Journal of Industrial and Systems Engineering
Vol. 1, No. 2, pp 151-170
Summer 2007

A Comparative Study of Exact Algorithms for the Two Dimensional Strip Packing Problem

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ABSTRACT

In this paper we consider a two dimensional strip packing problem. The problem consists of packing a set of rectangular items in one strip of width *W* and infinite height. They must be packed without overlapping, parallel to the edge of the strip and we assume that the items are oriented, i.e. they cannot be rotated. To solve this problem, we use three exact methods: a branch and bound method, a dichotomous algorithm and a branch and price method. The three methods were carried out and compared on literature instances.

Keywords: Strip packing, lower and upper bound, branch and bound, dichotomous search, column generation, branch and price.

1. INTRODUCTION

The two-dimensional strip packing problem (2SP) is a well-known combinatorial optimization problem. It has several industrial applications like the cutting of rolls of paper or textile material. Moreover, some approximation algorithms for Bin Packing problems are in two phases where the first phase aims at solving a strip packing problem (see Chung et al. (1982) and Berkey and Wang (1987)).

Consider a set of n rectangular items. Each item i has width w_i and height h_i $(i \in \{1,..., n\})$. The 2SP consists of packing all the items in a strip of width W and infinite height. Without loss of generality, we assume that the dimensions of the items and the strip are integers. The objective is to minimize the total height used to pack the items without overlapping. The orientation of items is assumed to be fixed, i.e., they cannot be rotated.

This problem is NP-hard in the strong sense (see Garey and Johnson (1979) and Martello et al. (2003)). For this reason, most of papers considering the 2SP problem are approximation algorithms. Fernandez de la Vega and Zissimopoulos (1998), Lesh et al.,(2003), Kenyon and Remila (1996) and Zhang et al. (2006) presented approximation algorithms for the strip packing problem, whereas Gomes and Oliveira (2006), Bortfeldt (2006) and Beltran et al., (2004) used meta-heuristics. Hopper and Turton (2001) provided an overview of meta-heuristic algorithms applied to 2D strip packing problem.To our knowledge, few papers studied exact algorithms to solve 2SP problem. Hifi

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(1998) introduced the cutting/packing problem with guillotine cut and he proposed two algorithms based on a branch and bound procedure and dichotomous search. Martello et al. (2003) proposed a new lower bound and used a branch and bound algorithm to solve the problem without guillotine constraint. Recently, three papers introduced the Strip Packing problem with guillotine constraint. Cui et al. (2006) proposed a recursive branch and bound algorithm to obtain an approximate solution. A new lower bound and a branch and bound algorithm were proposed by Bekrar et al. (2006b) for the guillotine problem. Finally, Cintra et al. (2006) used the column generation method and dynamic programming to solve another variant of the problem.

In this paper we study the two dimensional strip packing problem with non-guillotine pattern. We propose exact algorithms: branch and bound algorithm, branch and price procedure and dichotomous search to solve the problem. In Section 2 we present a lower bound and an upper bound used in a branch and bound approach. In Section 3, we use the column generation that gives a tighter lower bound. To obtain an optimal solution we present a specific branching scheme used in the branch and price procedure. Section 4 presents the dichotomous algorithm. The three algorithms are tested on literature instances and the computational results are described in Section 5. Finally, we present some perspectives of our work.

2. THE BRANCH AND BOUND ALGORITHM

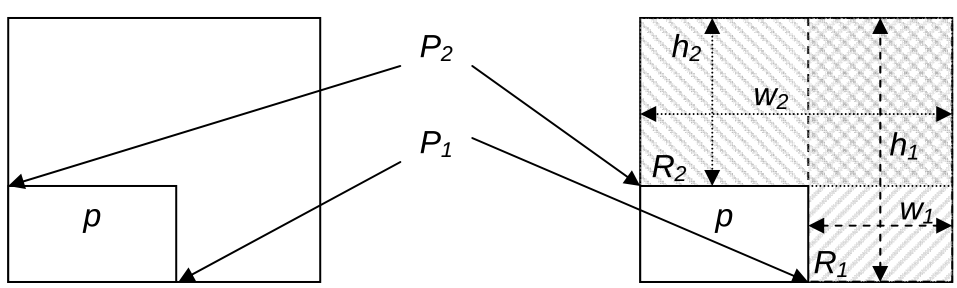
The first exact method that we propose is a branch and bound algorithm. It is proposed for the guillotine case (Bekrar et al. 2006b). We adapt this method to the non-guillotine case. The upper bound, the lower bound and the branching scheme used in this branch and bound are detailed in the following subsections.

2.1 The upper bound

Several papers proposed approximation algorithms for 2D Strip Packing Problem, with or without guillotine constraint (Fernandez de la Vega and Zissimopoulos (1998), Lesh et al., (2003), Kenyon and Remila (1996), Zhang et al. (2006) and Cui et al. (2006)).

We adopt the SHF heuristic proposed for the bin and strip packing problem with guillotine constraint. This heuristic was proposed previously by Ben Messaoud et al., (2003). We proposed some strategies to improve it in Bekrar et al., (2006a), and we called it BSHF. It is a generalization of the well-known two dimensional level algorithm.

The main idea of this algorithm is to exploit the non-used area in each shelf. It makes it possible to pack items one over another in the same shelf, which is not permitted in the level algorithms.In SHF, items are packed in *available rectangles (see Figure 1)*: the rectangles which have the bottom-left corner as an *available point (see Figure 1)*. Such a point can be the bottom-right or the top-left



P_1, P_2: Available Points *R_1, R_2: Available Rectangles*

Figure 1. Available Points and Available Rectangles

corner of an item already packed.

Items are sorted in non-increasing order of heights. The first item (the tallest one) is packed into the first *available rectangle* (the lowest). The leftmost item *initializes* a shelf with a height equal to the height of this item. After every item-packing, the set of *available rectangles* is updated. The updating consists of reducing the dimensions of *available rectangles* which overlap with the packed items. The item-packing creates two new *available rectangles*. This procedure is repeated until the last item is packed.

