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Jointly control of inventory and its pricing for a deteriorating item under multiple advance payments and delay in payments with partial backordering

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Abstract

Contrary to the past that inventory decisions and pricing are taken into consideration separately, due to the influence of these decisions on each other and thus profit, researchers have investigated these two issues simultaneously. Sometimes, wholesalers offer incentive financial policies to their customers in order to increase their sale. In this paper, a different combined model of inventory control and the way of its pricing for a deteriorating item with different incentive schemes including totally advance payment and partially advance, partially delayed payment are developed. We adopt a demand function jointly time and price-dependent and a backordering rate waiting time-dependent. Also shortage of allowable inventory considered. In each case, optimum price, replenishment cycle, the time with no shortage are obtained. Sensitivity analysis is performed and represented in several figures and tables. The results show that with increasing deterioration and backordering rates, the total annual profit is reduced.
Keywords: Inventory control, pricing, advance payment, deterioration products, delayed payment, variable demand function, partial backordering

1- Introduction

In the classic economic order quantity (EOQ) model, a simple inventory control based on some unrealistic assumptions is investigated. However, the assumptions that the quality of all products does not decrease throughout time, all customers leave the system in case of shortage and all the purchasing cost is settled up at the time of delivery are rarely met in reality. In addition, manufacturing systems with dererioration products appears in different industrial branches like food production, chemical and radioactive material manufacturing, and pharmaceutical industry that their disposal causes to some environmental issues (polotski et al, 2021).

According to what has been proclaimed in literature, deterioration is defined as impaired or poor in quality, operational or condition such as decay, loss, spoilage, vaporization, reduction, pilferage, loss of utility or loss of marginal value of a merchandise that results in reducing usefulness. Most of the physical items deteriorate rapidly over time, for example medicine, vaporization of liquids, blood banks, and so on (Wee, 1993). Each product has a certain life, in generally, products are damaged or destroyed or spoiled after certain periods.

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After this period of time, this type of product cannot be excellent in human usable conditions (khan et al, 2020). On the other hand, It is obvious that perishable goods lose their quality over time and demand for those products decreases, Therefore managing the inventory of declining goods in an uncertain environment and how to pricing those goods is one of the main concerns of company managers (Wu and Chan, 2014). Also, the price and the way of its payment is an important issue that consumers must pay to buying goods. As a result, it can be said that the demand for perishable products depends on their quality and price and type of its payment (Li and Teng, 2018).

Deteriorating of products is one of the crucial issues that the retailers are confronting since several years ago. The first effort to analyse the inventory problem for perishable goods was considered by Covert and Philip (1973) and Philip (1974). They presented the model with regard to considering a variable function for deterioration rate with a two-parameter and three-parameter Weibull distribution, respectively. Elsayed and Teresi (1983), and Rafaat et al (1991) have presented finite replenishment rate. Also, three excellent and comprehensive literature reviews intended detailed on deteriorating inventory items provided by Goyal and Giri (2001), Bakker et al. (2012) and Janssen et al. (2016). Diabat et al. (2017), designed a three inventory model based on economic order quantity (EOQ) in which perishability of a commodity in form of decay has been considered and demand is time depended. Also, shortage isn't allowed and back ordering and delayed payment is partial and full. Also, Khan et al. (2020), described the advertisement and advance payment as important factors in the popularity of goods, sales and their demand. They presented two inventory model for perishable products, that in one cases, they didn't consider shortage and the other, partial backlogged shortage has considered. Finally, for evaluating of suggested model, they solved three numerical examples and then analysed the optimality of the model with the proposed algorithm.

On the other hand, one of the important key roles in the success of the different organization is timing the payment. Besides, timing the payment has an important effect on the decision variables of inventory control systems. About this issue, three various basic strategies for paying acquisition cost are possible: (1) Delivery time payment, (2) delayed payment (3) advance payment. The combination of these strategies are also possible. In practice, suppliers offer some incentive schemes to persuade their retailers to buy more goods in order to increase their sales. Sometimes, to prevent from cancelling the orders and manage providing raw materials, the wholesalers would like their buyers to pay before delivery time as advance payment. Advance payment, as one of the components of most transactions in current market, has been addressed by many researchers. Maiti et al (2009) proposed an EOQ under prepayment in which demand is price dependent and selling price is related to prepayment amount. Gupta et al (2009) amended previous model by treating cost parameters as interval instead of fixed values. Taleizadeh et al (2011) presented a multiple-buyer multiple-vendor multi-constraint supply chain model for multi-product with stochastic demand and changeable lead time in which a portion of purchasing cost should be paid as advance payment. Also, Taleizadeh (2014a) developed an EOQ model with complete backordering under advance payments for an evaporating product. In Taleizadeh (2014b) the previous model was extended by considering partial backordering. Contrariwise, some sellers offer their buyers a specified period after time of delivery as delayed payment which is a type of price rebate. This special bonus on payment, because of paying later without extra interest, motivate retailers to order more quantities. The first traditional EOQ model under delayed payment was derived by Goyal (1985). Then, Jaggi and Aggarwal (1995) and Jamal et al (1997) extended this model by considering deterioration and shortages, respectively. Sarkar (2012a) presented an EOQ model for a deteriorating item with the various rate of deteriorating time under consideration of delayed payment. Then, Sarkar. (2012b) addressed an inventory model with stock related to demand in the presence of defective production under consideration of trade credit influences. Taleizadeh and Nematollahi. (2014) presented an inventory model for a perishable product and considered delayed payment and backordering over a limited planning time horizon in their model. Numerous relative papers are established in this topic such as Chang et al. (2009), Teng and Chang. (2009), Musa and Sani. (2012), Teng et al. (2013), Chen et al. (2014), Wu and Chan. (2014) and Wu et al. (2014). Also, combination of delayed and advance payment strategies has recently been addressed in the literature. Influences of advance payment strategy with two-level trade credits into an inventory model for perishable goods are discussed by Thangam (2012). Zhang et al (2014) considered inventory administrations with considering the advance payment strategy containing all payments in advance and partially advance and delayed payment. Zia and Taleizadeh

(2015) used a hybrid payment policy consists of linked to order partial-advanced-partial delayed payment as financial strategy and considered shortages in their model.

Li et al (2016) introduced a dynamic pricing and periodic ordering model for deteriorating products under uncertainly that inventory level is stochastic and follows the selling price and demand. Shortage and extra inventory in their model, allowed. They after the transformation of stochastic model to a Hamilton–Jacobi–Bellman (HJB) equation, with semi-smooth Newton method, solved the problem and obtained the optimal pricing strategy in optimal inventory level. Taleizadeh (2017) in a research has considered an inventory system in which disruption of system becomes in case of products defective and rejection of it by inspection before receipt. Therefore, to solve this problem, he designed a lot sizing model and he considered re-ordering and prepayment policy for applying the model in the real world. Finally, by presenting an algorithm, he has solved the problem and also has examined the results. Li et al (2017) designed a model with an optimal repayment strategy in which the main goal is to optimize the payment period and increase profits. They examined the effect of three factors: prepayment, cash payment and credit payment on the system's profit and showed that increasing the selling price increases the payment period. Duan et al (2018) presented a dynamic inventory, pricing and production model for deteriorating goods in restricted horizon period of time. Demand in their model is uncertain and stochastic and under the price changes. Their model composes of inventory, production, price and cost and after solving the problem they obtained optimal strategy for production, pricing and inventory. Considering duopoly retailers in joint pricing and inventory control model presented by Mahmoodi (2019). In that research duopoly retailers sell substitutable deteriorating products. They used game theory for solving and analyzing the result. In a competitive market, companies have recognized besides inventory management which is one of the important activities of supply chain operations and plays a major role in success of organizations, making correct decisions about pricing influences customer’s satisfaction. According to the most of literature in this topic, the total profit is a concave function regarding to the selling price; in other words, increasing price of products does not always increase profits. Furthermore, when the products of organizations are perishable, the importance of considering jointly these factors is increasing.

Consequently, in recent years, many researchers have addressed the problem of inventory management and pricing simultaneously especially for deteriorating goods such as electronics products, green, and fruits vegetables, and many others. Firstly, a pricing and lot-sizing model for a deteriorating item with variable rate for deterioration and allowing partial backlogging considered by Abad (1996, 2001). Chang et al (2006) developed a jointly pricing and ordering policy for a deteriorating item with partial backordering where demand is log-concave. Dye et al (2007) determined the optimal selling price and lot sizing policy for a deteriorating product under known and differentiable demand function of price and time with partial backordering. Taleizadeh et al (2015), in their article investigated a two-level supply chain which aimed to maximize the benefit the whole of the SC. In this way, they developed an inventory model in which product demand is based on price. They used Stackelberg approach for optimizing of price, replenishment rate of raw material, production rate and replenishment of product. A jointly inventory control and optimizing pricing for perishable product presented by Agi and Soni (2018). In their study, demand depends on price, inventory level and freshness condition. Then, they derived some good conditions for solving the problem and a suitable algorithm is presented for finding the optimal price, cycle length and quantity. Also, Chang et al. (2019), proposed a model for perishable goods with mixture of three payment in form of prepayment-cash-credit and based on economic production quantity. Their model is three echelon that includes supplier, manufacturer and customer and demand rate is depend on expire date and selling price. Tiwari et al. (2018), introduced a combined inventory and pricing model for deteriorating goods in a two-level supply chain. They assumed that shortage is allowed and the rate of deterioration did not increase over time, but it will be rotten after the expiration date. The main purpose of this research is to determine the optimal selling price and the optimal re-cycle time and the time required to reach zero inventory at the same time. Khan et al. (2019), used a pricing discount strategy for presenting their math model. They considered certain time for products and in that period, they tried to increase demand and sales by giving discounts. Also They two inventory model i) without shortage ii) with partial backlogged shortage for deteriorating products and evaluated the optimality each one with MATLAB software. an EOQ-based multi-part inventory model by determining the functional financial, marketing and operational relationships in a supply chain consisting of supplier-retailer and customer with the aim of presenting solutions to determine the optimal pricing, quantities specific in a variety of payments and prepayments was propose by Li et al (2020). They investigated the affection of payment methods based on the retailer's optimal selling prices on the supplier. According to their model, profit is made when payments and prepayments have validity and credit. Azadi et al (2019) designed a model for joint pricing and inventory replacement of perishable products. Their main purpose is reducing waste and increasing profit. They considered two stage for model and solved their problem with Benders algorithm. Polotski et al (2021) presented a new mathematical model for solving the trade-off between optimal inventory levels with considering machine failure. They showed that under limitations such as production capacity, repair processes, constant demand rate, and the optimal coverage level does not exceed the cumulative demand and lead to less disposal. Md mashud et al (2021) proposed a green inventory model for deterioration products with advance payment. They considered two section in their study. (i) They applied preservation technology for reducing goods deteriorations; and (ii) they proposed suitable green technology investment and preservation technology for decreasing both carbon emission and goods deterioration. In their model a 12% increase in total profit is shown. Xie et al (2021) in their research, considered inventory control for deteriorating products with high lifetime with aim of increase profits. They presented a model for mixed sales with two policies (i) frequently monitoring (ii) inspection during the cycle.

