Journal of Industrial and Systems Engineering
Vol. 13, No. 3, pp. 70-86
Summer (July) 2021

A sustainable inventory model for perishable items considering waste treatment for returning items

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Abstract

Perishing of the items in an inventory model has always been a crucial issue for beverage companies. Besides, some items not only perish but also have specific expiration dates. To manage the inventory-in-hand, proposing a delay in payment is usually an appropriate solution. Also, beverage companies encounter sustainable policies that are regulated to dispose of waste without damaging the environment. These regulations lead companies to establish waste treatment units in their companies to purify the obsolete items before disposing of them. Despite the importance of this challenge in practice, no research has been made to contribute to these issues. To fill the mentioned gap in the literature, this paper proposes an inventory model for perishable items which: (a) items perish continuously and have specific expiration dates, (b) a single-level trade credit is offered to the customer to stimulate the demand, and (c) waste treatment policy is considered for the beverage company to purify the returning items before disposing of them. To develop our work in practice, the case study of Behnoush Beverage Company is considered and a real domain dataset is utilized. To validate the proposed mathematical model, a sensitivity analysis is developed. Eventually, managerial implications are outlined and the findings are concluded.

Keywords: Inventory management, perishable items, delay in payment, wastewater treatment

1-Introduction

As the expectations from the customers in the food industry are increasing, the expiration date of items has become a vital factor in purchasing an item (Chen et al. 2016). When the item is near its expiration date, the demand approaches zero (Wu et al. 2016).

In practice, the accounts are not settled immediately and a credit period is offered to the buyer (Teng et al. 2016). One solution to the issue of selling the perishable items before their expiration dates are offering a trade credit period that stimulates the demand but increases the default risk (Mahata and De 2017).

In this study, green inventory management is concerned with the overarching question of how to efficiently manage inventories with respect to both costs and green aspects (Marklund and Berling 2017). Sustainability criteria such as carbon emission policies can be added to a perishable inventory model to restrict emissions (Shi et al. 2020).

The remainder of this paper is illustrated as follows. In section 2, a concise literature review is provided to indicate the impact of the expiration date, credit period, and sustainable disposal on an inventory system. In section 3, notations and assumptions required for developing a perishable inventory model are prepared and the mathematical model is developed. In section 4, a solution approach is illustrated. A special case for non-deteriorating items is developed in section 5. To validate the formulation, a case study is developed and a sensitivity analysis is proposed in sections 6 and 7 respectively. Finally, the findings are concluded in section 8.

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ISSN: 1735-8272,

2-Literature review

The most contribution models concerning perishable inventory and trade credit policy are as follows: Sarkar (2012) proposed the first inventory model considered a time-varying deterioration rate (i.e. expiration date). Besides, Teng and Lou (2012) developed a credit-dependent demand function, and Wang et al. (2014) prepared a model for items with expiration dates and credit-dependent demand. Then Wu et al. (2014) provided an extension with two levels of the credit period. Thereafter, some works were extended for partial credit period (Wu and Chan 2014), price-dependent demand (Feng et al. 2017; Tiwari et al. 2018a; Wu et al. 2017), and random expiration date (Tai et al. 2019). Khan et al. (2019) developed a perishable model with partial backordering and time-varying holding cost. Moreover, the demand rate can be dependent on the expiration date (Yang 2020).

On the other hand, among the many important studies on sustainable inventory and especially sustainable disposal, the following are the most highlighted.

Battini et al. (2014) considered disposing of the waste operation in a sustainable inventory system. This work was followed by Datta (2017) considering pricing decisions for defective items and Tsao et al. (2017) considering credit-dependent demand. Kazemi et al. (2018) developed a model for defective items with inspection considering carbon emission regulations and Tiwari et al. (2018b) expanded this work for deteriorating items. Mishra et al. (2019) developed an inventory model for perishable items considering a price/stock dependent demand and remanufacturing and disposing of processes for items. Mishra et al. (2020) proposed a model under carbon emission regulations considering disposing of the waste that was expanded by Taleizadeh et al. (2020) considering pricing decisions. Rout et al. (2020) provided a perishable model considering disposing of the waste and different emission policies.

