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Price Discount and Stochastic Initial Inventory in the Newsboy Problem

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ABSTRACT

Many extension of the newsboy problem have been solved in the literature. One of those extensions solves a newsboy problem with stochastic initial inventory, earlier extensions have focused on quantity discounts offered by suppliers. An important practical extension would address a combination of the two pervious extensions. In this paper we consider a newsboy problem in which the suppliers offer quantity discount and the initial inventory at the beginning of the period is a random variable. We obtain the optimal value of the order quantity which maximizes the total profit and then present the results for some practical distributions of both random variables, demand and initial inventory.

Keywords**:** Stochastic initial inventory, Newsboy problem, Inventory management, Supplier discount.

1. INTRODUCTION

The classical newsboy problem is to find product order quantity that maximizes the expected profit in a single period probabilistic demand framework. The classical newsboy model assumes that if the order quantity is larger than the realized demand, items left over at the end of period are sold at a salvage value or are disposed of. Further, in cases of a shortage, unsatisfied demand is lost with a charge for each unit (Hadley and Whitin, 1963; Johnson and Montgomery, 1974).

Several extensions to the newsboy problem have been proposed and solved in the literature. These extensions include dealing with random yields (Ehrhardt and Taube, 1987, Henig and Gerchak, 1990, Noori and Keller, 1986, Shih, 1980, and Jain and Silver, 1995), different states of information about demand (Moon and Choi, 1995 and Reyniers, 1991), and different news-vendor pricing policies (Khouja, 1995, Khouja, 1996, and Polatoglu, 1991). Some researchers address extensions with alternative objective functions such as maximizing expected utility instead of expected profit (Ismail and Louderback, 1979 and Lau, 1980), maximizing the market value of the firm using the capital asset pricing model framework (Anvari, 1987 and Magee, 1975), or maximizing the probability of achieving a target profit level (Ismail and Louderback, 1979, Lau, 1980, Sankarasubramanian and Kumaraswamy, 1983, and Shih, 1979).

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Other extensions consider more than one supplier (Kabak and Weinberg, 1972), price dependent demand (Lau and Lau, 1988), quantity discounts by suppliers (Pantumsinchai and Knowles, 1991, Jucker and Rosenblatt, 1985, and Lin and Kroll, 1997), and stochastic initial inventory (Haji and Bijari, 2003).

Khouja (1996) addresses an extension of the newsboy problem which is a combination of two of the above extensions, i.e., supplier discount extension due to Jucker and Rosenblat (1985) and progressive multiple discounts extension due to Khouja (1995).

An important practical extension would address a combination of the two previous extensions, i.e., quantity discount by the supplier (Jucker and Rosenblat, 1985) and stochastic initial inventory (Haji and Bijari, 2003).

The supplier quantity discounts are well known and studied in inventory management literature (Hadley and Whitin, 1963 and Love, 1979). For the newsboy problem, Jucker and Rosenblant (1985) and Pantumsinchai and Knowles (1991) solved the problem under different discounts conditions. While the case of supplier quantity is well known in the literature, the stochastic initial inventories are not, even though they are common in practice. Haji and Bijari (2003) proposed an extension to the newsboy problem in which the initial inventory is stochastic. Bijari and Haji (2004) also considered the stochastic initial inventory for the multi-product case. Among other applications, stochastic initial inventory may apply to the case where the decision about the order quantity must be made at a time long before the start of the period and the available inventory at the decision time may decrease stochastically due to several factors, such as deterioration, evaporation, consumption, etc. up to the start of the period.

In this paper, the newsboy problem with all-units quantity discount, stochastic initial inventory, and an objective of maximizing the expected profit is formulated. Next an algorithm to solve the proposed problem is developed and the results for some practical distributions for both random variables, i.e., demand in the period and the initial inventory at the start of the period are presented.

2. PROBLEM FORMULATION

In formulating the model, we assume that all the standard assumptions of the classical newsboy problem holds true. We also assume discounts are offered by the supplier for the purchase of large quantities. These discounts take the form of price breaks of the following type: there are given quantities $q_0 = 0$, $q_1, q_2, ..., q_n$, $q_j < q_{j+1}$, $j = 1, ..., n$ and $q_{n+1} = \infty$, such that if a quantity Q is purchased, $q_j \le Q < q_{j+1}$, then the unit cost of each of the units Q are C_j, i.e., the cost of Q units is $C_j Q$ and $C_{j+1} < C_j$. Furthermore, we assume that the demand for the product during the main period, with length T, is stochastic with a known distribution function. To satisfy the demand of this period an order equal to Q must be placed at a decision point A which is located L units of time before the start of the main period, Figure 1. At the ordering time A there is no knowledge about the size of inventory at the start of the main period, i.e., time B (Figure 1). For example, if the inventory on hand at time A is equal to some value I_M then during the lead time L this inventory may decrease due to several factors such as corrosion, evaporation, damage, etc. Therefore, the inventory at the start of the main period is a non-negative random variable with density function g.