Hybrid payment strategy which consists of multiple advance payments and delayed payment is one of the most up-to-date topics in the literature and most authors because of its complexity, do not apply it in their articles, but we use this strategy in this paper. To conclude, in dealing with the aforementioned studies, we investigate an integrated control of inventory and its pricing under multiple advance payments and delay in payments for deteriorating goods (e.g., fresh product) with partial backordering. In this model, the deterioration rate is constant, the demand function is dependent on both selling price and time and the backordering rate is waiting time dependent. The main objective function is to optimize the replenishment cycle, the time when inventory face no shortage and selling price simultaneously such that the total profit is maximized.

The rest outline of the paper is organized as follows. Section 2 represents the assumptions, notations and then the development of the mathematical model of each case. In section 3, we first establish some theorems for decision variables to be unique and exist. Next, we present an effective algorithm for optimizing selling price and replenishment schedules such that total profit is maximized in each case. In Section 4, we present a numerical example and some management insights. Finally, in section 5, a conclusion is presented and some future research directions are given.

2- Problem modelling

2-1- Problem description

In this paper, a jointly control of inventory and its pricing for a deteriorating item is investigated in order to optimize selling price, replenishment cycle and the length of the time with no shortage. The rate of deterioration is constant and the rate of demand is known and dependent on the price and time. Shortage is permitted and the variable backordering rate is dependent on waiting time up to next cycle. The payment policy applied in this mathematical model consists of multiple advance payments and delayed payment in which divided into three cases based on the length of delayed payment. Case 1 is the situation when only multiple advance payment is used, while the situations that delayed payment as well as multiple advance payment is incorporated into model are investigated such that if the time of delayed payment is during the time with presence of inventory defines Case 2 and if the time of delayed payment is during the time with shortage of inventory defines case 3. The problem is to find the best payment policy between case 1, 2 and 3 based on maximization of the total annual profit.
Furthermore, the following assumptions are made:

The horizon of planning is infinite.

1) The demand function, $D(p,t)=(a-bp)e^{\lambda t}$ is a linearly decreasing function of selling price and vary throughout time. As demand increases when price decreases in most of the markets, we adopt this function in order to reflect the relation between price and demand. Besides, the demand may either decrease or increase exponentially with time; thus, the form of multiplicative exponential time effect for demand rate is adopted which can reflect both increasing (where $\lambda > 0$) and decreasing (where $\lambda < 0$) effect.

2) Shortages are allowed where unsatisfied demand is backordered. The backordering rate is variable and dependent on waiting time up to next replenishment. $\beta(x)=k_0e^{-\delta x}$ Defines the backordered function, where x is the waiting time up to next cycle and δ is the backordering parameter. ($0\le\beta(x)\le 1, \beta(0)=1$)
3) The time which products deteriorate follows an exponential distribution with deterministic and constant parameter θ.
4) In this model, for payment policy, multiple advance payment which is divided into several equal sized portions and also delayed payment are incorporated and considered to be in three cases base on the value of delayed time.

2-2- Notations

To develop the mathematical model of joint inventory control and its pricing, the necessary notations are adopted as below:

Parameters	
C_o	the fixed ordering cost per order
C_p	the acquisition cost per unit
C_h	the holding cost per unit time
C_b	the backlogging cost per unit time
C_L	the lost sale per unit time
θ	the constant deterioration rate
α	the percentage of purchasing cost must be prepaid
N	the number of equal payments in advance
M	the length of delayed payment
L	the length of prepayments
i_e	the interest earned
i_p	the capital cost
Variables	
T	the replenishment cycle time
t_1	the length of time that demand do not face shortage
p	the selling price per unit
Q	the order quantity
TP	total annual profit
S	the backordered quantity

2-3-Model formulation

The inventory control is described as follows. At each cycle, Q units are arrived at the beginning and will decrease owing to demand rate and deterioration. Inventory drop to zero at time t_1 and then shortage will occur and unsatisfied demand will be backordered based on backordering rate up to time T. Hence, the change in inventory level with respect to time is given by the following differential equations:

$$\frac{dI_1(t)}{dt}+\theta I_1=-D(p,t)=-(a-bp)e^{\lambda t}, 0\le t\le t_1 \tag{1}$$

So the amount of inventory carried during first interval in replenishment cycle is calculated as follows:

$$I_1(t)=e^{-\theta t}\int_t^{t_1} e^{\theta t}\,(a-bp)e^{\lambda t}\,dt \to I_1=e^{-\theta t}\times(a-bp)\frac{e^{(\lambda+\theta)t}}{(\lambda+\theta)}\Big|_t^{t_1} \to$$
$$I_1(t)=\frac{(a-bp)e^{-\theta t}}{(\lambda+\theta)}\left[e^{(\lambda+\theta)t_1}-e^{(\lambda+\theta)t}\right] \tag{2}$$

And the maximum amount of inventory level is obtained and given as follows:

$$t=0\to I_0=\frac{(a-bp)}{(\lambda+\theta)}\left[e^{(\lambda+\theta)t_1}-1\right] \tag{3}$$

Furthermore, shortage occurred during interval $\left[t_1,T\right]$. So, the inventory level in second interval is governed by below differential equation:

$$\frac{dI_2(t)}{dt}=-D(p,t)\,\beta(T-t)=\frac{-(a-bp)e^{\lambda t}}{e^{\delta(T-t)}},\ \mathrm{t}_1\le \mathrm{t}\le T \tag{4}$$

By solving equation (4), with the boundary condition $I_2\left(t_1\right)=0$, we obtain

$$I_2(t)-\underbrace{I_2\left(t_1\right)}_{=0}=-\frac{(a-bp)e^{-\delta T}}{(\lambda+\delta)}\left[e^{(\lambda+\delta)t}-e^{(\lambda+\delta)t_1}\right] \tag{5}$$

The maximum amount of backorders will be obtained by setting $t=T$ into equation (5), so

$$S=-I_2(t=T)=\frac{(a-bp)e^{-dT}}{(l+d)}\ e^{(l+dT)}-\ e^{(l+dt_1)} \tag{6}$$

Consequently, the ordering quantity over the replenishment cycle can be computed by sum of initial on hand inventory and maximum amount of backorders, so

$$Q=I_0+S=(a-bp)\left[\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\frac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right] \tag{7}$$

We divide the components of total profit in two: identical and non-identical terms. Now, we derive identical relative costs and profit in each three possible cases. The identical components in three possible cases include: Ordering cost, purchasing cost, holding cost and sales revenue, which are computed as follows:

- Fixed ordering cost: $OC=C_o$
- Holding cost: $HC=C_h\int_0^{t_1} I_1(t)dt=C_h\int_0^{t_1}\frac{(a-bp)e^{-\theta t}}{(\lambda+\theta)}\left[e^{(\lambda+\theta)t_1}-e^{(\lambda+\theta)t}\right]$

$$=C_h\,\frac{(a-bp)\left[\lambda e^{(\lambda+\theta)t_1}-(\lambda+\theta)e^{\lambda t_1}+\theta\right]}{\lambda\theta(\lambda+\theta)}$$

- Purchasing cost: $PC=C_pQ=C_p(a-bp)\left[\frac{\left[e^{\lambda T}-e^{-\delta T+(\lambda+\delta)t_1}\right]}{\lambda+\delta}+\frac{\left[e^{(\lambda+\theta)t_1}-1\right]}{\lambda+\theta}\right]$

- Backordering cost: $BC = C_b \int_{t_1}^{T} -I_2(t)dt = C_b \int_{t_1}^{T} \frac{(a-bp)e^{-\delta T}}{(\lambda+\delta)}\left[e^{(\lambda+\delta)t} - e^{(\lambda+\delta)t_1}\right]$

$$= C_b \frac{(a-bp)}{(\lambda+\delta)^2}\left[-\left(1+(\lambda+\delta)(T-t_1)\right)e^{-\delta T+(\lambda+\delta)t_1} + e^{\lambda T}\right]$$

- Lost sale cost: $OC = C_L \int_{t_1}^{T} D(p,t)\left(1-\beta(T-t)\right)dt = C_L \int_{t_1}^{T} (a-bp)e^{\lambda t}\left[1-e^{-\delta(T-t)}\right]dt$

$$= C_L \frac{(a-bp)}{\lambda(\lambda+\delta)}\left[-\lambda e^{\lambda t_1} + \delta\left(e^{\lambda T} - e^{\lambda t_1}\right) + \lambda e^{-\delta T+(\lambda+\delta)t_1}\right]$$

$$SR = p\left[\int_0^{t_1} D(p,t)dt + S\right]$$

$$= p\left[\int_0^{t_1} (a-bp)e^{\lambda t}dt + \frac{(a-bp)e^{-\delta T}}{(\lambda+\delta)}\left[e^{(\lambda+\delta)T} - e^{(\lambda+\delta)t_1}\right]\right]$$

- Sales revenue: $= p(a-bp)\left[\frac{\left(e^{\lambda t_1}-1\right)}{\lambda} + \frac{\left[e^{\lambda T} - e^{-\delta T+(\lambda+\delta)t_1}\right]}{\lambda+\delta}\right]$

Then we derive non-identical terms including capital cost and interest earned for each case.