Despite important works associated with this field of study, no paper has considered a contribution of perishable items with delay in payment and disposing of the waste. An indication of the abovementioned papers and their contribution to our work is exposed in table 1.

Table 1. Literature review

Paper	Perishable items	Delay in payment	Disposing of waste	Demand			
				Constant	Price-dependent	Credit-dependent	Expiration date-dependent
Sarkar (2012)							
Teng and Lou (2012)		✓				✓	
Wang et al. (2014)	✓	✓				✓	
Wu et al. (2014)	✓	✓				✓	
Wu and Chan (2014)	✓	✓		✓			
Battini et al. (2014)			✓	✓			
Feng et al. (2017)	✓	✓			✓		✓
Wu et al. (2017)	✓	✓			✓		
Datta (2017)			✓		✓		
Tsao et al. (2017)			✓			✓	
Tiwari et al. (2018a)	✓	✓			✓		
Kazemi et al. (2018)			✓	✓			

Table 1. Continued

Paper	Perishable items	Delay in payment	Disposing of waste	Demand			
				Constant	Price-dependent	Credit-dependent	Expiration date-dependent
Tiwari et al. (2018b)	✓		✓	✓			
Khan et al. (2019)	✓				✓		
Tai et al. (2019)	✓			✓			
Mishra et al. (2019)	✓		✓		✓		
Yang 2020)	✓	✓					✓
(Mishra et al. (2020)			✓	✓			
Taleizadeh et al. (2020)			✓		✓		
Rout et al. (2020)	✓		✓	✓			
This paper	✓	✓	✓			✓	

3-Model development

3-1- Problem description

The abovementioned gap in the literature can be filled by the following proposed mathematical model. In the developed model, items are considered perishable with specific expiration dates, and the credit period is considered as a stimulator for the buyer to increase his/her demand. To consider sustainable responsibility, waste treatment for returning items is performed to decrease the possibility of disease related to obsolete items that are returned to the company. The mathematical model is formulated as follows.

3-2- Notations

Symbols used in the model as parameters and decision variables are presented in table 2.

Table 2. Notations

Notations	Descriptions
n	Trade credit period (in years) offered by the seller to the buyer
T	Buyer's replenishment cycle time (in years)
o	Ordering cost per order ($)
c	Unit purchasing cost ($)
h	The holding cost for each unit ($)
C_p	Unit waste treatment cost for returned items ($/unit)
COD_R	Amount of oxygen needed for oxidation on returned items ($\frac{mg}{litre}$)
COD_S	Amount of oxygen accepted by environment preservation organization for releasing items to wastewaters or irrigation uses ($\frac{mg}{litre}$)
α	The fraction of items which are returned
$I(t)$	Available inventory level at time t
$\theta(t)$	The time-dependent deterioration rate ($0 \le \theta \le 1$)
m	The maximum lifetime of items or expiration date (years)
$D(n)$	Annual demand rate as a function of the trade credit period
$F(n)$	Default risk as a function of the trade credit period

Table 2. Continued

Notations	Descriptions
Q	Seller's economic order quantity
$\Pi(n,T)$	Seller's total profit

3-3- Assumptions

1. The inventory system is studied for a single product.
2. We assume that the demand rate $D(n)$ is a positive exponential function of trade credit period n (Wu et al. 2014) as

$$D(n) = k(1 - ga)e^{an} \tag{1}$$

   Where k, a and g are positive constants with $0 \le a \le 1$ and $0 \le g \le 1$.
3. The longer the trade credit period means the higher the default risk to the seller (Wang et al. 2014; Wu et al. 2014). The rate of default risk giving the credit period n is assumed as

$$F(n) = 1 - e^{bn} \tag{2}$$

   Where b is a positive constant.
4. The products in inventory have expiration rates. The deterioration rate tends to 1 when the time approaches the maximum lifetime m (Sarkar 2012; Wang et al. 2014).

$$\theta(t) = \frac{1}{1 + m - t} \qquad 0 \le t \le T \le m \tag{3}$$

5. Returned items are the ones that are not sold in the target market by the buyers and are brought back to the companies as their expiration date has arrived. As these liquid items fully deteriorate, they cannot be abandoned in the environment and should be purified, before leaving them in urban wastewater.
6. The replenishment rate of the inventory is infinite.
7. Lead time is negligible.
8. Shortages are not allowed.