In this paper we use the following notation:

X	Quantity demanded during the main period, a random variable
$f(x)$	Probability density function of X

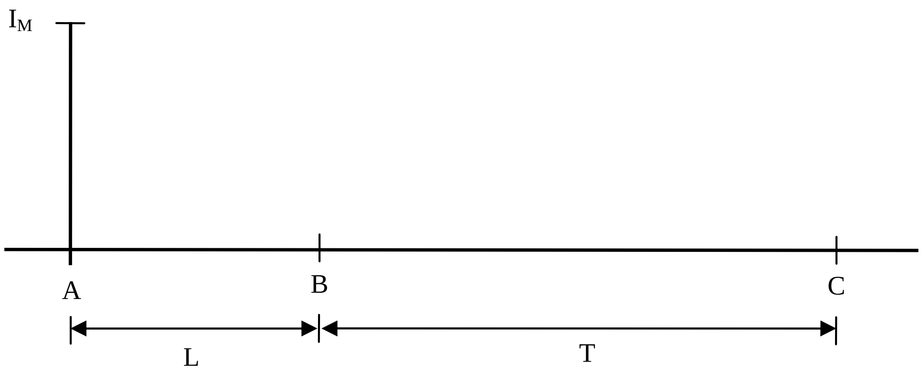

Figure 1. Decision time and main period.

F(x) Cumulative distribution function of *X*

Q Order quantity

C_j Purchase cost per unit of product when the order quantity lies in $q_j \le Q < q_{j+1}$, $j = 0,1,\dots,n$

V Selling price per unit of product

π Unit shortage cost

H_j Holding cost per unit of remaining product, including among others, the negative of salvage value when the unit purchase cost is C_j

Q Order quantity

I Initial inventory at the beginning of the main period before the arrival of order quantity *Q*, a random variable

g(i) Density function of initial inventory

R Inventory level at the beginning of the main period after order is received

For the classical newsboy problem when *I* is fixed and known, if C_j and H_j are cost per unit and holding cost per unit respectively for all values of Q, $0 \le Q < \infty$, then the expected profit per period denoted by $P(R)$ can be written as ((Hadley and Whitin, 1963 and Johnson and Montgomery, 1974):

$$P(R) = VE[X] - K(R)$$

where

$$K(R) = C_j(R-I) + H_j\int_0^R (R-x)f(x)dx + (V+\pi)\int_R^\infty (x-R)f(x)dx$$

or

$$K(R) = C_j(R-I) + H_jR - H_jE[X] + (H_j + V + \pi)\int_R^\infty (x-R)f(x)dx \qquad (1)$$

The expected profit as defined above will be maximized if *K*(*R*) is minimized.
The optimum *R*, say R^*, is the solution to:

$$F(R^*) = \frac{V + \pi - C_j}{V + \pi + H_j} \tag{2}$$

Therefore, the optimum policy is to order Q, where [4]:

$$\begin{cases} Q = R^* - I & \text{if} \quad R^* > I \\ Q = 0 & \text{if} \quad R^* \le I \end{cases}$$

Noting that $R=Q+I$ we can write (1) as:

$$K(R) = K(Q+I) = H_j I - H_j E[X] + (C_j + H_j)Q + (H_j + V + \pi)\int_{Q+I}^{\infty} (x - Q - I) f(x)dx$$

Now assuming that unit cost C_j applies only for those order quantities which lie in $q_j \le Q < q_{j+1}$, let $K_j(Q+I)$ be the value of $K(Q+I)$ for $q_j \le Q < q_{j+1}$, then

$$K_j(Q+I) = H_j I - H_j E[X] + (C_j + H_j)Q + (H_j + V + \pi)\int_{0}^{\infty} (x - Q - I) f(x)dx \tag{3}$$

and

$$K(Q+I) = K_j(Q+I) \qquad q_j \le Q < q_{j+1},\ j = 0,1,\ldots,n$$

When I is a random variable then to obtain the value of $K(Q+I)$ we use the conditional expectation formula (Ross, 1983) and condition on the value of the initial inventory. That is:

$$\begin{aligned} E\left[K(Q+I)\right] &= E\left[E\left[K(Q+I) \mid I\right]\right] \\ &= E\left[E\left[K_j(Q+I) \mid I\right]\right] \qquad q_j \le Q < q_{j+1} \quad j = 0,1,\ldots,n \end{aligned} \tag{4}$$

