and (40) are converted to

$$T(\tilde{N}_j^{Sl} \le \tilde{b}_l) \ge 1-\alpha \tag{42}$$

And

$$T(\tilde{N}_k^{Sh} \le \tilde{b}_h) \ge 1-\beta \tag{43}$$

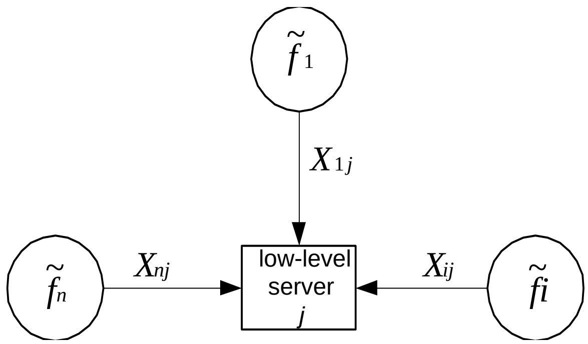

Figure 2. The set of demands which are approximately covered by the low-level server *j*

We first present the procedure for calculating $\tilde{N}_j^{Sl}$ for low-level servers and then using the results, we obtain the average number of customers at high-level servers ($\tilde{N}_k^{Sh}$**)**. In order to calculate $\tilde{N}_j^{Sl}$, we start by calculating the arrival rate of service demand to the low-level servers. Figure 2 shows the set of service calls which are covered by server *j*.

This set was initially defined as

$$\tilde{D}_j^l = \left\{ \frac{X_{1j}}{\tilde{f}_1}, \frac{X_{2j}}{\tilde{f}_2}, \ldots, \frac{X_{ij}}{\tilde{f}_i} \right\} \quad , \quad \forall j \tag{44}$$

Since $\tilde{f}_i$ in this fuzzy set is covered by the low-level server *j* with a degree of membership equal to 1 and the set itself is a convex fuzzy set, then $\tilde{D}_j^l$ is a discrete fuzzy number. To evaluate $\tilde{\lambda}_j^l$, we employ the centroid method (Sugeno, 1985 ; Lee, 1990) which is intended for transforming a fuzzy number to a classical (crisp) number. This method, however, will transform $\tilde{D}_j^l$ to a triangular fuzzy number because the elements of $\tilde{D}_j^l$ are all triangular fuzzy numbers.To employ the centroid method, let $\tilde{Z}$ stand for a discrete fuzzy number like

$$\tilde{Z} = \left\{ \frac{\mu_{\tilde{c}}(z_1)}{z_1}, \frac{\mu_{\tilde{c}}(z_2)}{z_2}, \ldots, \frac{\mu_{\tilde{c}}(z_i)}{z_i} \right\} \tag{45}$$

where the crisp numbers z_i 's are elements of $\tilde{Z}$ and $\mu_{\tilde{c}}(z_i)$ represents z_i 's degree of membership in $\tilde{Z}$. Using the centroid method, the fuzzy number $\tilde{Z}$ is transformed to the crisp number Z^* as

$$Z^* = \frac{\sum_i \mu_{\tilde{c}}(z_i) z_i}{\sum_i \mu_{\tilde{c}}(z_i)} \tag{46}$$

Now, using (46), the fuzzy number $\tilde{D}_j^l$ is transformed to a triangular fuzzy number, $\tilde{\lambda}_j^l$, as

$$\tilde{\lambda}_j^l = \frac{\sum_i \tilde{f}_i X_{ij}}{\sum_i X_{ij}} \quad , \quad \forall j \tag{47}$$

Since $\tilde{f}_i$'s are triangular fuzzy numbers, the number obtained from (47) is also a triangular fuzzy number in the from of

$$\tilde{\lambda}_j^l = (\lambda_j^{lp}, \lambda_j^{lm}, \lambda_j^{lo}) \quad , \quad \forall j \tag{48}$$

where,

$$\lambda_j^{lp} = \frac{\sum_i f_i^p X_{ij}}{\sum_i X_{ij}} \quad , \quad \lambda_j^{lm} = \frac{\sum_i f_i^m X_{ij}}{\sum_i X_{ij}} \quad , \quad \lambda_j^{lo} = \frac{\sum_i f_i^o X_{ij}}{\sum_i X_{ij}}$$