Case 1:

In this case, delayed payment is not involved with payment strategy and all the acquisition cost is paid as multiple prepayment in several equally portions before receiving an order. According to figure 1, no annual interest is earned, while capital cost is obtained as follows:

$$\begin{aligned} IP_1 &= i_p \frac{C_p Q}{N}\left[N\frac{L}{N} + (N-1)\frac{L}{N} + \ldots + \left(N-(N-1)\right)\frac{L}{N}\right] \\ &= i_p \frac{C_p Q}{N}\frac{L}{N}\left[N+(N-1)+\ldots+2+1\right] \\ &= i_p \frac{C_p Q}{N}\frac{L}{N}\frac{N(N+1)}{2} = i_p C_p Q \frac{L(N+1)}{2N} \end{aligned} \tag{8}$$

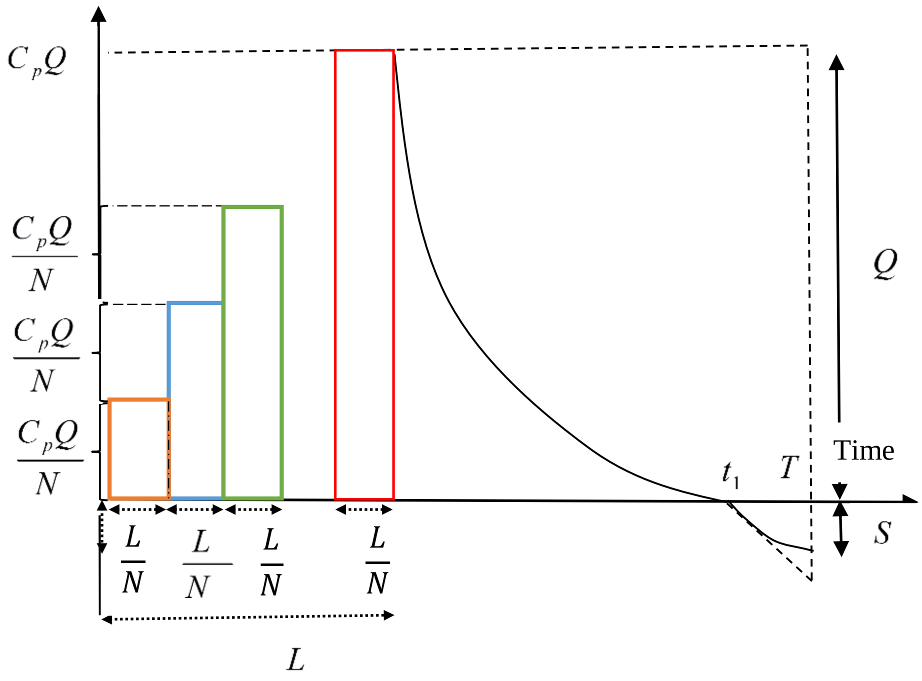

Fig 1. The annual capital cost for case 1

Hence, total annual profit of the inventory system in Case 1 is equal to:

$$TP_1\left(p,t_1,T\right)=\frac{CTP_1}{T}=\frac{SR-\left(A+OC+BC+HC+PC+IP_1-IE_1\right)}{T}$$

$$=\frac{1}{T}\left[\begin{array}{l} p\left(a-bp\right)\left[\dfrac{\left(e^{\lambda t_1}-1\right)}{\lambda}+\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}\right]- \\ C_o-C_h\dfrac{(a-bp)\left[\lambda e^{(\lambda+\theta)t_1}-(\lambda+\theta)e^{\lambda t_1}+\theta\right]}{\lambda\theta(\lambda+\theta)}- \\ C_b\dfrac{(a-bp)e^{-\delta T}}{(\lambda+\delta)^2}\left[-(\lambda+\delta)\left(T-t_1\right)e^{(\lambda+\delta)t_1}+e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]- \\ C_L\dfrac{(a-bp)}{\lambda(\lambda+\delta)}\left[-\lambda e^{\lambda t_1}+\delta\left(e^{\lambda T}-e^{\lambda t_1}\right)+\lambda e^{-\delta T+(\lambda+\delta)t_1}\right]- \\ C_p\left(a-bp\right)\left[\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\dfrac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]- \\ i_pC_p\left(a-bp\right)\left[\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\dfrac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]\dfrac{L\left(N+1\right)}{2N} \end{array}\right] \tag{9}$$

Case 2:

The payment policy used in this case is paying α percent of the purchasing cost as multiple prepayment in several equally portions and $1-\alpha$. Percent of the purchasing cost as delayed payment. The remaining amount of purchasing cost should be paid during interval $\left[0,t_1\right]$, i.e. when inventory exists in the system and shortage will not occur. As shown in figure 2, in this case, the capital cost and interest earned is equal to:

$$\begin{aligned} IP_2 &= \alpha i_p C_p Q \frac{L(N+1)}{2N} + i_p C_p \int_M^{t_1} I_1(t)\,dt \\ &= \alpha i_p C_p Q \frac{L(N+1)}{2N} + i_p C_p \frac{(a-bp)\left[-(\lambda+\theta)e^{\lambda t_1} + \lambda e^{-M\theta+(\lambda+\theta)t_1} + \theta e^{\lambda M}\right]}{\lambda\theta(\lambda+\theta)} \end{aligned} \tag{10}$$

$$\begin{aligned} IE_2 &= (1-\alpha) i_e p \int_0^M D(p,t)\,t\,dt = (1-\alpha) i_p p \int_0^M (a-bp)e^{\lambda t}\,t\,dt \\ &= (1-\alpha) i_e p (a-bp) \frac{\left[1+e^{\lambda M}(\lambda M-1)\right]}{\lambda^2} \end{aligned} \tag{11}$$

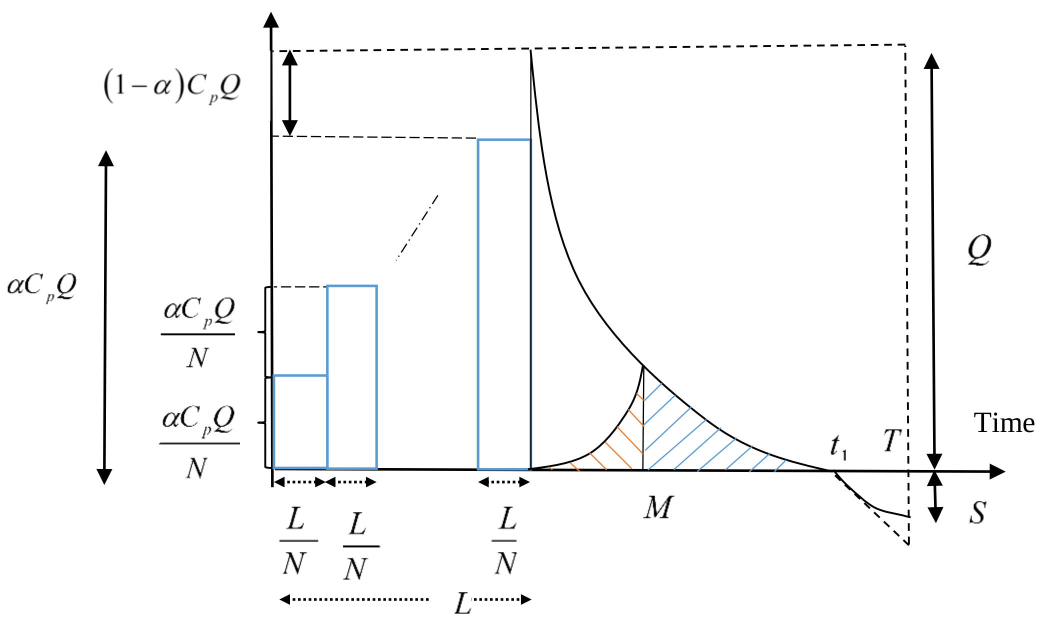

Fig 2. The annual capital cost and interest earned in case 2

Hence, total annual profit of the inventory system in case 2 is equal to:

$$
TP_2\left(p,t_1,T\right)=\frac{CTC_2}{T}=\frac{SR-\left(A+OC+BC+HC+PC+IP_2-IE_2\right)}{T}
$$

$$
=\frac{1}{T}\left[\begin{array}{l}
p\left(a-bp\right)\left[\frac{\left(e^{\lambda t_1}-1\right)}{\lambda}+\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{\left(\lambda+\delta\right)}\right]+ \\
\left(1-\alpha\right)p\left(a-bp\right)i_e\frac{\left[1+e^{\lambda M}\left(\lambda M-1\right)\right]}{\lambda^2}- \\
C_o-C_h\frac{\left(a-bp\right)\left[\lambda e^{(\lambda+\theta)t_1}-\left(\lambda+\theta\right)e^{\lambda t_1}+\theta\right]}{\lambda\theta\left(\lambda+\theta\right)}- \\
C_b\frac{\left(a-bp\right)e^{-\delta T}}{\left(\lambda+\delta\right)^2}\left[-\left(\lambda+\delta\right)\left(T-t_1\right)e^{(\lambda+\delta)t_1}+e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]- \\
C_L\frac{\left(a-bp\right)}{\lambda\left(\lambda+\delta\right)}\left[-\lambda e^{\lambda t_1}+\delta\left(e^{\lambda T}-e^{\lambda t_1}\right)+\lambda e^{-\delta T+(\lambda+\delta)t_1}\right]- \\
C_p\left(a-bp\right)\left[\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{\left(\lambda+\delta\right)}+\frac{\left[e^{(\lambda+\theta)t_1}-1\right]}{\left(\lambda+\theta\right)}\right]- \\
\alpha i_p C_p\left(a-bp\right)\left[\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{\left(\lambda+\delta\right)}+\frac{\left[e^{(\lambda+\theta)t_1}-1\right]}{\left(\lambda+\theta\right)}\right]\frac{L\left(N+1\right)}{2N}- \\
i_p C_p\frac{\left(a-bp\right)\left[-\left(\lambda+\theta\right)e^{\lambda t_1}+\lambda e^{-M\theta+(\lambda+\theta)t_1}+\theta e^{\lambda M}\right]}{\lambda\theta\left(\lambda+\theta\right)}
\end{array}\right] \tag{12}
$$

Case 3:

As same as Case 2, the payment policy used is paying α percent of the purchasing cost as multiple prepayment in several equally portions and $1-\alpha$ percent of the purchasing cost as delayed payment. However, in this case, the remaining amount of purchasing cost should be paid during interval $\left[t_1,T\right]$, i.e. when inventory does not exist in the system and demand will face shortage. As shown in figure 3, in this case, the capital cost and interest earned is equal to:

$$
IP_3 = a i_p C_p Q\frac{L\left(N+1\right)}{2N} \tag{13}
$$

$$
\begin{aligned}
IE_3 &= \left(1-\alpha\right)i_e p\left[\int_0^{t_1}D\left(p,t\right)tdt+\left(M-t_1\right)\int_0^{t_1}D\left(p,t\right)dt\right] \\
&= \left(1-\alpha\right)i_e p\left(a-bp\right)\left[\frac{\left[1+e^{\lambda t_1}\left(\lambda t_1-1\right)\right]}{\lambda^2}+\left(M-t_1\right)\frac{\left(e^{\lambda t_1}-1\right)}{\lambda}\right] \qquad (14) \\
&= \left(1-\alpha\right)i_e p\left(a-bp\right)\left[\frac{1+e^{\lambda t_1}\left(\lambda t_1-1\right)+\left(M-t_1\right)\lambda\left(e^{\lambda t_1}-1\right)}{\lambda^2}\right]
\end{aligned}
$$