In the guillotine version of this algorithm, we tested different rules of sorting items. We used the *Best Fit* rule to pack items and we proposed a new way to update the list of available rectangles. This algorithm gives good results on large instances in few seconds. The average waste rate is estimated at 9%.

2.2 Lower bounds for 2SP

In this section, we explain the lower bound used in our algorithms. First, we present some lower bounds proposed for the strip packing problem: the continuous lower bound L_c, a first lower bound of Martello et al., (2003) which we denote by L_{mmv1} and a second denoted by L_{mmv2}. In our algorithms, we use our new lower bound denoted by L_{BKCS}.

2.2.1 The continuous lower bound L_c

By splitting each item j into $w_j h_j$ unit squares, we obtain a lower bound L_c which is called *continuous lower bound*:

$$L_c = \left\lceil \frac{\sum_{i=1}^{n} w_i h_i}{W} \right\rceil$$

Let L_0 = max $\{L_c,\ max_{i=1...n}\ hi\}$. Martello et al. (2003) proved that the absolute worst case performance ratio of this bound is equal to 1/2.

2.2.2 First lower bound L_{mmv1} of Martello et al., (2003)

This first lower bound presented by Martello et al., (2003) is an extension of that proposed by Martello and Toth (1990) for the one-dimensional bin packing problem.

The idea is to subdivide the set of items into three sub-sets according to their dimensions. Let J be the set of items.

Let $\alpha \in [1, W/2]$

$J1 = \{ j \in J: w_j > W-\alpha \}$, the subset of large items,

$J2 = \{ j \in J : W -\alpha \geq w_j > W/2\}$, the subset of medium items,

$J3 = \{ j \in J : W/2 \geq w_j \geq \alpha \}$, the subset of small items,

The set of items such that $w_j < \alpha$ is ignored.

As we can see in Figure 2:

- Beside a large item we cannot pack any item of the other subsets,
- Two items of the subset *J2* cannot be packed one beside the other,
- Beside items of *J2* only items of the class *J3* can be packed.

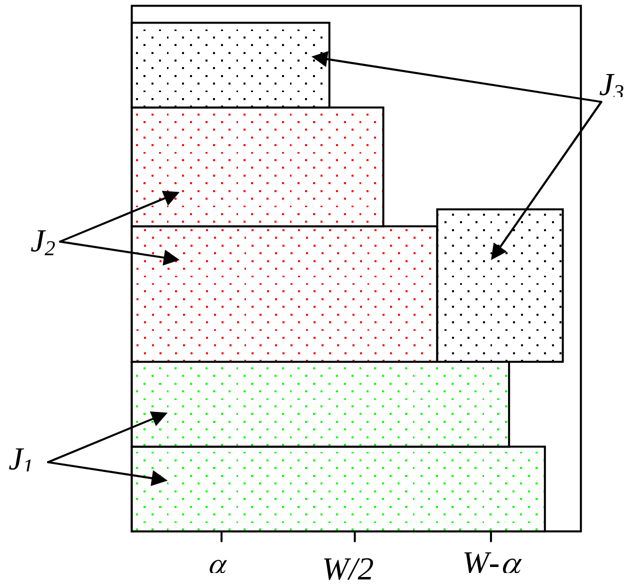


Figure 2. Different classes of items

Let:

$$L(\alpha) = \sum_{j \in J1 \cup J2} h_j + \max\left(0, \left\lceil \frac{\sum_{j \in J3} w_j h_j - (\sum_{j \in J2} (W - w_j) h_j)}{W} \right\rceil\right)$$

$L_{mmv1} = \max_{1 \le \alpha \le W/2}\{L(\alpha)\}$ is a valid lower bound for the 2SP problem that can be computed in *O(n log n)* time (for the proof see Martello et al., (2003)).

2.2.3 Second lower bound L_{mmv2} of Martello et al., (2003)

The second lower bound proposed by Martello et al., (2003) is based on a relaxation of the 2SP problem by cutting each item $j \in J$ into h_j unit-height *slices* of width w_j. The authors consider the *one dimensional contiguous bin packing problem (1CBP)*, where all slices must be packed in bins of size *W*. The h_j slices derived from the item *j* must be packed into h_j contiguous bins. The optimal solution of (1CBP) is a valid lower bound for the 2SP problem (denoted by L_{mmv2}). This solution is computed by a branch and bound algorithm. The authors proved that L_{mmv2} dominates all the previous lower bounds.

When the branch and bound algorithm fails to determine the optimal solution within a fixed time, a new instance of (1CBP) is created from the 2SP instance by cutting each item $j \in J$ into $\lfloor h_j/2 \rfloor$ slices of height 2, or into $\lfloor h_j/3 \rfloor$ slices of height 3, and so on, until a solvable (1CBP) instance is produced. L_{mmv2} is improved by using a specialized binary search procedure (for more details see Martello et al., (2003)). This lower bound gives the best results on the tested instances, but as we can remark, it is very time consuming.

2.2.4 New lower bound L_{BKCS}

The lower bound that we use in different algorithms was proposed in our previous work for the guillotine problem. It is based on the same decomposition presented in Section 2.2.2. This bound gives good results on the same instances of Martello et al., (2003) while the computational time is lower, and the implementation is easier.

Let us consider the decomposition presented in Section 2.2.2. As we can see (Figure 3):

- No item of the subsets can be packed with items of the subset *J*1,
- Two items of *J*2 cannot be packed side by side, only the items of *J*3 can be packed next to *J*2,
- $L = \sum_{j \in J1 \cup J2} h_j$, is a valid lower bound for 2SP problem.

Our idea is to calculate the maximum number of items from *J*3 that can be packed with items of *J*2, noted *nps*. This allowed us to compute an improved bound on the items of *J*3 that cannot be packed with *J*2. The latter bound is added to *L*.

J3 is sorted in non-decreasing order of area and $J_3^{'}$ denotes the subset of the ($|J_3| - nps$) smallest pieces of *J3*. The new lower bound is presented as:

$L_{BKCS} = \max_{1 \le \alpha \le W/2}\{L_{BKCS}(\alpha)\}$, where

$$L_{BKCS}(\alpha) = L + \max\left(\left\lceil \frac{\sum_{j \in J'_3} w_j h_j}{W} \right\rceil, h_{[|J3| - nps]}\right)$$

and $h_{[j]}$ is the height of the j^{th} item when sorting the items in non-decreasing order of height.