3-4- Mathematical model

During the replenishment cycle $[0, T]$, the seller's inventory is depleted due to demand and deterioration.

$$\frac{dI(t)}{dt} = D(n) - \theta(t)I(t) \tag{4}$$

With the boundary condition$I(T) = 0$, and solving the differential equation (4), we get

$$I(t) = e^{-\delta(t)} \int_t^T e^{\delta(t)} D(n)du \tag{5}$$

Where

$$\delta(t) = \int_0^t \theta(u)du = \ln\left(\frac{1 + m}{1 + m - t}\right) \tag{6}$$

Substituting (6) into (5) we will have the inventory level at time t as

$$I(t) = D(n)\frac{1 + m - t}{1 + m}\int_t^T \frac{1 + m}{1 + m - u}du = D(n)(1 + m - t)\ln\left(\frac{1 + m - t}{1 + m - T}\right) \tag{7}$$

Consequently, the seller's ordering quantity is

$$Q = I(0) = D(n)(1 + m)\ln\left(\frac{1 + m}{1 + m - T}\right) \tag{8}$$

Based on the above equations, the holding cost per cycle including capital cost after receiving Q units at time $t = 0$ is

$$HC = \frac{h}{T}\int_0^T I(t)dt = \frac{hD(n)}{T}\int_0^T (1+m-t)\ln\left(\frac{1+m-t}{1+m-T}\right)dt$$
$$= \frac{hD(n)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] \quad (9)$$

Meanwhile, the seller provides his/her buyer an upstream credit period of n years. Hence, the seller's net revenue received after the default risk is

$$SR = pD(n)[1-F(n)] = pk(1-g\alpha)e^{(a-b)n} \quad (10)$$

The chemical oxygen demand (COD) is an indicative measure of the amount of oxygen that can be consumed by reactions in a measured solution (Sawyer et al. 2003). COD is useful in terms of water quality by providing a metric to determine the effect an effluent will have on the receiving body. Therefore, the lower the COD of water, the healthier it is. Waste treatment cost which is the unit refinement cost for returning items to increase COD of items, which are healthy for drop-in nature or irrigation use, is calculated as

$$RC = \frac{I(0)}{T}C_p\alpha(COD_R - COD_S)$$
$$= \frac{ke^{an}(1-g\alpha)}{T}(1+m)\ln\left(\frac{1+m}{1+m-T}\right)C_p\alpha(COD_R - COD_S) \quad (11)$$

Finally, the seller's ordering cost per cycle is o, and the purchasing cost per cycle is $cI(0)$. Consequently, the seller's annual profit per unit time is

$$\Pi(n,T) = pD(n)[1-F(n)] - \frac{cI(0)}{T} - \frac{o}{T} - \frac{h}{T}\int_0^T I(t)dt - \frac{I(0)}{T}C_p\alpha(COD_R - COD_S)$$
$$= pk(1-g\alpha)e^{(a-b)n} - \frac{c}{T}ke^{an}(1-g\alpha)(1+m)\ln\left(\frac{1+m}{1+m-T}\right) - \frac{o}{T}$$
$$- \frac{hke^{an}(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right]$$
$$- \frac{ke^{an}}{T}(1-g\alpha)C_p\alpha(COD_R - COD_S)(1+m)\ln\left(\frac{1+m}{1+m-T}\right) \quad (12)$$

The seller's objective is to find the optimal credit period n^* and the optimal replenishment cycle time T^* such that the total annual profit per unit time $\Pi(n^*, T^*)$ is maximized. In the next section, we characterize the seller's optimal credit period and cycle time simultaneously.