Thus from (3), for $j=0,1,2,\ldots,n-1$ we have

$$E\left[K_j(Q+I) \mid I\right] = H_j I - H_j E[X] + (C_j + H_j)Q + (H_j + V + \pi)\int_{Q+I}^{\infty} (x - Q - I) f(x)dx \tag{5}$$

Unconditioning on I, we can write for $j=0,1,2,\ldots,n-1$

$$E\left[K_j(Q+I)\right] = H_j\left(E[I] - E[X]\right) + (C_j + H_j)Q + (H_j + V + \pi)\int_{0}^{\infty}\int_{Q+i}^{\infty} (x - Q - i) f(x)g(i)dxdi \tag{6}$$

3. OPTIMAL ORDER QUANTITY

The first derivative of $E\left[K_j(Q+I)\right]$ with respect to Q is:

$$\frac{dE\left[K_j(Q+I)\right]}{dQ} = C_j + H_j - (H_j + V + \pi)\int_{0}^{\infty} \left[1 - F(Q+i)\right] g(i)di$$

$$= C_j - V - \pi + (H_j + V + \pi)\int_0^\infty F(Q+i)g(i)di \tag{7}$$

The second derivative of $E\left[K_j(Q+I)\right]$ with respect to Q is:

$$\frac{d^2E\left[K_j(Q+I)\right]}{dQ^2} = (H_j + V + \pi)\int_0^\infty f(Q+i)g(i)di \tag{8}$$

Now it is clear that, from the non-negativity of density functions, the second derivative is positive. This implies that $E\left[K_j(Q+I)\right]$ is a convex function and has a unique solution in the range $0 \le Q < \infty$.

Let Q_j° denote the value of Q which minimizes $E\left[K_j(Q+I)\right]$ for $0 \le Q < \infty$. Then to obtain Q_j° we set the first derivative of this function equal to zero i.e.,

$$C_j - V - \pi + (H_j + V + \pi)\int_0^\infty F(Q_j^\circ + i)g(i)di = 0$$

This implies that

$$\int_0^\infty F(Q_j^\circ + i)g(i)di = \frac{V + \pi - C_j}{V + \pi + H_j} \tag{9}$$

or equivalently

$$P\left\{X \le Q_j^\circ + I\right\} = \frac{V + \pi - C_j}{V + \pi + H_j} \tag{10}$$

Q_j° can be obtained from (9). Note that, if $\int_0^\infty F(i)g(i)di \ge \frac{V + \pi - C_j}{V + \pi + H_j}$, i.e., if

$P\{X \le I\} \ge \frac{V + \pi - C_j}{V + \pi + H_j}$, then $Q_j^\circ = 0$.

Let

$Q_j^\bullet =$ The value of Q which minimizes $E\left[K_j(Q+I)\right]$ in its realizable region, i.e., in the region $q_j \le Q < q_{j+1}$. Thus

$$Q_j^\bullet = \begin{cases} q_j & if \quad 0 \le Q^\circ < q_j \\ Q_j^\circ & if \quad q_j \le Q^\circ < q_{j+1} \\ q_{j+1} & if \quad q_{j+1} \le Q^\circ < \infty \end{cases}$$

Also let

$Q^\bullet =$ The optimal value of Q which minimizes $E\left[K(Q+I)\right]$ for $0 \le Q < \infty$.

Then

$$E\left[K(Q^\bullet, I)\right] = \underset{0 \le j \le n}{Min}\, E\left[K_j(Q_j^\bullet, I)\right] \tag{11}$$

Note that since $C_{j+1} < C_j$, it is reasonable to assume that $H_{j+1} \le H_j$, $j = 0,1,2,...,n-1$. Thus we can easily show from (6) that the cost curves $E\left[K_j(Q,I)\right]$, $j = 0,1,\dots,n$, do not intersect for all Q, $0 \le Q < \infty$. That is,

$$E\left[K_{j+1}(Q,I)\right] < E\left[K_j(Q,I)\right] \text{ for } 0 \le Q < \infty\,,\ j = 0,1,\dots,n\,. \tag{12}$$

Therefore we can find the optimal quantity, $Q^{\bullet}$, by the following two step algorithm:

Step 1. Determine $Q_j^{\bullet}$ in a backward fashion, i.e., determine $Q_j^{\bullet}$ for $j = n$, $j = n-1$, …, until for some *j*, say $j = k$, $Q_k^{\bullet} = Q_k^{\circ}$. Then go to step 2.