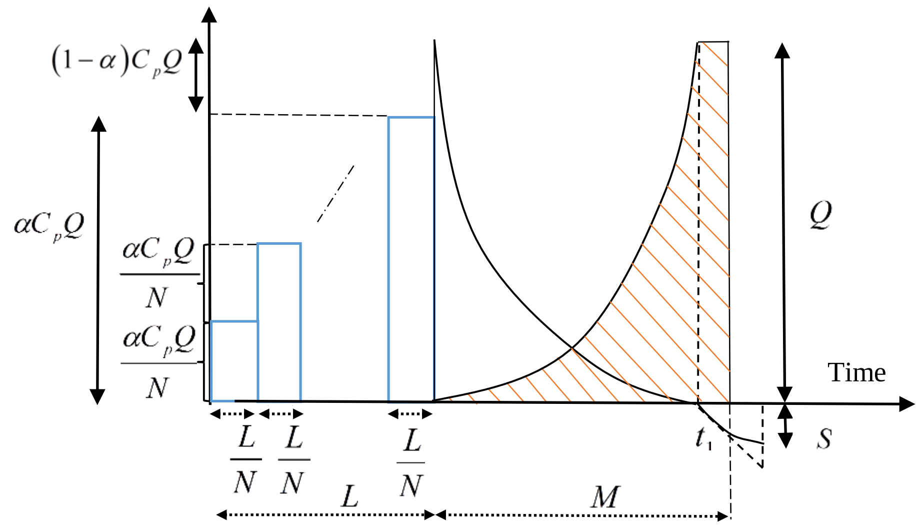

Fig. 3. The capital cost and interest earned in case 3

Hence, total annual profit of the inventory system in case 3 is equal to:

$$TP_3(p,t_1,T)=\frac{CTP_3}{T}=\frac{SR-(A+OC+BC+HC+PC+IP_3-IE_3)}{T}$$

$$=\frac{1}{T}\left[\begin{array}{l} p(a-bp)\left[\frac{\left(e^{\lambda t_1}-1\right)}{\lambda}+\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}\right]+ \\ (1-\alpha)i_e p(a-bp)\left[\frac{\left[1+e^{\lambda t_1}(\lambda t_1-1)\right]+(M-t_1)\lambda\left(e^{\lambda t_1}-1\right)}{\lambda^2}\right]- \\ C_o-C_h\frac{(a-bp)\left[\lambda e^{(\lambda+\theta)t_1}-(\lambda+\theta)e^{\lambda t_1}+\theta\right]}{\lambda\theta(\lambda+\theta)}- \\ C_b\frac{(a-bp)e^{-\delta T}}{(\lambda+\delta)^2}\left[-(\lambda+\delta)(T-t_1)e^{(\lambda+\delta)t_1}+e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]- \\ C_L\frac{(a-bp)}{\lambda(\lambda+\delta)}\left[-\lambda e^{\lambda t_1}+\delta\left(e^{\lambda T}-e^{\lambda t_1}\right)+\lambda e^{-\delta T+(\lambda+\delta)t_1}\right]- \\ C_p(a-bp)\left[\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\frac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]- \\ \alpha i_p C_p(a-bp)\left[\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\frac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]\frac{L(N+1)}{2N} \end{array}\right] \tag{15}$$

3- Solution methodology

Now, in order to optimize the selling price and ordering policies such that proposed annual total profit function is maximized in each case, the following steps are used. In all cases, first, we establish the necessary conditions for maximization of total profit function to prove that for any given p, the optimal solutions of T and t_1 not only exist but also are unique. Second, after carrying out some analysis we demonstrate that for any given T and t_1, there exists a unique p which maximizes the annual total profit.

Case 1:

To maximize total annual profit for any given p, it is required to solve the following equations simultaneously. So

$$\frac{\delta TP_1(T,p,t_1)}{\delta t_1}=\frac{(a-bp)}{T}\begin{bmatrix} e^{\lambda t_1}\left(p+\frac{C_h}{\theta}+C_L\right)-e^{(\lambda+\theta)t_1}\left(\frac{C_h}{\theta}+C_p+i_pC_p\frac{L(N+1)}{2N}\right) \\ -e^{-\delta T+(\lambda+\delta)t_1}\left(p+C_L-C_b(T-t_1)-C_p-i_pC_p\frac{L(N+1)}{2N}\right)\end{bmatrix}=0 \tag{16}$$

$$\frac{\partial TP_1(p,t_1,T)}{\partial T}=\frac{TCTP_1'-CTP_1}{T} \tag{17}$$

In which

$$CTP_1'=\begin{bmatrix} p(a-bp)\left(\frac{\lambda e^{\lambda T}+\delta e^{-\delta T+(\lambda+\delta)t_1}}{(\lambda+\delta)}\right)- \\ C_p(a-bp)\left[\frac{\lambda e^{\lambda T}+\delta e^{-\delta T+(\lambda+\delta)t_1}}{(\lambda+\delta)}\right]- \\ C_L\frac{(a-bp)}{(\lambda+\delta)}\left[\delta\left(e^{\lambda T}-e^{-\delta T+(\lambda+\delta)t_1}\right)\right]- \\ \frac{C_b(a-bp)}{(\lambda+\delta)^2}\left[\lambda\left(e^{\lambda T}-e^{-\delta T+(\lambda+\delta)t_1}\right)+\delta(\lambda+\delta)(T-t_1)e^{-\delta T+(\lambda+\delta)t_1}\right]- \\ i_pC_p(a-bp)\frac{L(N+1)}{2N}\left[\frac{\lambda e^{\lambda T}+\delta e^{-\delta T+(\lambda+\delta)t_1}}{(\lambda+\delta)}\right]\end{bmatrix} \tag{18}$$

Theorem 1. For any given p, we have

(a) The system of equation 16 and 17 has a unique solution.

(b) The solution in (a) satisfies the second-order conditions for the optimum.

Proof. Please see Appendix A for details.

Next, we study the condition which p is unique and also exists. For any given T and t_1 the first-order necessary condition for total profit maximization is given as follows

$$\frac{\delta TP_1\left(T,t_1,p\right)}{\delta p}=\frac{1}{T}\left[\begin{array}{l}\left(a-2bp\right)\left[\frac{\left[e^{\lambda t_1}-1\right]}{\lambda}+\frac{\left[e^{\lambda T}-e^{-\delta T+\left(\lambda+\delta\right)t_1}\right]}{\left(\lambda+\delta\right)}\right]+\\ bC_h\frac{\left[\lambda e^{\left(\lambda+\theta\right)t_1}-\left(\lambda+\theta\right)e^{\lambda t_1}+\theta\right]}{\lambda\theta\left(\lambda+\theta\right)}+\\ bC_b\frac{\left[-\left(\lambda+\delta\right)\left(T-t_1\right)e^{-\delta T+\left(\lambda+\delta\right)t_1}+e^{\lambda T}-e^{-\delta T+\left(\lambda+\delta\right)t_1}\right]}{\left(\lambda+\delta\right)^2}+\\ bC_L\frac{\left[-\left(\lambda+\delta\right)e^{\lambda t_1}+\delta e^{\lambda T}+\lambda e^{-\delta T+\left(\lambda+\delta\right)t_1}\right]}{\lambda\left(\lambda+\delta\right)}+\\ bC_p\left[\frac{\left[e^{\lambda T}-e^{-\delta T+\left(\lambda+\delta\right)t_1}\right]}{\left(\lambda+\delta\right)}+\frac{\left[e^{\left(\lambda+\theta\right)t_1}-1\right]}{\left(\lambda+\theta\right)}\right]+\\ bi_pC_p\left[\frac{\left[e^{\lambda T}-e^{-\delta T+\left(\lambda+\delta\right)t_1}\right]}{\left(\lambda+\delta\right)}+\frac{\left[e^{\left(\lambda+\theta\right)t_1}-1\right]}{\left(\lambda+\theta\right)}\right]\frac{L\left(N+1\right)}{2N}\end{array}\right] \tag{19}$$

Furthermore, the second-order derivative of $TP_1\left(T,t_1,p\right)$ with respect to is

$$\frac{\delta^2 TP_1\left(T,t_1,p\right)}{\delta p^2}=\frac{-2b\left[\frac{\left(e^{\lambda t_1}-1\right)}{\lambda}+\frac{e^{-\delta T}\left[e^{\left(\lambda+\delta\right)T}-e^{\left(\lambda+\delta\right)t_1}\right]}{\left(\lambda+\delta\right)}\right]}{T}\leq 0 \tag{20}$$

With the assumptions of $\lambda<0$ and $\left|\lambda\right|>\theta,\delta$, (as in real market, the exponential parameter of demand function is much larger than deterioration and backordering rate), the $\frac{\delta^2 TP_1\left(T,t_1,p\right)}{\delta p^2}$ is negative and $TP_1\left(T,t_1,p\right)$ is a concave function of p for a given T and t_1. Hence, the value of p obtained from following equation is unique.

$$p^{*}=\frac{a}{2b}+\frac{\left[\begin{array}{l}\dfrac{e^{(\lambda+\theta)t_1}}{(\lambda+\theta)}\left(\dfrac{C_h}{\theta}+C_p+i_pC_p\dfrac{L(N+1)}{2N}\right)-\dfrac{e^{\lambda t_1}}{\lambda}\left(\dfrac{C_h}{\theta}+C_L\right)\\ -\dfrac{e^{-\delta T+(\lambda+\delta)t_1}}{(\lambda+\delta)}\left(C_p+i_pC_p\dfrac{L(N+1)}{2N}+\dfrac{C_b}{(\lambda+\delta)}+C_b(T-t_1)-C_L\right)\\ +\dfrac{e^{\lambda T}}{(\lambda+\delta)}\left(\dfrac{C_b}{(\lambda+\delta)}+\dfrac{C_L\delta}{\lambda}+C_p+i_pC_p\dfrac{L(N+1)}{2N}\right)\\ +\dfrac{C_h}{\lambda(\lambda+\theta)}-\dfrac{\left(C_p+i_pC_p\dfrac{L(N+1)}{2N}\right)}{(\lambda+\theta)}\end{array}\right]}{2\left[\dfrac{\left(e^{\lambda t_1}-1\right)}{\lambda}+\dfrac{\left[e^{\lambda T}-e^{-\delta T+(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}\right]} \tag{21}$$