4-Solution procedure

For any given n In order to find an optimal solution T^*, taking the first and second-order derivatives of $\Pi(n,T)$ with respect to T and rearranging terms, we can get

$$\frac{d\Pi(n,T)}{dn} = pk(a-b)(1-g\alpha)e^{(a-b)n} - \frac{k}{T}ae^{an}(1-g\alpha)C_p\alpha(COD_R - COD_S)(1+m)\ln\left(\frac{1+m}{1+m-T}\right) - \frac{c}{T}ake^{an}(1-g\alpha)(1+m)\ln\left(\frac{1+m}{1+m-T}\right)$$
$$- \frac{hake^{an}(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] \quad (13)$$

and

$$\frac{d^2\Pi(n,T)}{dn^2} = pk(a-b)^2(1-g\alpha)e^{(a-b)n} - \frac{k}{T}a^2e^{an}(1-g\alpha)C_p\alpha(COD_R - COD_S)(1 + m)\ln\left(\frac{1+m}{1+m-T}\right) - \frac{c}{T}a^2ke^{an}(1-g\alpha)(1 + m)\ln\left(\frac{1+m}{1+m-T}\right) - \frac{ha^2ke^{an}(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] \tag{14}$$

For any given T In order to find an optimal solution n^*, taking the first- and second-order derivatives of $\Pi(n,T)$ with respect to n and rearranging, terms we will obtain

$$\frac{d\Pi(n,T)}{dT} = \frac{o}{T^2} - \frac{cke^{an}(1-g\alpha)(1+m)}{(1+m-T)T} + \frac{cke^{an}(1-g\alpha)(1+m)}{T^2}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{ke^{an}(1-g\alpha)C_p\alpha(COD_R - COD_S)(1+m)}{(1+m-T)T} + \frac{ke^{an}(1-g\alpha)C_p\alpha(COD_R - COD_S)(1+m)}{T^2}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{hke^{an}(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2(1+m-T)\mathrm{T}} + \frac{T}{4} - \frac{(1+m)^2}{2T}\ln\left(\frac{1+m}{1+m-T}\right)\right] \tag{15}$$

and

$$\frac{d^2\Pi(N,T)}{dT^2} = -\frac{2o}{T^3} - cke^{an}(1-g\alpha)\left[-\frac{2(1+m)}{(1+m-T)T^2} + \frac{(1+m)}{T(1+m-T)^2} + \frac{2(1+m)\ln\left(\frac{1+m}{1+m-T}\right)}{T^3}\right] - ke^{an}(1-g\alpha)C_p\alpha(COD_R - COD_S)\left[-\frac{2(1+m)}{(1+m-T)T^2} + \frac{(1+m)}{T(1+m-T)^2} + \frac{2(1+m)\ln\left(\frac{1+m}{1+m-T}\right)}{T^3}\right] - hke^{an}(1-g\alpha)\left[\frac{2(1+m)^2}{T^3}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{2(1+m)^2}{(1+m-T)T^3} + \frac{(1+m)^2}{(1+m-T)^2T^2} - \frac{(1+m)^2}{(1+m-T)T^2}\right] < 0 \tag{16}$$

In order to obtain theoretical results from (13) - (16), we use two following theorems. For simplicity, we define a statement $\Delta(n)$ as

$$\Delta(n) = p(a-b)(1-g\alpha) - \frac{1}{T}a(1-g\alpha)C_p\alpha(COD_R - COD_S)(1+m)\ln\left(\frac{1+m}{1+m-T}\right) - \frac{c}{T}a(1-g\alpha)(1+m)\ln\left(\frac{1+m}{1+m-T}\right) - \frac{ha(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right) + \frac{T^2}{4} - \frac{(1+m)T}{2}\right] \quad (17)$$