Due to the fact that the demand for service at each node follows a Poisson process, the service calls' arrival rate to server *j* also obeys a Poisson process. Server *j*'s capacity to render service follows an exponential distribution with parameter $\tilde{\mu}_j^l$ too. Since the parameters of such distributions are fuzzy in nature, the queueing model at each server will be an FM/FM/1 model (FM$\equiv$fuzzy Markovian). Now, to evaluate $\tilde{N}_j^{Sl}$, we use the fuzzy Little relations which are proposed by Jo, Tsujimura, Gen, and Yamazaki (1993), i.e.,

$$\tilde{N}_j^{Sl} = \frac{\tilde{\lambda}_j^l}{\tilde{\mu}_j^l - \tilde{\lambda}_j^l} \tag{49}$$

Because $\tilde{\lambda}_j^l$ and $\tilde{\mu}_j^l$ are triangular fuzzy numbers, $\tilde{N}_j^{Sl}$ obtained from (49) will also be a triangular fuzzy number, i.e.,

$$\tilde{N}_j^{Sl} = (N_j^{Slp}, N_j^{Slm}, N_j^{Slo}) \tag{50}$$

where,

$$\tilde{N}_j^{Sl} = \left(\frac{\sum_i f_i^p X_{ij}}{\mu_j^{lo}\sum_i X_{ij} - \sum_i f_i^o X_{ij}} \, , \, \frac{\sum_i f_i^m X_{ij}}{\mu_j^{lm}\sum_i X_{ij} - \sum_i f_i^m X_{ij}} \, , \, \frac{\sum_i f_i^o X_{ij}}{\mu_j^{lp}\sum_i X_{ij} - \sum_i f_i^p X_{ij}}\right)$$

Reasoning in a similar manner, the average number in system for high-level servers, $\tilde{N}_k^{Sh}$, is obtained as

$$\tilde{N}_k^{Sh} = (N_k^{Shp}, N_k^{Shm}, N_k^{Sho}) \tag{51}$$

where,

$$\tilde{N}_k^{Sh} = \left(\frac{\sum_i \beta_i f_i^p Y_{ik}}{\mu_k^{ho}\sum_i Y_{ik} - \sum_i \beta_i f_i^o Y_{ik}}, \frac{\sum_i \beta_i f_i^m Y_{ik}}{\mu_k^{hm}\sum_i Y_{ik} - \sum_i \beta_i f_i^m Y_{ik}}, \frac{\sum_i \beta_i f_i^o Y_{ik}}{\mu_k^{hp}\sum_i Y_{ik} - \sum_i \beta_i f_i^p Y_{ik}}\right)$$

By transforming the constraints (39) and (40) to deterministic form, the model is converted to a mixed integer programming model. To make this possible, we use the following lemma that is proved by Shavandi and Mahlooji (2004).

Lemma : Given two triangular fuzzy numbers $\tilde{I} = (I^p, I^m, I^o)$ and $\tilde{J} = (J^p, J^m, J^o)$, we have

a) $T(\tilde{I} \le \tilde{J}) = 1 \;\Leftrightarrow\; I^m \le J^m$ (52)

b) $T(\tilde{I} \le \tilde{J}) \ge 1 - \alpha \;\Leftrightarrow\; I^m \le J^o - (1-\alpha)(J^o - J^m)$ (53)

Using (52) we can transform constraint (42) to a linear form as

$$T\left(\tilde{N}_j^{Sl} \le \tilde{b}_l\right) \ge 1 - \alpha \equiv N_j^{Slm} \le b_l^o - (1-\alpha)\left(b_l^o - b_l^m\right) \tag{54}$$