Case 2:
To maximize total annual profit for any given p, it is necessary to solve the following equations simultaneously. So

$$\frac{\partial TP_2(p,t_1,T)}{\partial t_1}=\frac{(a-bp)}{T}\left[\begin{array}{l}e^{-\delta T+(\lambda+\delta)t_1}\left(\begin{array}{l}-p-C_L+C_b(T-t_1)\\ +C_p+\alpha i_pC_p\dfrac{L(N+1)}{2N}\end{array}\right)\\ -e^{(\lambda+\theta)t_1}\left(\dfrac{C_h}{\theta}+C_p+\alpha i_pC_p\dfrac{L(N+1)}{2N}+\dfrac{i_pC_p}{\theta}\right)\\ +e^{\lambda t_1}\left(p+\dfrac{C_h}{\theta}+C_L+\dfrac{i_pC_pe^{M\theta}}{\theta}\right)\end{array}\right]=0 \tag{22}$$

$$\frac{\partial TP_2(p,t_1,T)}{\partial T}=\frac{TCTP_2'-CTP_2}{T} \tag{23}$$

In which

$$CTP_2' = \begin{bmatrix} p\left(a-bp\right)\left(\dfrac{\lambda e^{\lambda T} + \delta e^{-\delta T + (\lambda+\delta)t_1}}{(\lambda+\delta)}\right) - \\ C_p\left(a-bp\right)\left[\dfrac{\lambda e^{\lambda T} + \delta e^{-\delta T + (\lambda+\delta)t_1}}{(\lambda+\delta)}\right] - \\ C_L\dfrac{(a-bp)}{(\lambda+\delta)}\left[\delta\left(e^{\lambda T} - e^{-\delta T + (\lambda+\delta)t_1}\right)\right] - \\ \dfrac{C_b\left(a-bp\right)}{(\lambda+\delta)^2}\begin{bmatrix} \lambda\left(e^{\lambda T} - e^{-\delta T + (\lambda+\delta)t_1}\right) + \\ \delta(\lambda+\delta)(T-t_1)e^{-\delta T + (\lambda+\delta)t_1} \end{bmatrix} - \\ i_p C_p\left(a-bp\right)\dfrac{L(N+1)}{2N}\left[\dfrac{\lambda e^{\lambda T} + \delta e^{-\delta T + (\lambda+\delta)t_1}}{(\lambda+\delta)}\right] \end{bmatrix} \tag{24}$$

Theorem 2. For any given p, we have

(a) The system of equation 22 and 23 has a unique solution.

(b) The solution in (a) satisfies the second-order conditions for the optimum.

Proof. To avoid redundancy, a relatively approach to prove which is similar to the one in Appendix A is applied. Next, we study the condition which p is unique and also exists. For any given T and t_1, the first-order necessary condition for total profit maximization is given as follows:

$$\frac{\delta TP_2(T, t_1, p)}{\delta p} = \begin{bmatrix} (a-2bp)\left[\dfrac{\left(e^{\lambda t_1}-1\right)}{\lambda} + \dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T} - e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}\right] + \\ (1-\alpha)i_e(a-2bp)\dfrac{\left[1+e^{\lambda M}(\lambda M - 1)\right]}{\lambda^2} \\ +bC_h\dfrac{\left[\lambda e^{(\lambda+\theta)t_1} - (\lambda+\theta)e^{\lambda t_1} + \theta\right]}{\lambda\theta(\lambda+\theta)} + \\ bC_b\dfrac{e^{-\delta T}}{(\lambda+\delta)^2}\left[-(\lambda+\delta)(T-t_1)e^{(\lambda+\delta)t_1} + e^{(\lambda+\delta)T} - e^{(\lambda+\delta)t_1}\right] + \\ bC_L\dfrac{\left[-\lambda e^{\lambda t_1} + \delta\left(e^{\lambda T} - e^{\lambda t_1}\right) + \lambda e^{-\delta T + (\lambda+\delta)t_1}\right]}{\lambda(\lambda+\delta)} + \\ bC_p\left[\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T} - e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)} + \dfrac{\left[e^{(\lambda+\theta)t_1} - 1\right]}{(\lambda+\theta)}\right] + \\ b\alpha i_p C_p\left[\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T} - e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)} + \dfrac{\left[e^{(\lambda+\theta)t_1} - 1\right]}{(\lambda+\theta)}\right]\dfrac{L(N+1)}{2N} + \\ bi_p C_p\dfrac{\left[-(\lambda+\theta)e^{\lambda t_1} + \lambda e^{-M\theta + (\lambda+\theta)t_1} + \theta e^{\lambda M}\right]}{\lambda\theta(\lambda+\theta)} \end{bmatrix} \tag{25}$$

Furthermore, the second-order derivative of $TP_1(T, t_1, p)$ with respect to p is

$$\frac{\delta^2 TP_2(T,t_1,p)}{\delta p^2} = \frac{-2b\left[\frac{\lambda\left(e^{\lambda t_1}-1\right)+(1-\alpha)i_e\left[1+e^{\lambda M}(\lambda M-1)\right]}{\lambda^2}+\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}\right]}{T} \le 0 \tag{26}$$

With the assumptions of $\lambda < 0$ and $|\lambda| > \theta, \delta$, (as in real market, the exponential parameter of demand function is much larger than deterioration and backordering rate), the $\frac{\delta^2 TP_1(T,t_1,p)}{\delta p^2}$ is negative and $TP_1(T,t_1,p)$ is a concave function of p for a given T and t_1. Hence, the value of p obtained from following equation is unique.

$$p^* = \frac{a}{2b} + \frac{\left[\begin{array}{l} C_h \frac{\left[\lambda e^{(\lambda+\theta)t_1}-(\lambda+\theta)e^{\lambda t_1}+\theta\right]}{\lambda\theta(\lambda+\theta)}+ \\ C_b \frac{e^{-\delta T}}{(\lambda+\delta)^2}\left[-(\lambda+\delta)(T-t_1)e^{(\lambda+\delta)t_1}+e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]+ \\ C_L \frac{\left[-\lambda e^{\lambda t_1}+\delta\left(e^{\lambda T}-e^{\lambda t_1}\right)+\lambda e^{-\delta T+(\lambda+\delta)t_1}\right]}{\lambda(\lambda+\delta)}+ \\ C_p\left[\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\frac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]+ \\ \alpha i_p C_p\left[\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\frac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]\frac{L(N+1)}{2N}+ \\ i_p C_p \frac{\left[-(\lambda+\theta)e^{\lambda t_1}+\lambda e^{-M\theta+(\lambda+\theta)t_1}+\theta e^{\lambda M}\right]}{\lambda\theta(\lambda+\theta)} \end{array}\right]}{2\left[\frac{\left(e^{\lambda t_1}-1\right)}{\lambda}+\frac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+(1-\alpha)i_e\frac{\left[1+e^{\lambda M}(\lambda M-1)\right]}{\lambda^2}\right]} \tag{27}$$

Case 3:

To maximize total annual profit for any given p, it is necessary to solve the following equations simultaneously. So

$$\frac{\partial TP_3\left(p,t_1,T\right)}{\partial t_1}=\frac{\left(a-bp\right)}{T}\begin{bmatrix} e^{-\delta T+\left(\lambda+\delta\right)t_1}\begin{pmatrix} -p-C_L+C_b\left(T-t_1\right) \\ +C_p+\alpha i_p C_p\dfrac{L\left(N+1\right)}{2N}\end{pmatrix} \\ -e^{\left(\lambda+\theta\right)t_1}\left(\dfrac{C_h}{\theta}+C_p+\alpha i_p C_p\dfrac{L\left(N+1\right)}{2N}\right) \\ +e^{\lambda t_1}\left(p+\dfrac{C_h}{\theta}+C_L+\left(1-\alpha\right)i_e p\dfrac{\left(M\lambda-1\right)}{\lambda}\right) \\ +\dfrac{\left(1-\alpha\right)i_e p\left(a-bp\right)}{\lambda}\end{bmatrix}=0 \tag{28}$$

$$\frac{\partial TP_3\left(p,t_1,T\right)}{\partial T}=\frac{TCTP_3'-CTP_3}{T} \tag{29}$$

In which

$$CTP_3'=\begin{bmatrix} p\left(a-bp\right)\left(\dfrac{\lambda e^{\lambda T}+\delta e^{-\delta T+\left(\lambda+\delta\right)t_1}}{\left(\lambda+\delta\right)}\right)- \\ C_p\left(a-bp\right)\left[\dfrac{\lambda e^{\lambda T}+\delta e^{-\delta T+\left(\lambda+\delta\right)t_1}}{\left(\lambda+\delta\right)}\right]- \\ C_L\dfrac{\left(a-bp\right)}{\left(\lambda+\delta\right)}\left[\delta\left(e^{\lambda T}-e^{-\delta T+\left(\lambda+\delta\right)t_1}\right)\right]- \\ \dfrac{C_b\left(a-bp\right)}{\left(\lambda+\delta\right)^2}\left[\lambda\left(e^{\lambda T}-e^{-\delta T+\left(\lambda+\delta\right)t_1}\right)+\delta\left(\lambda+\delta\right)\left(T-t_1\right)e^{-\delta T+\left(\lambda+\delta\right)t_1}\right]- \\ i_p C_p\left(a-bp\right)\alpha\dfrac{L\left(N+1\right)}{2N}\left[\dfrac{\lambda e^{\lambda T}+\delta e^{-\delta T+\left(\lambda+\delta\right)t_1}}{\left(\lambda+\delta\right)}\right]\end{bmatrix} \tag{30}$$

Theorem 3. For any given p , we have

(a) The system of equation 28 and 29 has a unique solution.

(b) The solution in (a) satisfies the second-order conditions for the optimum.