To compute *nps*, we use the property presented by Boschetti and Mingozzi (2003) for calculating an upper bound on the number of items that can be packed in a bin. Bekrar et al., (2006b), proposed two methods to determine the bin in which the items of *J*3 can be packed. A third method based on solving a knapsack problem is also used. For *nps*, we take the best result computed by the three methods.

2.3 The branching scheme

To solve the 2SP problem to optimality, we employ the same branching-scheme used by Martello et al. (2003). The Depth First strategy is adopted in the tree search.

At each decision, a partial solution is composed of the set of items already packed $I \subset J$. This solution is increased by selecting each item $j \in I \backslash J$ for branching, and packing them in a "feasible" position.

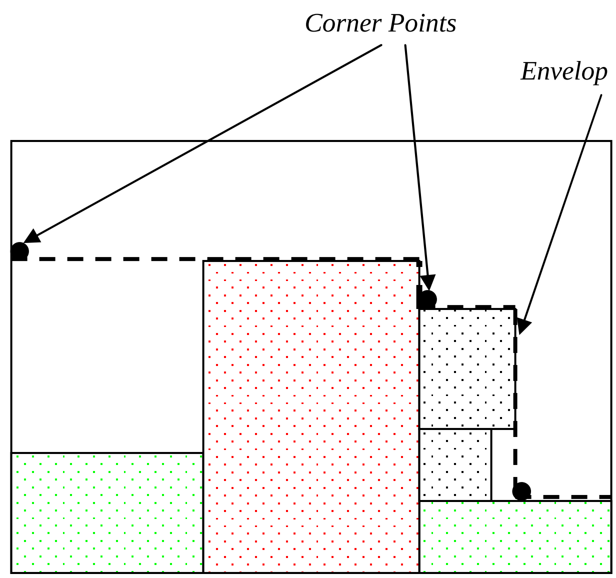


Figure 3. Envelop and Corner Points.

Martello et al. (2000) proved that the items of I define an "envelop" that separates the two regions where the items of $J \backslash I$ may (resp. may not) be placed, and it is enough to generate nodes where j is placed in the *"corner points"* (black points in Figure 3). The *"corner points"* corresponding to points where the slope of the envelop changes from vertical to horizontal, and it can be determined in $O(|I| log |I|)$ time.

From the partial solution in each node, we construct a new instance of 2SP problem. This new instance is composed from the non-packed items and items created from the packed items. The lower bounds described above are applied on this new instance to give an evaluation of the partial solution.

A new property to eliminate symmetrical pattern

In the bin packing and strip packing problem, the problem of equivalence and dominance patterns is always encountered (for more details about equivalence and dominance problem see Scheithauer(1998)). To avoid symmetrical patterns, we apply the technique used by Martello et al (2003) and our technique based on the following property:
Let $n1$ and $n2$ be two potential nodes in the search tree. If the set of the non-packed items in the two nodes is the same, and the envelops of the two nodes are identical, then the two patterns are equivalent. Before generating a new node we check if any equivalent node exists in the same level.

3. THE BRNACH AND PRICE ALGORITHM

In this section, we present another exact algorithm for the two dimensional strip packing problem. First, we use the column generation to obtain a tight lower bound, and then a specific branching scheme is used to obtain an optimal solution. The used method is an adaptation of the one proposed by Pisinger and Sigurd (2005) for solving the bin-packing case.

The column generation was applied for the first time in 1963 by Gilmore and Gomory for the one dimensional cutting stock problem. Since then it has been used for different variants of the cutting and packing problems. Vanderbeck (1999) proposed different applications of the column generation for the one dimensional cutting and packing problems. The approach was used for the three staged two dimensional bin packing problem by Puchinger and Raidl (2004) and for the two-dimensional bin packing problem with variable bin sizes and costs by Pisinger and Sigurd (2005).

The two dimensional strip packing problem can be formulated as a Set Covering problem as follows:

Let P be the set of all feasible patterns in a single bin. A feasible pattern $p \in P$ is a configuration in which a set of items are packed in a single bin of width W and height H_p. The different constraints cited above have to be respected. We introduce the binary variable x_p that is equal to 1 iff the pattern p is selected in the final solution and 0 otherwise. The objective is to minimize the total height used to pack all items. The problem is formulated by the following model:

$$(LP)\begin{cases} Minimize & \sum_{p\in P} H_p x_p & \\ st & \sum_{p\in P} y_{ip} x_p \geq 1 & \forall i \in \{1,\dots,n\} \\ & x_p \in \{0,1\} & \forall p \in P \end{cases}$$

where y_{ip} is a constant equal to 1 iff the pattern p contains item i and 0 otherwise. The LP-relaxation,

$$(RLP)\begin{cases} Minimize & \sum_{p\in P} H_p x_p & \\ st & \sum_{p\in P} y_{ip} x_p \geq 1 & \forall i \in \{1,\dots,n\} \\ & 0 \leq x_p \leq 1 & \forall p \in P \end{cases}$$

presented by the model (*RLP*), gives a tight lower bound on the integer optimal solution.
The number of feasible packing may be exponential even if the number of items is small. For that reason, we use the column generation approach to solve *(RLP)*. This method adds a new variable (pattern) at each iteration.

3.1 A column generation for the 2SP problem

In the column generation approach, we start by solving *(RLP)* for a restricted set of patterns. This is called *restricted master problem*. The restricted set is determined by solving the two dimensional strip packing problem by the heuristic presented in section 2.1. After solving the restricted problem, we look for a new feasible pattern with a minimum reduced cost. If a column with a negative reduced cost is found, it is added to the restricted master problem and the process is reiterated.

Otherwise, the current master solution is optimal (for more details about column generation and branch and price approaches, see Barnhart et al. (1998)).