By substituting the equivalent of N_j^{Slm} from (50) and doing appropriate mathematical manipulations we will arrive at the linear form

$$\sum_{i=1}^{n} (\beta_i^l - \gamma_j^l) X_{ij} \le 0 \quad , \quad \forall j \tag{55}$$

where,

$$\beta_i^l = f_i^m + b_l^o f_i^m - (1-\alpha)\left(b_l^o - b_l^m\right) f_i^m \quad , \quad \forall i \tag{56}$$

$$\gamma_j^l = b_l^o \mu_j^{lm} - (1-\alpha)(b_l^o - b_l^m)\mu_j^{lm} \quad , \quad \forall j \tag{57}$$

Likewise, constraint (43) will be transformed to the following form

$$\sum_{i=1}^{n} (\beta_i^h - \gamma_k^h) Y_{ik} \le 0 \quad , \quad \forall k \tag{58}$$

where,

$$\beta_i^h = \beta_i f_i^m + b_h^o \beta_i f_i^m - (1-\beta)\left(b_h^o - b_h^m\right)\beta_i f_i^m \quad , \quad \forall i \tag{59}$$

$$\gamma_k^h = b_h^o \mu_k^{hm} - (1-\beta)(b_h^o - b_h^m)\mu_k^{hm} \quad , \quad \forall k \tag{60}$$

Therefore, the final referral FHQ-MCLP model can be written as follows:

$$\max Z = \sum_{i,j} a_i X_{ij} + \sum_{i,k} a_i Y_{ik}$$

s.t.:

$$X_{ij} \le W_j \quad , \forall i, j$$
$$V_{jk} \le Z_k \quad , \forall j, k$$
$$V_{jk} \le W_j \quad , \forall j, k$$
$$Y_{ik} \le X_{ij} \quad , \forall i, j, k$$
$$Y_{ik} \le V_{jk} \quad , \forall i, j, k$$
$$X_{ij} \le S_{ij}^{dl} \quad , \forall i, j$$
$$Y_{ik} \le S_{ik}^{dh} \quad , \forall i, k$$
$$V_{jk} \le S_{jk}^{lh} \quad , \forall j, k$$
$$\sum_{j=1}^{n} W_j = P_l$$
$$\sum_{k=1}^{n} Z_k = P_h$$
$$\sum_{i=1}^{n} (\beta_i^l - \gamma_j^l) X_{ij} \le 0 \quad , \forall j$$
$$\sum_{i=1}^{n} (\beta_i^h - \gamma_k^h) Y_{ik} \le 0 \quad , \forall k$$
$$0 \le X_{ij} \le 1 \,,\; 0 \le V_{jk} \le 1 \,,\; 0 \le Y_{ik} \le 1 \,,\; W_j = 0,1 \,,\; Z_k = 0,1$$

3.2. The FHQ-MCLP for the nested systems

In the nested hierarchical systems, the high-level servers are capable of rendering service at lower levels as well, while the low-level servers offer the low-level service only. Since for the purpose of developing this model, all the parameters defined in section (3) are still valid, we just define the needed variables and fuzzy sets.

The decision variables for the nested FHQ-MCLP are as follows:

X_{ij} : The degree of membership for node *i* to be covered by the low-level server *j*.
V_{ik} : The degree of membership for node *i* to be covered by the high-level server *k*.

The variables W_j and Z_k have the same interpretations as before. Except for the fuzzy sets of demands covered by the high-level servers, the other sets are defined as before. To define the fuzzy set of demands covered by the high-level servers in the nested system we reason as follows.

From each node *i*, two calls for service with different degrees of membership arrive at the high-level server *k*. The low-level service with rate $\tilde{f}_i$ is covered by the high-level server *k* with the degree of membership X_{ik}. The high-level service with rate $\beta_i \tilde{f}_i$ is covered by the high-level server *k* with the degree of membership V_{ik}. On the basis of the fuzzy algebraic relations, these rates can be added together with the sum having a degree of membership equal to the minimum of (X_{ik} , V_{ik}).