Proof. Again, to avoid redundancy, a relatively approach to prove which is similar to the one in Appendix A is applied. Next, we study the condition which p is unique and also exists. For any given T and t_1, the first-order necessary condition for total profit maximization is given as follows:

$$\frac{\delta TP_3(T,t_1,p)}{\delta p}=\frac{1}{T}\left[\begin{array}{l}(a-2bp)(1-\alpha)i_e\dfrac{\left[1+e^{\lambda t_1}(\lambda t_1-1)+(M-t_1)\lambda\left(e^{\lambda t_1}-1\right)\right]}{\lambda^2}+\\(a-2bp)\left[\dfrac{\left(e^{\lambda t_1}-1\right)}{\lambda}+\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}\right]+\\bC_h\dfrac{\left[\lambda e^{(\lambda+\theta)t_1}-(\lambda+\theta)e^{\lambda t_1}+\theta\right]}{\lambda\theta(\lambda+\theta)}+\\bC_b\dfrac{e^{-\delta T}}{(\lambda+\delta)^2}\left[-\left(1+(\lambda+\delta)(T-t_1)\right)e^{(\lambda+\delta)t_1}+e^{(\lambda+\delta)T}\right]+\\bC_L\dfrac{\left[-\lambda e^{\lambda t_1}+\delta\left(e^{\lambda T}-e^{\lambda t_1}\right)+\lambda e^{-\delta T+(\lambda+\delta)t_1}\right]}{\lambda(\lambda+\delta)}+\\bC_p\left[\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\dfrac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]+\\b\alpha i_pC_p\left[\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\dfrac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]\dfrac{L(N+1)}{2N}\end{array}\right] \tag{31}$$

Furthermore, the second-order derivative of $TP_1(T,t_1,p)$ with respect to p is

$$\frac{\delta^2 TP_3(T,t_1,p)}{\delta p^2}=\frac{-2b\left[\begin{array}{l}\dfrac{\left[e^{\lambda T}-e^{-\delta T+(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\dfrac{\left(e^{\lambda t_1}-1\right)}{\lambda}\\ \dfrac{+(1-\alpha)i_e\left[1+e^{\lambda M}(\lambda M-1)\right]}{\lambda^2}\end{array}\right]}{T}\le 0 \tag{32}$$

With the assumptions of $\lambda<0$ and $|\lambda|>\theta,\delta$, (as in real market, the exponential parameter of demand function is much larger than deterioration and backordering rate), the $\dfrac{\delta^2 TP_1(T,t_1,p)}{\delta p^2}$ is negative and $TP_1(T,t_1,p)$ is a concave function of p for a given T and t_1. Hence, the value of p obtained from following equation is unique.

$$p^{*}=\frac{a}{2b}+\frac{\begin{bmatrix} C_h\dfrac{\left[\lambda e^{(\lambda+\theta)t_1}-(\lambda+\theta)e^{\lambda t_1}+\theta\right]}{\lambda\theta(\lambda+\theta)}+\\ C_b\dfrac{e^{-\delta T}}{(\lambda+\delta)^2}\left[-(\lambda+\delta)(T-t_1)e^{(\lambda+\delta)t_1}+e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]+\\ C_L\dfrac{\left[-\lambda e^{\lambda t_1}+\delta\left(e^{\lambda T}-e^{\lambda t_1}\right)+\lambda e^{-\delta T+(\lambda+\delta)t_1}\right]}{\lambda(\lambda+\delta)}+\\ C_p\left[\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\dfrac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]+\\ \alpha i_p C_p\left[\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+\dfrac{\left[e^{(\lambda+\theta)t_1}-1\right]}{(\lambda+\theta)}\right]\dfrac{L(N+1)}{2N}\end{bmatrix}}{2\left[\dfrac{\left(e^{\lambda t_1}-1\right)}{\lambda}+\dfrac{e^{-\delta T}\left[e^{(\lambda+\delta)T}-e^{(\lambda+\delta)t_1}\right]}{(\lambda+\delta)}+(1-\alpha)i_e\dfrac{\left[1+e^{\lambda M}(\lambda M-1)\right]}{\lambda^2}\right]} \tag{33}$$

3-1- The algorithm to find the optimal solutions

In order to jointly optimize (T, t_1, p), some steps should be followed. Step 1-4, 5-8 and 9-12 returned the optimal solutions of case 1, case 2 and Case 3, respectively.

Step 1. Start with initial trial value of price (i.e. $p_i = C_p + i_p C_p \dfrac{L(N+1)}{2N}$).

Step 2. By applying equation 16 and 17 the optimal solution of (T, t_1) is obtained for given selling price p_i.

Step 3. By substituting the (T, t_1) into equation 21, determine the optimal selling price p_{i+1}.

Step 4. If the stopping condition, which is getting sufficiently small difference between p_i and p_{i+1} is satisfied, return the optimal solutions and go to step 5; otherwise, substitute new value of selling price with initial selling price and return to step 2.

Step 5. Start with initial trial value of selling price (i.e. $p_j = C_p + i_p C_p \alpha \dfrac{L(N+1)}{2N}$).

Step 6. By applying equation 22 and 23 the optimal solution of (T, t_1) is obtained for given selling price p_j.

Step 7. By substituting the (T, t_1) into equation 27, determine the optimal selling price p_{j+1}.

Step 8. If the stopping condition, which is getting sufficiently small difference between p_j and p_{j+1} is satisfied, return the optimal solutions and go to step 9; otherwise, substitute new value of selling price with initial selling price and return to step 6.

Step 9. Start with initial trial value of selling price (i.e. $p_k = C_p + i_p C_p \alpha \dfrac{L(N+1)}{2N}$).

Step 10. By applying equation 28 and 29 the optimal solution of (T, t_1) is obtained for given selling price p_k.

Step 11. By substituting the (T, t_1) into equation 33, determine the optimal selling price p_{k+1}.

Step 12. If the stopping condition, which is getting sufficiently small difference between p_k and p_{k+1} is satisfied, return the optimal solutions and go to step 13; otherwise, substitute new value of selling price with initial selling price and return to step 10.

Step 13. Calculate and compare the total profit in each case by amount of optimal decision variables obtained in step 4, 8 and 12. Return the maximum amount of total profit as the optimum solution with the most profitable payment strategy.

4- Numerical examples

We use MATHEMATICA 10.2 and Matlab R2012 to apply the algorithm to solve several numerical examples and then illustrate the effectiveness of solution procedure. Let $C_o = 250$ \$ per order, $C_p = 20$ \$ per unit, $C_h = 20$ \$ per unit, $C_b = 10$ \$ per unit, $C_l = 25$ \$ per unit, $\alpha = 0.2$, $N = 5$, $\theta = 0.1$, $i_p = 0.15$ /\$/year and $L = 15/365$ and define demand function as $D(p,\mathrm{t}) = (200-4p)e^{-0.8t}$ and backordering rate as $\beta(x) = e^{-0.1x}$.
We adopt two different length of delayed payment for Case 2 and Case 3 which are $M = 4$ months and $M = 8$ months.
As shown in figures 4, 5 and 6, the numerical results reveal that in each case, the total profit function is strictly concave. Thus, these figures make us assure that the local maximum obtained by the proposed algorithm is the global maximum solution.

Case 1:

$$(p,t_1,T) = (38.05, 0.5432, 1.4712) \text{ and } TP_1 = 281.2641$$

Case 2:

$$(p,t_1,T) = (37.91, 0.5176, 1.4432) \text{ and } TP_2 = 289.8842$$

Case 3:

$$(p,t_1,T) = (37.82, 0.5399, 1.4624) \text{ and } TP_3 = 310.5869$$

Optimal solution:

$$TP^*(p,t_1,T) = \max\{TP_1(p,t_1,T), TP_2(p,t_1,T), TP_3(p,t_1,T)\} \rightarrow$$

$$(p^*,t_1^*,T^*) = (37.82, 0.5399, 1.4624)$$

$$TP^* = 310.5869$$

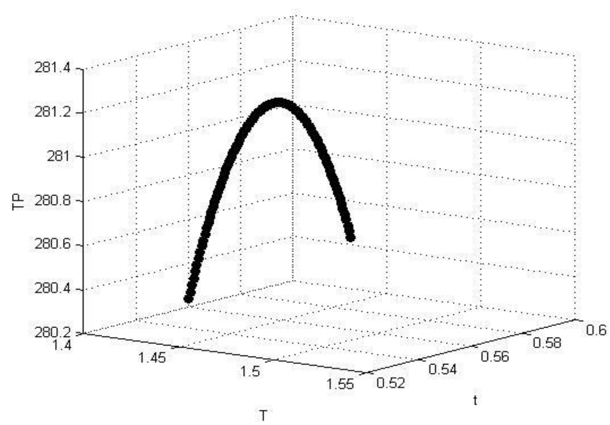

Fig 4. The total profit per unit time of case 1

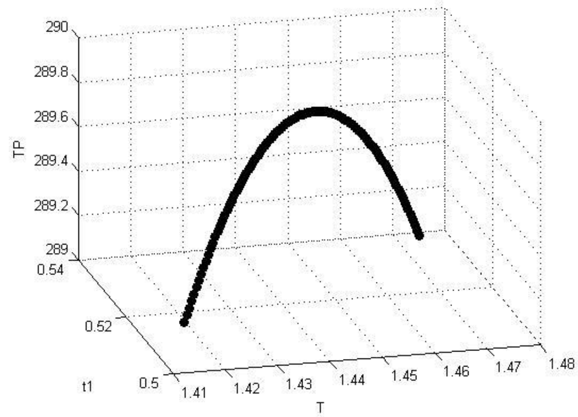

Fig 5. The total profit per unit time of case 2

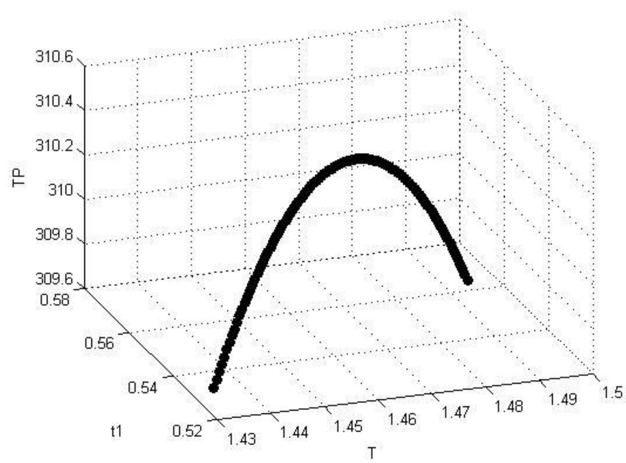

Fig 6. The total profit per unit time of case 3

Then, by using the same data, sensitivity analysis has been done to study the impact of some crucial parameters value. First, as identical costs of inventory system is worth considering, table 1, 2 and 3 display the impact of increasing and decreasing each of the relevant identical cost on the optimal solutions in each case. The other parameters which should be analysed are rate of backordering and rate of deterioration in each case. The results are shown in table 4, 5 and 6. Finally, the problem is resolved for different amount of M to evaluate the impact of length of delayed payment on the optimal solutions. The result are shown in table 7.