Let π_i, $i \in \{1,..., n\}$ be the dual variables associated to the model *(RLP)*. The dual program of *(RLP)* is given by (*DRLP*):

$$(DRLP)\begin{cases} Maximize & \sum_{p \in P} \pi_p \\ st & \sum_{i=1}^{n} y_{ip}\pi_i \leq H_p \qquad \forall p \in P \\ & \pi_i \geq 0 \qquad \forall i \in \{1,...,n\} \end{cases}$$

The reduced cost of the packing $p \in P$ is computed by the equation:

$$C_p = H_p - \sum_{i=1}^{n} y_{ip}\pi_i \ .$$

If $Cp \geq 0$ for all $p \in P'$, where P' is the set of the remaining patterns, then the solution is optimal for *(RLP)*. Otherwise, if the reduced cost is negative, the new column can be added to the primal model. This is equivalent to add the violated constraint to the dual model *(DRLP)*.

The problem of finding a new column with minimum reduced cost is called the *pricing problem*. For the strip packing problem, we look for a new feasible pattern that has a negative reduced cost. This corresponds to the two dimensional knapsack problem with respect to the bin packing constraints (no overlapping, orthogonal packing, no item rotation). We associate to each item $i \in \{1,..., n\}$ a profit π_i. The objective is to maximise the profit of the packed items in a single bin of width W and height H_p.

To solve the pricing problem, we try to construct a feasible solution with a negative reduced cost by a greedy heuristic. If this heuristic fails to find such a solution, we apply the exact algorithm described thereafter.

Greedy heuristic to solve the two-dimensional knapsack problem

The items are sorted in non-decreasing order of the profits. At every iteration we pack the current item in the best available rectangle. Let H_i be the height of the current solution after packing item i. If the reduced cost of the current solution is less than that of the best solution found until now, then the current solution is saved. The heuristic is sketched in *Algorithm 1*below.
When the heuristic PSHF fails to find a pattern with a negative reduced cost, we solve the two dimensional knapsack problem to optimality by using the algorithm described as follows.

An exact algorithm to solve the two-dimensional knapsack problem

The *two-dimensional Knapsack Problem* (*2KP*), whose input consists of a set N={1,…,n} of “small” rectangles (*items*), the i^{th} having a *width* w_i, a *height* h_j and a *profit* p_i, and a “large”

Algorithm 1 *PSHF Algorithm*

- Sort the list of the rectangles in non-decreasing order of profits.
- *RC = 0;* $H_s = 0$ *;% RC: reduced cost, Hs: height of the saved solution*
- for (i = 1 to *n*) do
  - pack *i* in the best available rectangle
  - let H_i be the height of the current solution, p_i be the profit of item *i*, and *S* be the set of the packed items
  - if $(H_i - \sum_{k \in S} p_k) < RC$ then
    - $RC = H_i - \sum_{k \in S} p_k$,
    - $H_s = H_i$,
    - Save the current solution,
- End_for
- End.

rectangle (*knapsack*) of width *W* and height *H*. The objective is to pack a maximum-profit subset of the items into the knapsack, with the constraint that items do not overlap and each of them must have its edge parallel to the edge of the knapsack. *2KP* is a generalization of the famous *one-dimensional Knapsack Problem* (KP) and it is proved to be strongly NP-hard (see Capara and Monaci (2004)). The 2KP has been widely studied in the literature. Beasley (1985), Hadjiconstantinou and Christofides (1995), Boschetti et al. (2002) and Fekete and Schepers (1997) present different upper bounds embedded into different branch-and-bound algorithms.

To solve this problem we use the technique of Fekete and Schepers (2001). It consists of splitting the problem into two sub-problems:

- An optimization problem: one dimensional knapsack problem in which we consider each item $i \in N$ as an object of capacity $w_i h_i$. The objective is to maximize the profits of objects packed in a knapsack of capacity equal to WH_p.
- A decision problem: we check if the items obtained by the resolution of the optimization problem can be packed in a single bin of width *W* and height H_p. The different constraints of bin packing must be respected.

The one dimensional knapsack problem is formulated as follows:

$$\begin{aligned} &Maximize \quad \sum_{i=1}^{n} p_i x_i \\ &st \quad \sum_{i=1}^{n} w_i h_i x_i \le WH_p \\ &\qquad x_i \in \{0,1\} \qquad \forall i \in \{1,\ldots,n\} \end{aligned}$$

Where p_i is the profit of the item *i*, H_p is the height of the pattern *p*. In this model, x_i is a binary variable equal to 1 if the item *i* is selected in the knapsack and 0 otherwise.

The Decision Problem

The decision problem consists of checking the feasibility of the solution obtained when solving the one dimensional knapsack problem. This problem is called orthogonal packing problem (2OPP). The (2OPP) is an NP-complete decision problem (Garey and Johnson (1978)); it consistsof determining if a set of rectangles (items) can be packed into one rectangle of a fixed size (bin). Several papers studied this problem. Fekete and Schepers (2007) proposed an exact algorithm based on the graph theory. Clautiaux et al., (2006a, 2006b) proposed two approaches to solve 2OPP to optimality. The first one is a two step algorithm where they determined the x-coordinates of items in the first step, then they checked the feasibility of the obtained configurations in the second step (Clautiaux et al., (2006a)). Clautiaux et al., (2006b), used another approach based on constraint programming. The constraint programming approach was also used by Pisinger and Sigurd (2007). We choose to use the last approach in our algorithm because we have experimentally checked that it has less computational time.

The (2OPP) is solved by a constraint satisfaction program (CSP). For each pair of items *i, j* the domain M_{ij} is associated. M_{ij} = *{left, right, below, above}* determine the possible relative placements among which we should choose at least one. The different relations between items can be defined as:

$r_{ij} \in M_{ij} \qquad i, j \in \{1, \dots, n\}, i \neq j$

$r_{ij} = left \Rightarrow x_i + w_i \leq x_j \qquad i, j \in \{1, \dots, n\}, i \neq j$

$r_{ij} = right \Rightarrow x_j + w_j \leq x_i \qquad i, j \in \{1, \dots, n\}, i \neq j$

$r_{ij} = below \Rightarrow y_i + h_i \leq y_j \qquad i, j \in \{1, \dots, n\}, i \neq j$

$r_{ij} = above \Rightarrow y_j + h_j \leq y_i \qquad i, j \in \{1, \dots, n\}, i \neq j$

$0 \leq x_i \leq W - w_i \qquad i \in \{1, \dots, n\}$

$0 \leq y_i \leq H - h_i \qquad i \in \{1, \dots, n\}$

Initially all relations r_{ij} are set to undefined.