Thus we need to define the variable Z_{ik} as:

Z_{ik} : The degree of membership for node i to be covered by the high-level server k which covers both low-level and high-level services, i.e.,

$$Z_{ik} = \min (X_{ik} , V_{ik}) \tag{61}$$

The fuzzy set of demands which are approximately covered by the high-level server k is defined as

$$\tilde{D}_k^h = \left\{ \frac{z_{1k}}{\tilde{f}_1 + \beta_1 \tilde{f}_1}, \frac{z_{2k}}{\tilde{f}_2 + \beta_2 \tilde{f}_2}, ..., \frac{z_{ik}}{\tilde{f}_i + \beta_i \tilde{f}_i} \right\} \tag{62}$$

We can equivalently write (61) in the form of the following two constraints which will be added to the model.

$$Z_{ik} \le X_{ik} \tag{63}$$
$$Z_{ik} \le V_{ik} \tag{64}$$

Thus the fuzzy queueing constraint on the high-level servers will change to

$$\sum_{i=1}^{n} (\beta_i^h - \gamma_k^h) Z_{ik} \le 0 \tag{65}$$

where,

$$\beta_i^h = \left[1 + b_h^o - (1-\alpha)(b_h^o - b_h^m)\right](1+\beta_i) f_i^m , \forall i \tag{66}$$
$$\gamma_k^h = \left[b_h^o - (1-\alpha)(b_h^o - b_h^m)\right] \mu_k^{hm} \quad , \quad \forall k \tag{67}$$

We now present the final FHQ-MCLP model for the nested system. In this model, λ_l and λ_h stand for the weights of populations being covered by the low- and high-level services.

$$\max \ Z = \lambda_l \sum_{i,j} a_i X_{ij} + \lambda_h \sum_{i,k} a_i V_{ik}$$

s.t.:

$$X_{ij} \le W_j + Z_j \quad , \forall i, j$$
$$V_{ik} \le Z_k \quad , \forall i, k$$
$$W_j + Z_j \le 1 \quad , \quad \forall j$$
$$X_{ij} \le S_{ij}^{dl} \quad , \forall i, j$$
$$V_{ik} \le S_{ik}^{dh} \quad , \forall i, k$$
$$Z_{ik} \le X_{ik}$$
$$Z_{ik} \le V_{ik}$$
$$\sum_{j=1}^{n} W_j = P_l$$
$$\sum_{k=1}^{n} Z_k = P_h$$

$$\sum_{i=1}^{n}(\beta_i^l-\gamma_j^l)X_{ij}\le 0 \quad , \quad \forall j$$

$$\sum_{i=1}^{n}(\beta_i^h-\gamma_k^h)Z_{ik}\le 0 \quad , \quad \forall k$$

$$0\le X_{ij}\le 1,\ 0\le V_{ik}\le 1\ ,\ 0\le Z_{ik}\le 1\ , W_j=0,1\ \ ,\ \ Z_k=0,1$$

4. AN EXAMPLE

In this section we present the results obtained from solving a typical problem for the probabilistic HIQ_MCLP as well as the FHQ_MCLP, and compare the results. To solve the problem, we used the branch and bound method and IBM OSL v3, on a Pentium 2, 333 MHZ. Table 1 illustrates the parameter values for the problem and Tables 2 and 3 display the results of solving the probabilistic HIQ_MCLP and FHQ_MCLP, respectively.