Step 2. Compute $K_j(Q_j^{\bullet})$ for $j = k, k+1,\dots,n$. The optimal $Q^{\bullet}$ is that value of $Q_l^{\bullet}$ for which $K_l(Q_l^{\bullet}) = \underset{k\le j\le n}{Min}\, K_j(Q_j^{\bullet})$ and $Q^{\bullet} = Q_l^{\bullet}$.

4. HANDLING SPECIAL CASES OF DEMAND AND INITIAL INVENTORY

In this section we provide some examples to illustrate the solution of relation (9) for some special cases. Let

$$\beta_j = \frac{V + \pi - C_j}{V + \pi + H_j} \tag{13}$$

Then from (9)

$$\int_0^{\infty} f(Q_j^{\circ} + i) g(i) di = \beta_j \tag{14}$$

Case 1

When demand is exponentially distributed with mean $1/\lambda$ and the initial inventory follows a general distribution *G*, for each value of β_j we can write

$$\begin{aligned}\beta_j &= \int_0^{\infty} F(Q_j^{\circ} + i) g(i) di \\ &= \int_0^{\infty} (1 - e^{-\lambda(Q_j^{\circ}+i)}) g(i) di \\ &= 1\text{-}e^{-\lambda Q_j^{\circ}} \int_0^{\infty} e^{-\lambda i} g(i) di\end{aligned}$$

Or

$$\beta_j = 1\text{-}e^{-\lambda Q_j^{\circ}} \tilde{G}(\lambda)$$

$$e^{-\lambda Q_j^{\circ}} = \frac{1-\beta_j}{\tilde{G}(\lambda)} \tag{15}$$

Where $\tilde{G}(s)$ stands for the Laplace transform of *G*, and is known for the most well known distributions. It is straightforward to solve (15) for Q_j°. For instance, if *I* is exponential with mean $1/\mu$ then $\tilde{G}(\lambda) = \dfrac{\mu}{\lambda + \mu}$ and from (15) we can write

$$e^{-\lambda Q_j^\circ} = \frac{(1-\beta_j)(\lambda-\mu)}{\mu}$$

Case 2

When demand is uniformly distributed in the range (*a, b*) and the initial inventory follows a general distribution *G*, again for any value of β_j we can write

$$\beta_j = \int_0^\infty \frac{Q_j^\circ + i - a}{b-a} g(i)di$$

Or equivalently

$$\beta_j = \frac{Q_j^\circ + E[I] - a}{b-a}$$

Or

$$Q_j^\circ = \beta_j(b-a) + a - E[I] \tag{16}$$

In which *E*[*I*] is the expected value of the initial inventory. It is clear from (16) that in this case we do not need the form of the distribution function of *I*, only a knowledge about the mean of this random variable is sufficient.

Case 3

When random variables *X* and *I* are both normally distributed with parameters ((μ_x, σ_x) and (μ_i, σ_i) respectively, then we can write

$$P\left\{X \le Q_j^\circ + I\right\} = P\left\{X - I \le Q_j^\circ\right\} = P\{U \le a\}$$

Since *X* and *I* are independent and normally distributed random variables then *U* is also normally distributed with mean $\mu_u = \mu_x - \mu_i$ and variance $\sigma_u^2 = \sigma_x^2 + \sigma_i^2$. Thus

$$\beta_j = P\left\{U \le Q_j^\circ\right\} = P\left\{Z \le \frac{Q_j^\circ - \mu_u}{\sigma_u}\right\} \tag{17}$$
$$= P\{Z \le k\}$$

Where *Z* is a standard normal random variable and $k = \frac{Q_j^\circ - \mu_u}{\sigma_u}$. Knowing the value of μ_x, σ_x, μ_i and σ_i one can easily find Q_j° for any given value of β_j.

Remark

We note that from (15), (16), and (17) Q_j° is a monotone increasing function of β_j. In general, for the case when both *X* and *I* have arbitrary distributions we can establish this fact as follows

Let $Y(Q)=\int_0^\infty F(Q+i)g(i)di$ .Then $Y(Q)$ is an increasing function of Q. This is due to the fact that the derivative of $Y(Q)$ is positive, that is,

$$\frac{dY(Q)}{dQ}=\int_0^\infty f(Q+i)g(i)di>0$$

This fact provides a useful hint in solving relation (9) for complicated cases.