In every iteration of the algorithm, two rectangles *i* and *j* are considered and a value is assigned to r_{ij} from M_{ij}. It is then checked whether a feasible assignment of coordinates exists respecting the current relations r_{ij}. If the coordinate verification fails, the algorithm backtracks. Otherwise a recursive call is done.

3.2 The branch and price algorithm

When we apply the column generation algorithm, it is possible that an integer solution is obtained. In such a case this solution is optimal for the two dimensional strip packing problem. Otherwise, to obtain an optimal solution for the problem, we use the branch and price approach. The branching scheme used in the branch and price algorithm must work well with the pricing problem.

Ryan and Foster (1981) used a special scheme for crew scheduling. Branching is done by dividing the solution space into two sets. In the first set two tasks *r* and *s* appear together and in the second set they must appear separately. We adapt this scheme to our problem. In the first node, two rectangles *i* and *j* must be in the same pattern, in which case they can be combined into one object when solving the knapsack in the pricing problem. In the second node, we consider that the rectangles *i* and *j* will be in two different patterns, then a constraint $x_i + x_j \leq 1$ is added to the knapsack model. In each node of the branch and price tree, we apply the column generation procedure as described above until finding an optimal solution.

4. THE DICHOTOMOUS ALGORITHM

The dichotomous approach for solving packing problems was initially proposed in the well-known paper of Hifi (1998). This type of algorithm starts by computing a lower bound denoted by *BI*. An upper bound, denoted by *BS*, is then computed. If the upper bound is equal to the lower bound, then this is the optimal solution. Otherwise, we search the optimal length included in the interval [*BI*,

Algorithm 2 Dichotomical Algorithm

```
INPUT: n (number of items), listP (list of dimensions of items), W (Width of the strip),
TimeLimit: TL
OUTPUT: The optimal length of the used strip
Subroutine: The algorithm OPP (n, L, W1, H1) that returns TRUE if the set L of n items can be
packed in the bin of width W1 and height H1
BI := L_BKCS(n, listP,W)
BS := BSHF(n, listP,W)
if (BI == BS) then
        Return BI; STOP.
else If (OPP (n, listP, W, BI) == TRUE) then
        Return BI; STOP.
else
        while ((BI < BS) and (Time < TL)) do
                If (BI - BS == 1) then
                        Return BS;   STOP.
                else    S := ⌊(BS + BI)/2⌋
                        If (OPP(n, listP, W, S) == TRUE) then
                                BS := S
                        else       BI := S
                        end if
                end if
        end while
end if
end if
```

BS] for which the items can be packed. To reduce the search space, we use a dichotomous search. The Dichotomous Algorithm is sketched in (Algorithm 1). Here, *BI* is initialized by L_{BKCS} and BS by the heuristic *BSHF* described in the previous sections.

When a height *S* is taken from the interval [*BI*, *BS*], a decision problem is generated: *could the set of items be packed in the bin of width W and height S?* The problem of determining if a set of rectangles can be packed in a larger rectangle of a fixed size, is known as the two dimensional orthogonal packing problem (2OPP).

To solve (2OPP) we use the algorithm of Pisinger and Sigurd 2006). The problem and the .algorithm used to solve it are described in a previous section

5. Computational Results

All the proposed algorithms were coded in C++ and tested on a Pentium M 1.7 GHz with 1G of RAM. For the column generation and linear programs we used CPLEX and Concert Technology.

The first tests were carried out on a series of small and medium instances from the literature.Those instances are available at the website of M. Hifi (http://www.laria.u-picardie.fr/hifi/OR-Benchmark/).

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Table1. Results obtained by the three algorithms for Hifi instances

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Name	n	W	L_{BKCS}	BSHF	S_{bb}	T_{bb}	*nodes*	S_{da}	T_{da}	S_{cg}	T_{cg}	S_{bp}	T_{bp}
SCP 1	10	5	13	13	13	0.00	1	13	0.00	13	0.01	13	0.00
SCP 2	11	4	40	40	40	0.00	1	40	0.00	40	20.00	40	0.00
SCP 3	15	6	14	19	14	7.1	230269	14	403.83	13.31	333.28	19	OM
SCP 4	11	6	19	23	20	2.45	79180	20	160.35	18.83	390.00	22	OM
SCP 5	8	20	20	20	20	0.00	1	20	0.00	20	0.08	20	0.00
SCP 6	7	30	38	38	38	0.00	1	38	0.00	38	0.00	38	0.00
SCP 7	8	15	12	15	14	0.01	3383	14	0.06	13.5	0.02	14	1.01
SCP 8	12	15	17	20	17	1.57	76755	17	10.45	17	4.09	17	0.00
SCP 9	12	27	68	77	68	0.14	945	68	0.01	68	2.00	68	0.00
SCP 10	8	50	78	80	80	0.18	1030	80	0.01	80	1.00	80	0.00
SCP 11	10	27	47	52	48	4.89	138513	48	0.42	48	1.00	48	0.00
SCP 12	18	81	34	38	38	3600	15914244	34	0.01	38	3600.00	38	3600.0
SCP 13	7	70	42	57	50	0.38	29361	50	0.15	50	2.00	50	0.00
SCP 14	10	100	60	83	69	15.57	471111	69	196.65	69	600.34	69	0.00
SCP 15	14	45	34	35	34	2688.13	16080100	34	445.94	34	891.51	34	0.00
SCP 16	14	6	32	35	33	606.12	23365070	33	216.83	32.5	992.03	35	OM
SCP 17	9	42	39	43	34	2.07	3808	34	0.00	34	1.40	34	0.00
SCP 18	10	70	89	104	100	9.43	261879	100	97.03	97.5	43.15	100	88.25
SCP 19	12	5	26	27	26	0.01	46	26	16.36	24.2	2.74	27	OM
SCP 20	10	15	19	24	20	1.73	40905	20	1.32	20	8.50	20	0.00
SCP 21	11	30	135	150	140	24.31	1058902	140	63.29	140	121.10	140	0.00
SCP 22	22	90	34	43	42	3600	45739106	39	1597.91	42.3	3600.00	43	3600.0
SCP 23	12	15	33	39	34	37.97	1675964	34	24.07	35	3600.00	39	0.00
SCP 24	10	50	103	142	109	19.34	387932	109	71.05	109	566.32	109	0.00
SCP 25	15	25	35	43	36	3600	15839980	36	657.15	40	3600.00	43	3600.0
Ave.			43.24	50.4	45.48	568.85	4855939	45.2	158.51	45.40	735.32	46.4	
N_Opt					22			25				18	

The second tests were carried out on instances of different sizes and different types. They are available at the website of DEIS laboratory (http://www.or.deis.unibo.it/research.html).