Table 1. Impact of changing identical components on case 1

Parameter	change%	Decision variables				change%			
		p	T	t_1	TP_1	p	T	t_1	TP_1
C_h	75	38.16	1.2597	0.328	230.217	0.29	-14.38	-39.62	-18.15
	25	38.10	1.3754	0.4457	258.741	0.13	-6.51	-17.95	-8.01
	0	38.05	1.4712	0.5432	281.264	0	0	0	0
	-25	37.97	1.6206	0.6953	314.535	-0.21	10.15	28.00	11.83
	-75	37.48	2.4896	1.5813	470.855	-1.50	69.22	191.11	67.41
C_p	75	44.41	0.9288	0.3247	122.520	16.71	-36.87	-40.22	-143.5
	25	40.22	1.2604	0.4552	100.923	5.70	-14.33	-16.20	-64.12
	0	38.05	1.4712	0.5432	281.264	0	0	0	0
	-25	35.84	1.6289	0.6133	511.552	-5.81	10.72	12.91	81.88
	-75	31.34	1.9308	0.7576	1132.01	-17.63	31.24	39.48	302.48
C_b	75	38.39	1.2911	0.6139	227.417	0.89	-12.24	13.02	-19.14
	25	38.19	1.3969	0.5713	259.321	0.37	-5.05	5.17	-7.80
	0	38.05	1.4712	0.5432	281.264	0	0	0	0
	-25	37.87	1.5961	0.5085	309.525	-0.47	8.49	-6.39	10.05
	-75	37.24	1.9012	0.4051	400.812	-2.13	29.23	-25.42	42.50
C_l	75	38.16	1.4100	0.5653	263.879	0.29	-4.16	4.07	-6.18
	25	38.09	1.4491	0.5511	275.091	0.11	-1.50	1.45	-2.19
	0	38.05	1.4712	0.5432	281.264	0	0	0	0
	-25	38.01	1.4944	0.5352	287.866	-0.11	1.58	-1.47	2.35
	-75	37.92	1.5478	0.5174	302.564	-0.34	5.21	-4.75	7.57
N	75	38.05	1.4712	0.5432	281.455	0	0	0	0.07
	25	38.05	1.4712	0.5432	281.349	0	0	0	0.03
	0	38.05	1.4712	0.5432	281.264	0	0	0	0
	-25	38.05	1.4578	0.5377	280.913	0	-0.91	-1.01	-0.12
	-75	38.07	1.4682	0.542	279.230	0	-0.20	-0.22	-0.72
L	75	38.07	1.4548	0.5364	278.961	0.05	-1.11	-1.25	-0.82
	25	38.06	1.4630	0.5398	280.499	0.03	-0.56	-0.63	-0.27
	0	38.05	1.4712	0.5432	281.264	0	0	0	0
	-25	38.04	1.4660	0.5411	282.028	-0.03	-0.35	-0.39	0.27
	-75	38.02	1.4736	0.5443	283.558	-0.08	0.16	0.20	0.82

Table 2 Impact of changing identical components on case 2

Parameter	change%	Decision variables				change%			
		p	T	t_1	TP_2	p	T	t_1	TP_2
C_h	75	38.05	1.2582	0.3279^	240.46	0.37	-12.82	-36.65	-17.05
	25	37.97	1.3615	0.4339	268.572	0.16	-5.66	-16.17	-7.35
	0	37.91	1.4432	0.5176	289.884	0	0	0	00
	-25	37.82	1.564	0.6415	320.043	-0.24	8.37	23.94	10.40
	-75	37.41	2.1401	1.2318	444.042	-1.32	48.29	137.98	53.18
C_p	75	44.22	0.847	0.2996^	-116.512	16.64	-41.31	-42.12	-140.2
	25	40.05	1.2671	0.4496	109.570	5.64	-12.20	-13.14	-62.20
	0	37.91	1.4432	0.5176	289.884	0	0	0	0
	-25	35.73	1.0601	0.5872	519.787	-5.75	-26.54	13.45	79.31
	-75	31.29	1.9123	0.7415	1140.45	-17.46	32.50	43.26	293.42
C_b	75	38.22	1.2538	0.5794	234.371	0.82	-13.12	11.94	-19.15
	25	38.04	1.3653	0.5422	267.307	0.34	-5.40	4.75	-7.79
	0	37.91	1.4432	0.5176	289.884	0	0	0	0
	-25	37.74	1.5455	0.4872	318.872	-0.45	7.09	-5.87	10.00
	-75	37.14	1.8911	0.3962	411.862	-2.03	31.04	-23.45	42.08
C_l	75	38.01	1.379	0.5369	271.995	0.26	-4.45	3.73	-6.17
	25	37.94	1.4197	0.5244	283.539	0.08	-1.63	1.31	-2.19
	0	37.91	1.4432	0.5176	289.884	0	0	0	0
	-25	37.87	1.4673	0.5106	296.664	-0.11	1.67	-1.35	2.34
	-75	37.78	1.5227	0.4948	311.739	-0.34	5.51	-4.40	7.54
N	75	37.91	1.4432	0.5176	289.929	0.00	0.00	0.00	0.02
	25	37.91	1.4432	0.5176	289.913	0.00	0.00	0.00	0.01
	0	37.91	1.4432	0.5176	289.884	0	0	0	0
	-25	37.91	1.4432	0.5176	289.816	0.00	0.00	0.00	-0.02
	-75	37.91	1.4432	0.5176	289.476	0.00	0.00	0.00	-0.14
L	75	37.91	1.4306	0.5128	289.417	0.00	-0.87	-0.93	-0.16
	25	37.91	1.4432	0.5176	289.731	0.00	0.00	0.00	-0.05
	0	37.91	1.4432	0.5176	289.884	0	0	0	0
	-25	37.91	1.4389	0.516	290.038	0.00	-0.30	-0.31	0.05
	-75	37.9	1.4383	0.5158	290.343	-0.03	-0.34	-0.35	0.16

^ This is the undefined solution which is not in the specified range

Table 3. Impact of changing identical components on case 3

Parameter	change%	Decision variables				change%			
		p	T	t_1	TP_3	p	T	t_1	TP_3
C_h	75	38.02	1.2565	0.3271	253.965	0.53	-14.08	-39.41	-18.23
	25	37.91	1.3693	0.4437	286.133	0.24	-6.37	-17.82	-7.87
	0	37.82	1.4624	0.5399	310.586	0.00	0.00	0.00	0.00
	-25	37.69	1.6078	0.6898	344.709	-0.34	9.94	27.76	10.99
	-75	37.17	2.4675	1.6622	478.662	-1.72	68.73	143	54.12
C_p	75	44.26	0.8425	0.2933	-111.67	17.03	-42.39	-45.68	-135.9
	25	39.98	1.2813	0.4637	125.278	5.71	-12.38	-14.11	-59.66
	0	37.82	1.4624	0.5399	310.586	0.00	0.00	0.00	0.00
	-25	35.64	1.6222	0.6108	544.281	-5.76	10.93	13.13	75.24
	-75	31.21	1.9229	0.7543	1168.05	-17.48	31.49	39.71	276.08
C_b	75	38.1	1.2796	0.6084	256.496	0.74	-12.50	12.69	-17.42
	25	37.93	1.3866	0.5671	288.529	0.29	-5.18	5.04	-7.10
	0	37.82	1.4624	0.5399	310.586	0.00	0.00	0.00	0.00
	-25	37.66	1.5613	0.5058	338.624	-0.42	6.76	-6.32	9.03
	-75	37.11	1.8979	0.4042	428.767	-1.88	29.78	-25.13	38.05
C_l	75	37.91	1.4002	0.5613	293.087	0.24	-4.25	3.96	-5.63
	25	37.85	1.4398	0.5475	304.307	0.08	-1.55	1.41	-2.02
	0	37.82	1.4624	0.5399	310.586	0.00	0.00	0.00	0.00
	-25	37.78	1.4855	0.532	317.060	-0.11	1.58	-1.46	2.08
	-75	37.7	1.5394	0.5145	331.701	-0.32	5.27	-4.70	6.80
N	75	37.82	1.4624	0.5399	310.517	0.00	0.00	0.00	-0.02
	25	37.82	1.4624	0.5399	310.500	0.00	0.00	0.00	-0.03
	0	37.82	1.4624	0.5399	310.586	0.00	0.00	0.00	0.00
	-25	37.82	1.4624	0.5399	310.402	0.00	0.00	0.00	-0.06
	-75	37.82	1.4624	0.5399	310.056	0.00	0.00	0.00	-0.17
L	75	37.83	1.4492	0.5345	309.958	0.03	-0.90	-1.00	-0.20
	25	37.83	1.4624	0.5399	310.334	0.03	0.00	0.00	-0.08
	0	37.82	1.4624	0.5399	310.586	0.00	0.00	0.00	0.00
	-25	37.82	1.4579	0.538	310.650	0.00	-0.31	-0.35	0.02
	-75	37.81	1.4573	0.5378	310.927	-0.03	-0.35	-0.39	0.11

^ This is the undefined solution which is not in the specified range

Based on the result on table 1, 2 and 3, by increasing the relevant costs, the optimal total profit decreases. This trend is so logical since the total profit is negatively dependent on identical costs. On the other hand, due to the increase of purchasing cost total profit gets much smaller. In the other words, total profit is more sensitive to purchasing cost than other costs. Moreover, as N gets larger, the optimal value of total profit increase. However, the other decision variables are not really sensitive to changes in N . Furthermore, a slightly decrease in the value of total profit is observed as increase of length of advance payment.