5. CONCLUSION

In this paper we considered an extension of the single period inventory problem. We assumed that the supplier offers quantity discounts for the amount purchased. We also assumed that the initial inventory at the beginning of the period is a random variable with a known distribution function. This assumption among other applications , may be applied to the case where the decision about the order quantity must be made at a time long before the start of the period and the available inventory may decrease stochastically due to several factors, such as deterioration, evaporation, etc, up to the start of the period. With these assumptions we obtained the optimal order quantity that maximizes the total profit for the single period inventory model.

REFERENCES

[1] Anvari M. (1987), Optimality criteria and risk in inventory models: The case of the newsboy problem; *Journal of the Operational Research Society* 38; 625-632.

[2] Bijari M., Haji R. (2004), The single-period (news-vendor) problem with stochastic initial inventory; *International Journal of Engineering Science* 15(2); 47-54.

[3] Ehrhardt R., Taube L. (1987), An inventory model with random replenishment quantity; *Int. J. Prod. Res*. 25; 1795-1803.

[4] Hadley G., Whitin T. M. (1963), Analysis of inventory systems; Prentice-Hall, Englewood Cliffs, NJ.

[5] Haji R., Bijari M. (2003), The newsboy problem with stochastic initial inventory; Proceedings of the 8th Annual International Conference on Industrial Engineering, Nov. 10-12, 822-827.

[6] Henig M., Gerchak Y. (1990), The structure of periodic review policies in the presence of random yield; *Opr. Res.* 38; 634-643.

[7] Ismail B., Louderback J. (1979), Optimizing and satisfying in stochastic cost-volume-profit analysis; *Decision Sciences* 10; 205-217.

[8] Jain K., Silver E.A. (1995), The single period procurement problem where dedicated supplier capacity can be reserved, *Naval Res. Logist* 42; 915-934.

[9] Johnson L.A., Montgomery D.C. (1974), Operations research in production planning, scheduling and inventory control; John Wiley, New York.

[10] Jucker J.V., Rosenblatt M.J. (1985), Single-period inventory models with demand uncertainty and quantity discounts; Behavioral implications and a new solution Procedure, *Naval Research Logistics Quarterly* 32; 537-550.

[11] Kabak I., Weinberg C. (1972), The generalized newsboy problem, contract negotiations and secondary vendors; *IIE Transactions* 4; 154-157.

[12] Khouja M. (1995), The newsboy problem under progressive multiple discounts, *Eur. J. Op. Res.* 84; 458-466.

[13] Khouja M. (1996), The newsboy problem with progressive retailer discounts and supplier quantity discounts, *Decis. Sci.* 27; 589-599**.**

[14] Lau H. (1980), The newsboy problem under alternative optimization objective, *Journal of the Operational Research Society* 31; 525-535.

[15] Lau, A., Lau, H. (1988), The newsboy problem with price-dependent demand Distribution; *IIE Transaction* 20, 168-175.

[16] Lin C., Kroll D.E. (1997), The single-item newsboy problem with dual performance measures and quantity discounts; *European Journal. of Operations Research* August; 562-565.

[17] Love S. (1979), Inventory control; McGraw-Hill, New York.

[18] Magee R.P. (1975), Cost-volume-profit analysis, uncertainty and capital market equilibrium, *Journal of Accounting Research* 13; 257-266.

[19] Moon I. Choi S. (1995), The distribution free newsboy problem with balking, *J. Opl. Res. Soc*. 46; 537-542.

[20] Nahmias S. (2005), Production and Operations Analysis, 5th ed.; McGraw-Hill, New York.

[21] Noori A.H., Keller G. (1986), One-period order quantity strategy with uncertain mach between the amount received and quantity requisitioned, *INFOR*, 24; 1-11.

[22] Pantumsinchai P., Knowles T.W. (1991), Standard container size discounts and the single-period inventory problem, *Decision Sciences* 22; 612-619.

[23] Polatoglu L.H. (1991), Optimal order quantity and pricing decisions in single-period inventory systems, *Int. J. Prod. Econ*. 23; 175-185.

[24] Reyniers D. (1991), A high-low search for a newsboy problem with delayed information feedback, *Oper. Res.* 38; 838-846.

[25] Ross S.M. (1983), Stochastic processes, John Wiley, New York.

[26] Sankarasubramanian E., Kumaraswamy S. (1983), Optimal order quantity for pre-determined level of profit, *Management Science* 29; 512-514.

[27] Shih W. (1979), A general decision model for cost-volume-profit analysis under uncertainty, *The Accounting Review* 54; 687-706.

[28] Shih W. (1980), Optimal inventory policies when stockouts result from defective products, *Int. J. Prod. Res*. 18; 677-686.