We compare the three methods: the branch and bound algorithm, the branch and price procedure and the dichotomous search. For each problem, Tables 1 and 2 give the following information:

Table2. Results obtained by the three algorithms for Martello et al. instances

Name	n	W	L_{BKCS}	$BSHF$	S_{bb}	T_{bb}	*nodes*	S_{da}	T_{da}	S_{cq}	T_{cq}	S_{bp}	T_{bp}
NGCUT01	10	10	20	25	23*	1.01	18610	23*	23.20	25	3600	25	0.00
NGCUT02	17	10	28	33	30	3600	43836217	30*	1052.72	23	127.54	33	3600
NGCUT03	21	10	28	34	28*	1621.31	19687731	28*	519.70	28	1095.91	28*	0.00
NGCUT04	7	10	17	23	20*	0.01	1953	20*	0.02	20	0.01	20*	0.00
NGCUT05	14	10	36	37	36*	0.01	204	36*	119.58	36	2275.14	36*	0.00
NGCUT06	15	10	29	36	31*	3579.01	103839866	31*	1079.26	36	3600	37	0.00
NGCUT07	8	20	20	21	20*	0.01	18176	20*	0.00	20	0.01	20*	0.00
NGCUT08	13	20	32	44	33*	55.01	999353	33*	178.06	34.14	3600	21	0.00
NGCUT09	18	20	49	64	50	3600.00	19863917	50*	1269.83	57.89	1631.31	44	OM
NGCUT10	13	30	58	84	80*	198.21	13082227	80*	1152.83	67.88	1790.85	64	OM
NGCUT11	15	30	50	60	52*	1058.6	15171529	52*	733.18	54	3600	84	0.00
NGCUT12	22	30	79	96	87*	3600	64961552	87*	866.32	90	3600	60	0.00
HT01	16	20	20	20	20*	0.00	1	20*	0.00	20	1712.3	20*	0.00
HT02	17	20	20	23	20*	7593	74454013	20*	378.81	21.4	3600	23	0.00
HT03	16	20	20	25	20*	25.23	624842	20*	197.53	22.56	3600	25	0.00
HT04	25	40	15	17	16	3600.00	78101161	15*	874.05	17	3600	17	0.00
HT05	25	40	15	17	15	3600.00	8027995	15*	571.65	15	2550.95	15*	0.00
HT06	25	40	15	15	15*	0.00	1	15*	0.00	15	1081.6	15*	0.00
HT07	28	60	30	33	33	3600	9672085	31*	2045.34	34.64	3600	33	0.00
HT08	29	60	30	36	35	3600	14676076	31*	1905.08	30	1546	36	3600
HT09	28	60	30	30	30*	0.00	1	30*	0.00	30	1745	30*	0.00
CGCUT01	16	10	23	25	23*	521.31	14533561	23*	323.54	23.99	2957	25	3600
CGCUT02	23	70	63	82	71	3600	32227282	67*	1219.41	78.02	3600	82	0.00
CGCUT03	62	70	636	728	676	3600	635542	676	3600.00	728	3600	728	0.00
GCUT01	10	50	1016	1016	1016*	0.00	1	1016*	0.00	1016	47	1016*	0.00
GCUT02	20	250	1133	1349	1275	3600	23863965	1251*	2214.62	1349	3600	1349	0.00
GCUT03	30	250	1803	1810	1810	3600	16936397	1810	3600.00	1810	3600	1810	0.00
GCUT04	50	250	2934	3214	3189	3600	1463787	3214	3600.00	3214	3600	3214	0.00
BENG01	20	25	30	35	35	3600	40359226	30*	608.41	35	3600	35	0.00
BENG02	40	25	57	60	60	3600	2641751	58	3600.00	60	3600	60	0.00
BENG03	60	25	84	89	89	3600	775077	86	3600.00	89	3600	89	0.00
BENG04	80	25	107	112	109	3600	381697	109	3600.00	112	OM	112	0.00
BENG05	100	25	134	138	138	3600	209982	135	3600.00	138	OM	138	0.00
BENG06	40	40	36	39	39	3600	4972538	37	2858.63	39	3600	39	0.00
BENG07	80	40	67	72	72	3600	341251	72	3600.00	72	OM	72	0.00
BENG08	120	40	101	108	108	3600	228359	102	3600.00	108	OM	108	0.00
BENG09	160	40	126	126	126*	0.00	1	126*	0.00	130	OM	126	0.00
BENG10	200	40	156	158	158	3600	243611	158	0.00	158	OM	158	0.00
Average			240.71	261.4	254.95	2375.07	15969777.32	254.13	1383.99	259.41	2605.02	259.1	300.00
N_Opt					17			27				9	

- Problem name, number of items n and the width of the strip W,
- Value of the lower bound L_{BKCS} and values of the upper bound $BSHF$,
- Value of the branch and bound solution (S_{bb}) and the number of nodes (*nodes*),

- Value of the branch and price solution (S_{bp}) and the value obtained by the column generation (S_{cg}),
- Value of the dichotomous algorithm solution (S_{da}),
- The computational time spent by the branch and bound algorithm T_{bb}, that of the branch and price procedure T_{bp} and that of the dichotomous algorithm T_{da} (in seconds),
- **Average:** The average of time spent by each algorithm. The average value of the lower bound, the upper bound and the number of nodes,
- **N_Opt:** The number of times when each algorithm reached the optimal solution,
- **OM**: *Out of Memory* given by CPLEX when solving the linear program in the branch and price algorithms is impossible.