Suppose this example relates to health care services where the low-level servers provide primary health care and the high-level servers provide high-level health care services. In this problem we have a network with 15 nodes that represent different regions, each region has a population (a_i), and the estimation of approximate demand rate for low-level services is given by $\tilde{f}_i=(f_i^p, f_i^m, f_i^o)$. The number of low-level servers to be located, P_l, is 3, and the number of high-level centers to be located, P_h, is 2. The distance between two nodes is measured and treated in terms of the distance standard for low-level and high-level services and on the basis of such treatment the membership degrees are determined. In this problem, it is assumed that the distance standards are the same for low-level and high-level services, so that the membership degrees for the distance between nodes are identical for both low-level and high-level services, i.e., $s_{ij}^{dl}=s_{jk}^{lh}=s_{ik}^{dh}$. The maximum allowable number of customers are determined approximately for both levels and are assumed to be $\tilde{b}_l$ = (2, 3, 4), and $\tilde{b}_h$ = (1, 2, 3). The service rate at each level of servers is determined by $\tilde{\mu}_l$ = (30, 40, 50), and $\tilde{\mu}_h$ = (10, 20, 30). The percentage of low-level service demands that are referred to the high-level centers, is β_i = 0.2, for all nodes. So under these circumstances we seek to locate the servers and allocate the demand nodes to the servers in such a way that the population covered approximately around the distance standards is maximized. To achieve this purpose, we use the branch and bound method to solve the following small-scaled typical problem. We compare the optimal solutions obtained for the probabilistic model proposed by Marianov and Serra (2001) as well as the fuzzy model.

On the basis of the results obtained, a comparison of the probabilistic and fuzzified models is appropriate. Table 2 shows the optimal solution for the probabilistic HIQ_MCLP. In this problem the low-level servers are located at nodes 1, 2, 5, and the high-level servers are located at nodes 8, and 10. In the probabilistic version, node *i*, can be covered by the low-level server *j* and the high-level server *k*, only when it's distance from servers is less than or equal to the distance standard. So the nodes 3, 4, 9, and 12, are not covered by any low-level or high-level servers. For example, among the covered nodes, node 6 is covered by the low-level server 5, and the high-level server 10. Therefore in the probabilistic HIQ_MCLP each demand node can be covered by just one server. So one demand node can not select a server from the available list of servers and it must ask for service at the specified server that is determined for it.

In reality, it does not sound that acceptable to restrict each node to receive service from just one server. Besides, it does not seem real to deprive a demand node from receiving service on the basis that its distance from a server is *somewhat* larger than the distance standard.

In the fuzzified hierarchical models that are developed in this paper, each node can be covered by any low- or high-level server with a degree of membership. On the other hand, the models are equipped to consider priorities to ask for and to render services. In fact, in these models each server provides service on the basis of its own priorities (degree of membership) in the same way that each demand node chooses to receive service from servers according to its own priorities. When the conditions of rendering service are identical for all servers, distance becomes the measure on the basis of which a demand node assigns priorities to servers. In this way, each demand node prefers to go to its nearest server and if this server is occupied, to go to the next nearest server, and so on.

In FHQ_MCLP model, each demand node assigns a priority to each low- and high-level server on the basis of the degree of membership for its own distance from each one of them ($s_{ij}^{dl}, s_{jk}^{lh}, s_{ik}^{dh}$). As Table 3 indicates, low-level servers for FCHQ_MCLP are located at nodes 1, 8, and 10 and high-level servers are located at nodes 1 and 10. All of the demand nodes are covered by servers according to the degrees of membership. For example, the demand node 5 is covered by the low-level servers 1, and 8 with the degrees of membership 1, and 0.8 , and for high-level services is covered by the high-level servers 1, and 10 with the degrees of membership 1, and 0.9. This means that node 5 gives the highest priority to the low-level server 1 and less priority to the low-level server 8. FHQ_MCLP model makes it possible for servers to assign their own priorities as well. This is accomplished by X_{ij}^{l}, and X_{ik}^{h} which stand for the degrees of membership for covering nodes. For instance, for the low-level server j = 1, nodes 1, 2, 5, and 8 have the highest priority for receiving service, node 10 has the second highest priority, and so on. As can be seen in Table 3, each demand node may be covered by various servers and there is a possibility that none of the nodes is deprived from receiving service. This obviously is the advantage of a fuzzy treatment of the problem.

5. CONCLUSIONS AND FUTURE EXTENSIONS

This work presents two new fuzzified queueing maximal covering location models for the referral and nested hierarchical systems which are named referral FHQ_MCLP and nested FHQ_MCLP. The parameters of these models are estimated approximately and defined as fuzzy numbers. The constraints related to the service quality are also assumed to be fuzzy in nature. The allocation variables assume real numbers between zero and one known as the degrees of membership, which explain how demand nodes are covered by servers. So in these models partial coverage of each demand is made possible and it is no longer required for a demand to call just one server for service. Therefore, in these models demand nodes can set their own priorities in selecting the appropriate server from a list by the degree of membership based on their distance from servers. So it seems the models which are presented here are in more agreement with the real situation.