Table 4. Impact of changing θ and δ on case 1

Parameters		p	T	t_1	TP_1
θ=0.1	δ=0.1	38.05	1.4712	0.5432	281.2641
	δ=0.3	38.24	1.3069	0.6094	228.2217
	δ=0.5	38.39	1.2082	0.6503	197.4505
	δ=0.7	38.50	1.1456	0.6797	177.3917
θ=0.25	δ=0.1	38.07	1.3924	0.4637	264.3244
	δ=0.3	38.25	1.2159	0.5182	207.5493
	δ=0.5	38.39	1.1094	0.5515	174.3573
	δ=0.7	38.50	1.0414	0.5755	152.6087
θ=0.4	δ=0.1	38.09	1.3341	0.4048	251.0659
	δ=0.3	38.26	1.1490	0.4510	191.3630
	δ=0.5	38.40	1.0373	0.4792	156.2709
	δ=0.7	38.50	0.9652	0.4993	133.1951
θ=0.65	δ=0.1	38.12	1.2644	0.3340	234.3899
	δ=0.3	38.28	1.0694	0.3709	170.9971
	δ=0.5	38.41	0.9516	0.3933	133.5090
	δ=0.7	38.51	0.8754	0.4093	108.7592

Table 5. Impact of changing θ and δ on case 2

Parameters		p	T	t_1	TP_2
θ=0.1	δ=0.1	37.91	1.4432	0.5176	289.8842
	δ=0.3	38.07	1.2702	0.5753	235.2874
	δ=0.5	38.2	1.1662	0.6108	203.4288
	δ=0.7	38.29	1.0995	0.636	182.5795
θ=0.25	δ=0.1	37.94	1.3767	0.4501	274.0265
	δ=0.3	38.09	1.194	0.4986	216.1986
	δ=0.5	38.21	1.0838	0.5281	182.2511
	δ=0.7	38.31	1.0133	0.5495	159.9455
θ=0.4	δ=0.1	37.97	1.3259	0.3983	261.2592
	δ=0.3	38.11	1.136	0.44	200.8267
	δ=0.5	38.23	1.0216	0.4655	165.196
	δ=0.7	38.32	0.9477	0.4836	141.7168
θ=0.65	δ=0.1	38.01	1.2628	0.3339	244.7533
	δ=0.3	38.14	1.0645	0.3678	180.9458
	δ=0.5	38.25	0.9448	0.3883	143.1345
	δ=0.7	38.34	0.8674	0.403	118.1356

Table 6. Impact of changing θ and δ on case 3

Parameters		p	T	t_1	TP_3
θ=0.1	δ=0.1	37.82	1.4624	0.5399	310.5869
	δ=0.3	37.95	1.2956	0.6037	257.6081
	δ=0.5	38.06	1.1954	0.6428	226.6423
	δ=0.7	38.14	1.1316	0.6707^	206.3571
θ=0.25	δ=0.1	37.87	1.3859	0.4617	292.3235
	δ=0.3	38	1.2077	0.5146	235.9791
	δ=0.5	38.1	1.0999	0.5465	202.8585
	δ=0.7	38.18	1.0310	0.5694	181.0901
θ=0.4	δ=0.1	37.91	1.3290	0.4035	277.6719
	δ=0.3	38.04	1.1428	0.4486	218.5709
	δ=0.5	38.14	1.0300	0.4758	183.6868
	δ=0.7	38.21	0.9571	0.4950	160.7016
θ=0.65	δ=0.1	37.96	1.2606	0.333	258.9137
	δ=0.3	38.08	1.0647	0.3695	196.1969
	δ=0.5	38.18	0.9462	0.3912	159.0002
	δ=0.7	38.25	0.8693	0.4065	134.4161

As tables 4, 5 and 6 display, by increasing the deterioration rate, total profit of the inventory system and replenishment cycle significantly decreases. That is because as most of products deteriorate throughout time, their demand value gets smaller and then the amount of sells decreases. Moreover, as backordering rate gets larger, the total profit and the replenishment cycle decrease, while the time with no shortage increases. These trends are observed in each case.

Table 7. The impact of changing length of delayed payment on optimality

parameters	Optimal Case	p^*	T^*	t_1^*	TP^*
M=0.0416	Case 1	38.05	1.4712	0.5432	281.2641
M=0.116	Case 1	38.05	1.4712	0.5432	281.2641
M=0.332	Case 2	37.91	1.4432	0.5176	289.8842
M=0.448	Case 2	37.87	1.4545	0.5303	297.2738
M=0.5	Case 1	38.05	1.4712	0.5432	281.2641
M=0.666	Case 3	37.82	1.4624	0.5399	310.5869

As table 7 projects, since larger value of length of delayed payment, much more opportunity the retailer has to deposit the sales in an interest beating in, the optimal total profit almost increases by increasing M. In detail, for first four values of M profit increase happens by increasing M. Then, there is a decrease in switching the optimality from Case 2 to Case 1 and after that again increase in total profit is observed. It is recommended choosing the larger value of delayed payment period to gain more profit.

5- Conclusion

In this paper, an integrated pricing and inventory control model proposed for deterioration items. The payment scheme involves multiple advance payments also delayed payment. Shortages are allowed and

backordering rate is waiting time dependent and demand function is dependent on both selling price and time. The main objective function is to maximize the total profit, the selling price, replenishment cycle and the time with no shortage simultaneously. Moreover, the concavity of the total profit functions based on price is proven. Then, we offer an efficient algorithm to find the optimal solutions of the purposed model. Finally, several numerical examples are extracted in order to illustrate the procedure of solution in solving the presented variables. Also, the effect of deterioration rate, backordering rate, delay in payment and multiple advance payments are discussed. The results show that when delayed payment is offered, by higher value of delayed payment time, the retailer’s total profit does increase.
This is the first work that optimizes price and replenishment policy for a deterioration item under hybrid payment strategy and partial backordering shortages. Future research can extend the mathematical model by either considering practicable assumptions or solving by other solution procedures. For instance, some crucial parameters such as length of delayed payment or advance payment can be considered as decision variables. Also, the current demand function which is deterministic can be extended to be stochastic. Finally, one can amend this model for non-instantaneous deterioration product with variable deterioration rate.

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Appendix A.

(a) Because of high complexity of the equations due to three dimensions system, a straightforward proof of Theorem 1 by Hessian Matrix does not exist. However, the simplified kind of this equations system which inspire us can be found in Chang et al. (2006), Dye (2007) and Yang et al. (2009).

(b) We apply the simple procedure used in Chang et al. (2006) which is, first, for any given p, we take the second partial derivatives of $TP_1(T, t_1, p)$ with regard to t_1 and we get:

$$\frac{\delta^2 TP_1(T,p,t_1)}{\delta t_1^{\,2}} = \frac{(a-bp)}{T}\begin{bmatrix} -C_b\left[1-(T-t_1)(\lambda+\delta)\right]e^{-\delta T+(\lambda+\delta)t_1} \\ e^{\lambda t_1}\begin{bmatrix} +\lambda\left(C_p + i_p C_p \dfrac{L(N+1)}{2N}\right)\left(e^{-\delta(T-t_1)} - e^{\theta t_1}\right) \\ -\left(C_p + i_p C_p \dfrac{L(N+1)}{2N}\right)\left(\theta e^{\theta t_1} - \delta e^{-\delta(T-t_1)}\right) \\ -C_h e^{\theta t_1} + \dfrac{\lambda C_h}{\theta}\left(1-e^{\theta t_1}\right) \end{bmatrix} \\ +\left(p+C_L\right)\left(\lambda e^{\lambda t_1} - (\lambda+\delta)e^{-\delta T+(\lambda+\delta)t_1}\right) \end{bmatrix} \tag{A1}$$

We have assumed that $\lambda < 0$ and $|\lambda| > \theta, \delta$, (because of practicality in the real market). Thus, $TP_1(T, t_1, p)$ gets negative value and is a strictly concave function in t_1. Next, in order to investigate the concavity of the function with respect to T, we take the first partial derivatives of $TP_1(T, t_1, p)$ with regard to T and we get:

$$\frac{\partial TP_1(T,p,t_1)}{\partial T} = \frac{-CTP_1 + T\cdot CTP_1'}{T^2} \tag{A2}$$

Where

$$CTP_1' = \begin{bmatrix} p(a-bp)\left(\dfrac{\lambda e^{\lambda T} + \delta e^{-\delta T+(\lambda+\delta)t_1}}{(\lambda+\delta)}\right) - \\ C_p(a-bp)\left[\dfrac{\lambda e^{\lambda T} + \delta e^{-\delta T+(\lambda+\delta)t_1}}{(\lambda+\delta)}\right] - \\ C_L \dfrac{(a-bp)}{(\lambda+\delta)}\left[\delta\left(e^{\lambda T} - e^{-\delta T+(\lambda+\delta)t_1}\right)\right] - \\ \dfrac{C_b(a-bp)}{(\lambda+\delta)^2}\left[\lambda\left(e^{\lambda T} - e^{-\delta T+(\lambda+\delta)t_1}\right) + \delta(\lambda+\delta)(T-t_1)e^{-\delta T+(\lambda+\delta)t_1}\right] - \\ i_p C_p (a-bp)\dfrac{L(N+1)}{2N}\left[\dfrac{\lambda e^{\lambda T} + \delta e^{-\delta T+(\lambda+\delta)t_1}}{(\lambda+\delta)}\right] \end{bmatrix} \tag{A3}$$

For notational convenience we substitute equation A.4 which is into equation A.2 and we get the first partial derivatives of $Q(T)$ with regard to T as follows:

$$Q(T) = -CTP_1 + T \cdot CTP_1' \tag{A4}$$

$$\frac{\partial Q(T)}{\partial T} = -CTP_1' + T \cdot CTP_1'' + CTP_1' = T \cdot CTP_1'' \tag{A5}$$

Where

$$CTP_1'' = \left(\frac{a-bp}{\lambda+\delta}\right)\begin{bmatrix} -\left(p - C_p - i_p C_p \dfrac{L(N+1)}{2N}\right)\left(\delta^2 e^{-\delta T + (\lambda+\delta)t_1} - \lambda^2 e^{\lambda T}\right) \\ -\lambda^2\left(\dfrac{C_L \delta}{\lambda} + \dfrac{C_b}{(\lambda+\delta)}\right)e^{\lambda T} \\ -\delta\left[\delta C_L + \dfrac{C_b \lambda}{(\lambda+\delta)} + C_b\left(1 - \delta(T - t_1)\right)\right]e^{-\delta T + (\lambda+\delta)t_1} \end{bmatrix} \tag{A6}$$

After some algebraic manipulation, we can prove that the Eq. A6 gets negative value if $p - C_p - i_p C_p \frac{L(N+1)}{2N} > 0$. i.e. the selling price is much bigger than purchasing cost and the capital cost before receiving products which is mostly occur in real market. Hence with this assumption, as the first derivation of $Q(T)$ is negative, $Q(T)$ is decreasing function of T. Using L'Hospital's Rule, we get:

$$\lim_{T \to \infty} Q(T) = -\infty \tag{A7}$$

$$Q(0) \approx C_o + C_b (a - bp) t_1^2 > 0 \tag{A8}$$

So there exists a unique T in interval $[0, \infty]$ such that $TP_1(T, t_1, p)$ is maximized.