Table 3. Data of the instance SCP16.

n=14, W = 6

items	h_i	w_i
1	2	2
2	3	3
3	5	2
4	6	2
5	11	4
6	9	3
7	5	4
8	2	2
9	3	3
10	5	2
11	5	2
12	5	2
13	5	2
14	6	2

$L_{BKCS} = 32$, $BSHF = 35$, $OPT = 33$

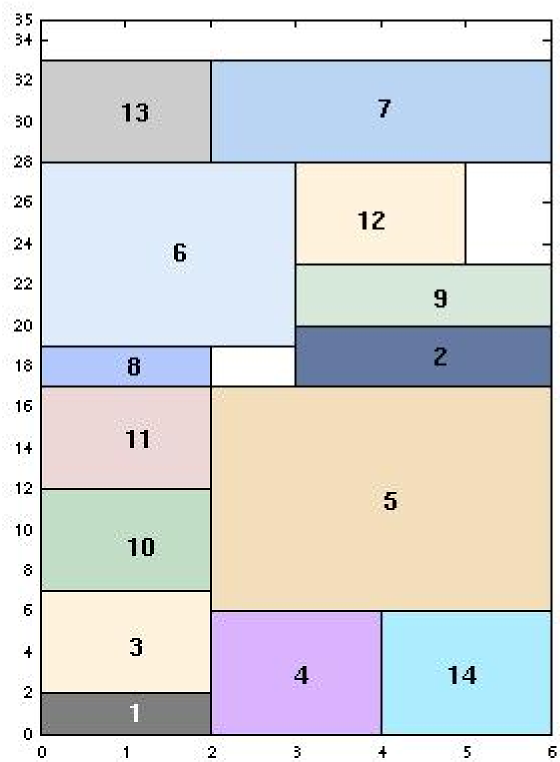

Figure 4. Optimal solution of the instance SCP16

As we can see in Table 1, the dichotomical algorithm outperforms the two other algorithms in the number of solved problems and the computational time. This algorithm solved all the instances in an average time equal to 158.51 seconds for each problem. The branch and bound algorithm solved 22 instances in an average time equal to 568.86 seconds for each problem and the branch and price solved 18 problems in an average time equal to 1291.56 seconds for each problem.

In Table 2, we present the results obtained by the different algorithms carried out on the instances of Martello et al. (2003). In this case, the performance of the dichotomous algorithm is also confirmed. As these instances are harder than the previous ones, the dichotomous algorithm could solve 27 out of 38. The branch and bound algorithm could solve 17 instances. The less efficient algorithm on these instances is the branch and price that could solve only 9 instances out of 38.

As illustration, Figure 4 presents the optimal solution of the instance SCP16. The data of such an instance are presented in Table 3.

To analyze the performance of the algorithms to more extent, we compare the branch-and-bound method and the dichotomous algorithm on instances randomly generated. We adapt the classes of instances proposed by Berkey and Wang (1987) and Martello and Vigo (1998) for the two dimensional bin-packing problem to the strip packing problem. These instances consist of ten classes of problems. For each class, there are 40 instances: 10 with 10 rectangles, 10 with 15 rectangles, 10 with 20 rectangles and 10 with 25 rectangles. The first six classes have been proposed by Berkey and Wang (1987):

j = 1,…, n. n = {10, 15, 20, 25},
Class I: wj and hj uniformly random in [1, 10], W=10;
Class II: wj and hj uniformly random in [1, 10], W=30;
Class III: wj and hj uniformly random in [1, 35], W=40;
Class IV: wj and hj uniformly random in [1, 35], W=100;
Class V: wj and hj uniformly random in [1, 100], W=100;
Class VI: wj and hj uniformly random in [1, 100], W=300;

The remaining four classes were inspired b Martello and Vigo (1998). The items are classified into the following four types:

Type 1: wj uniformly random in [2/3W, W], hj uniformly random in [1, W/2];
Type 2: wj uniformly random in [1, W/2], hj uniformly random in [2/3W, W];
Type 3: wj uniformly random in [W/2, W], hj uniformly random in [W/2, W];
Type 4: wj uniformly random in [1, W/2], hj uniformly random in [1, W/2];

The strip widths are W = 100 for all these classes, while the items are as follows:

Class VII: type 1 with probability 70%, type 2, 3, 4 with probability 10% each;
Class VIII: type 2 with probability 70%, type 1, 3, 4 with probability 10% each;
Class IX: type 3 with probability 70%, type 1, 2, 4 with probability 10% each;
Class X: type 4 with probability 70%, type 1, 2, 3 with probability 10% each;

In Table 4, we present the results obtained by testing the two algorithms on randomly generated instances. For each 10 instances, we present the average values of:

- **LB:** lower bound,
- **UB:** upper bound,
- ***BS/LB***: the ratio of the best solution obtained by each algorithm on the lower bound. Each instances was carried out for 3600 seconds (if no optimal solution was is achieved, we take the best one),
- ***#Opt*** : the number of found optimum,
- ***Time***: the average time spent for solving the instance.

The results shown in Table 4, confirm the results obtained previously. The dichotomous algorithm outperforms the branch-and-bound method in the number of optimal solutions

Table 4. Results obtained by dichotomous and branch and bound algorithms on random instances