The extensions to this paper include developing heuristic solution methods for solving the problems, generalizing the proposed idea to maximal availability location problem (MALP), and developing hierarchical models with up to two service levels.

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Appendix : Tables

Table 1. Parameter values for the example

Number of nodes (n) = 15															
N	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Ai	937	503	524	654	585	597	580	679	782	914	582	628	854	695	912
f^p	2	3	7	6	5	2	4	1	1	7	9	8	4	3	5
f^m	4	5	9	8	7	4	6	3	3	9	11	10	6	5	7
f^o	5	8	11	12	9	6	8	5	5	13	13	12	9	8	10
C_j	100	120	110	980	850	760	950	115	125	102	130	90	80	92	105
K_k	250	220	185	159	145	220	200	215	198	212	211	196	168	175	185
							$s_{ij}^{dl} = s_{jk}^{lh} = s_{ik}^{dh}$								
1	1	1	.2	.5	1	0	.6	1	0	.9	.7	.14	.51	.3	0
2		1	.2	0	.5	.14	.32	1	.25	.64	.9	.15	.62	0	.7
3			1	0	.3	.5	.18	.51	.61	.71	.2	.02	0	1	.9
4				1	.21	.51	.54	.61	.12	.15	0	1	.29	.84	.17
5					1	1	.9	.8	.14	.21	.51	.3	0	1	.24
6						1	.2	1	0	.3	.6	.9	.4	.7	.6
7							1	.2	0	.9	.4	.61	.72	.1	.2
8								1	.6	0	.9	.8	.4	.7	.61
9									1	1	.3	.8	.47	.16	.92
10										1	.2	.7	.8	.14	.61
11											1	.9	.2	.4	.31
12												1	.2	.1	.09
13													1	.8	.12
14														1	0
15															1

$\tilde{b}_l = (2, 3, 4)$	$\tilde{b}_h = (1, 2, 3)$	$\tilde{\mu}_l = (30, 40, 50)$	$\tilde{\mu}_h = (10, 20, 30)$
$\alpha = 0.05$	$\beta_i = 0.2$	$p_l = 3$	$p_h = 2$

Table 2. The optimum solution for the referral HIQ-MCLP

Low-level servers locations : 1, 2, 5 High-level servers locations : 8, 10															
Covering the nodes by the low -and- high level servers (X_{ijk})															
Low-level server	Nodes														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	0	0	0	0	0	1	0	1	0	0	0	0	0
2	0	1	0	0	0	0	0	0	0	0	1	0	1	0	1
5	0	0	0	0	1	1	1	0	0	0	0	0	0	1	0
High-level servers															
8	1	1	0	0	1	1	0	1	0	0	1	0	0	1	0
10	0	0	0	0	0	0	1	0	0	1	0	0	1	0	1
Optimal objective function value : 7838															

Table 3. The optimum solution for the referral FHQ-MCLP

Low-level servers locations : 1, 8, 10 High-level servers locations : 1, 10															
Covering the nodes by the low-level servers (X_{ij})															
Low-level server	Nodes														
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	1	0.2	0.5	1	0.2	0.6	1	0	0.9	0.7	0.1	0.5	0.3	0
8	1	1	0.5	0.6	0.8	1	0.2	1	0.6	0	0.9	0.8	0.4	0.7	0.6
10	0.9	0.6	0.7	0.1	0	0.3	0.9	0	1	1	0.2	0.7	0.8	0.1	0.6

The degree of membership for referring the high-level services from low-level servers to high-level servers (V_{jk})		
Low-level server	High-level server	
	1	10
1	1	0.9
8	1	0
10	0	1
Optimal objective function value : 22535.79		