				Dichotomous Algorithm			B&B Algorithm		
Class	***n***	***LB***	***UB***	***BS/LB***	***#Opt***	***Time(sec)***	***BS/LB***	***#Opt***	***Time(sec)***
I	10	35,1	36,4	1.0114	10	0,01	1.0114	10	73.86
	15	45,8	49,4	1.0175	10	246,45	1.0197	10	934.74
	20	62,6	67,2	1.0431	9	1315,46	1.0495	5	1955.88
	25	81,8	85,4	1.0269	7	1758,08	1.0391	4	1800.71
8		56,33	59,60	1.0271	9,00	830,00	1.0337	7,25	1191.30
II	10	9,8	11,9	1.0918	10	0,01	1.0918	10	212.36
	15	16,2	19,6	1.0617	9	181,20	1.1296	5	1200.40
	20	19,2	21,8	1.0833	8	612,83	1.0885	3	2829.70
	25	26,8	29,5	1.0560	9	1122,34	1.0933	3	2961.28
Avg		18,00	20,70	1.0694	9,00	479,09	1.1000	5,25	1800.93
III	10	31,9	39	1.0784	10	0,26	1.0784	10	484.55
	15	142,8	158	1.0315	9	225,92	1.0455	5	1383.41
	20	166,2	177	1.0397	8	809,81	1.0554	4	2750.40
	25	217,2	233,6	1.0451	6	1779,74	1.0681	2	1190.60
Avg		139,53	151,90	1.0419	8,25	703,93	1.0591	5,25	1452.24
IV	10	35,2	45,8	1.1591	10	1,35	1.1591	10	960.26
	15	48,7	60,5	1.1170	9	633,73	1.1396	4	1752.30
	20	58,2	68,6	1.0962	8	1482,41	1.1718	4	2380.80
	25	84,2	99	1.0784	6	2547,14	1.1544	4	2811.80
Avg		56,58	68,48	1.1038	8,25	1166,16	1.1564	5,50	1976.29
V	10	305,6	338,8	1.0517	10	2,12	1.0517	10	91.70
	15	399,7	439,4	1.0435	9	580,76	1.0545	5	726.80
	20	564,9	593,7	1.0287	8	1285,34	1.0320	5	682.90
	25	624,2	696	1.0676	4	3566,67	1.1041	0	3600.00
Avg		473,59	516,98	1.0483	7,75	1358,72	1.0637	5,00	1275.35
VI	10	103,8	131,6	1.1397	10	0,25	1.1397	10	425.90
	15	118	146,5	1.1008	7	486,46	1.1585	2	2052.90
	20	163,2	200,4	1.1134	7	2005,12	1.1857	1	2922.10
	25	220	265,2	1.1264	4	3474,58	1.1368	1	2806.30
Avg		151,25	185,93	1.1202	7,00	1491,60	1.1547	3,50	2051.80
VII	10	337,3	340,9	1.0033	10	3,57	1.0033	10	4.30
	15	532	532,4	1.0000	10	2,20	1.0000	10	2.50
	20	731,3	739,1	1.0093	10	626,70	1.0093	10	258.30
	25	703,1	719,1	1.0209	7	1839,95	1.0110	7	1286.00
Avg		575,93	582,88	1.0098	9,25	618,10	1.0068	9,25	387.78
VIII	10	229.2	289.3	1.1514	10	4.801	1.1514	10	717.8
	15	386.9	436.3	1.0843	9	1327.03	1.1194	6	2186.2
	20	506.13	581.75	1.1022	6	3244.92	1.1094	2	3294.5
	25	623	704.83	1.1134	2	3577.98	1.1263	1	1804.9
Avg		436.31	503.05	1.1087	6.75	2038.68	1.1232	4.75	2000.85
IX	10	537.6	549.3	1.0113	10	3.11	1.0113	10	25.8
	15	866.3	871.2	1.0057	10	36	1.0057	10	56.7
	20	1155.5	1160.5	1.0000	10	0.014	1.0036	9	400.2
	25	1464.5	1479	1.0076	8	794.798	1.0099	8	733.5
Avg		1005.98	1015.00	1.0055	9.50	208.48	1.0074	9.25	304.05
X	10	162.4	177.1	1.0499	10	1.01	1.0499	10	253.8
	15	251	273.3	1.0319	9	278.257	1.0562	4	2246.5
	20	344.1	384.1	1.0668	6	1897.34	1.0898	4	2858.8
	25	386.11	416.56	1.0590	3	2717.82	1.0869	1	3471.4
Avg		285.90	312.76	1.0541	7.00	1223.61	1.0758	4.75	2207.64

found and in the average time to compute a solution. The dichotomous algorithm has been able to solve on average more than 8 instances from 10 in every class, while the branch-and bound procedure has been able to solve only the half of the instances.As for the two dimensional bin packing problem, the classes of problems numbered VI, VIII and X are the hardest problems to solve. For example, in the class VIII with n = 25 items, the dichotomous algorithm solved only 2 problems out of 10 and the branch and bound solved only one problem.

The easiest instances to solve are those of the classes: I, II, VII and IX. On average, 9 problems among 10 are solved by the dichotomous algorithm for the class I (resp., 7 problems solved by the branch-and-bound) and 9 among 10 for the problems in class II (resp., 5 for the B&B algorithm). This can be explained by the fact that the two first classes contain small items generated randomly in the interval *[1, 10]*.

The problems in the classes VII and IX are also easy to solve because they are composed of 70 % of large items, i.e. items which have a width larger than the half of the width of the strip. Our method used to compute the lower bound and the heuristic used to calculate an upper bound provide very slight results for the type of these problems. The two algorithms have been able to solve on the average more than 9 out of 10 problems in less than 10 minutes for each problem.

6. CONCLUSION AND PERSPECTIVES

In this paper we proposed three exact algorithms to solve the two dimensional non-guillotine strip packing problem. The first method is a branch and bound algorithm based on the branching scheme used by Martello et al., (2003). In this algorithm, we use a new lower bound, an upper bound and an elimination property to avoid symmetrical patterns. The second approach is a dichotomical algorithm. This method uses the lower bound and the upper bound to define an interval that includes the optimal solution. By a dichotomical search, we find a length of a bin in which all items can be packed. At each iteration, a feasibility problem is solved. The last algorithm used to solve the 2SP problem is the branch and price method. We apply a column generation procedure that gives a tight lower bound. To obtain an optimal solution, we use a specific branching scheme.

The different algorithms are compared on a literature instances. The obtained results show that the dichotomical algorithm outperforms the two other algorithms. It could solve most problems in a reasonable time. We notice also that the branch and price method is the less efficient of algorithms for the tested instances.In this study, we observed the importance of the feasibility problem. As perspective, we aim to study this problem and to extend this work to solve other types of packing problems.

ACKNOWLEDGMENTS

Research supported in part by Champagne-Ardenne Regional Council (district grant) and the European Social Fund.

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