And also we define a statement $\Delta(T)$ as

$$\Delta(T) = o - cke^{an}(1-g\alpha)(1+m)(m-\ln(1+m)) - ke^{an}(1-g\alpha)C_p\alpha(COD_R - COD_S)(1+m)(m-\ln(1+m)) - \frac{hke^{an}(1-g\alpha)}{4}[m^2 - 2(1+m)^2\ln(1+m) + 2(1+m)^2] \quad (18)$$

4-1- Theorem 1

For any given $T > 0$, if $(a-b)^2p - a^2c - C_pa^2\alpha(COD_R - COD_S) \leq 0$, then we can get

1. If $\Delta(n) \leq 0$, then $\Pi(n,T)$ is maximized at $n^* = 0$.
2. If $\Delta(n) > 0$, there exists a unique $n^* > 0$ such that maximizes $\Pi(n,T)$.

See appendix A for proof.

4-2- Theorem 2

For any given $n^* \geq 0$, if

$\ln\left(\frac{1+m}{1+m-T}\right) \geq \frac{(2+2m-3T)T}{2(1+m-T)^2}$

And

$\ln\left(\frac{1+m}{1+m-T}\right) \geq \frac{(2+2m-3T)+(1+m-T)T}{2(1+m-T)^2}$

Then we will have

1. If $\Delta(T) \geq 0$, then $\Pi(n,T)$ is maximized at $T^* = m$.
2. If $\Delta(T) < 0$, there exists a unique $T^* \in (0,m)$ such that $\Pi(n,T)$ is maximized.

See appendix B for proof.

5-Special case

For numerical experiments, we consider a special case in which the expiration date is approaching infinity as follows.

$$\frac{1+m}{T}\ln\left(\frac{1+m}{1+m-T}\right) = \frac{\ln\left(\frac{1+m}{1+m-T}\right)}{\frac{T}{1+m}} \quad (19)$$

Then

$$\frac{d}{dm}\ln\left(\frac{1+m}{1+m-T}\right) = \frac{1+m-T}{1+m}\left[\frac{1}{1+m-T} - \frac{1+m}{(1+m-T)^2}\right] = \frac{1+m-T}{1+m}\left(\frac{-T}{1+m-T}\right) \quad (20)$$

And because

$$\frac{d}{dm}\frac{T}{1+m-T}=\frac{-T}{(1+m-T)^2} \tag{21}$$

Therefore

$$\lim_{m\to\infty}\frac{1+m}{T}\ln\left(\frac{1+m}{1+m-T}\right)=\lim_{m\to\infty}\frac{\frac{d}{dm}\ln\left(\frac{1+m}{1+m-T}\right)}{\frac{d}{dm}\left(\frac{T}{1+m}\right)}=\lim_{m\to\infty}\frac{1+m-T}{1+m}=1 \tag{22}$$

Consequently, the order quantity of the seller can be proposed as

$$Q=I(0)=D(n)(1+m)\ln\left(\frac{1+m}{1+m-T}\right)=D(n)T \tag{23}$$

Similarly, we can obtain the following results

$$\begin{aligned}\lim_{m\to\infty}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)-\frac{1+m}{2}T\right]&=\frac{1}{2}\lim_{m\to\infty}\left[\frac{\ln\left(\frac{1+m}{1+m-T}\right)-\frac{T}{1+m}}{\frac{1}{(1+m)^2}}\right]\\&=\frac{1}{2}\lim_{m\to\infty}\left[\frac{\frac{-T}{(1+m-T)(1+m)}+\frac{T}{(1+m)^2}}{\frac{-2}{(1+m)^3}}\right]\\&=\frac{1}{2}\lim_{m\to\infty}\left[\frac{T^2(1+m)}{2(1+m-T)}\right]=\frac{1}{4}\lim_{m\to\infty}T^2=\frac{T^2}{4}\end{aligned} \tag{24}$$

We conclude that the holding cost of the seller at each replenishment cycle is obtained as

$$\lim_{m\to\infty}hD(n)\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{1+m}{2}T\right]=\frac{hD(n)T^2}{2} \tag{25}$$

Therefore, the seller's annual profit is given as

$$\begin{aligned}\Pi(n,T)=pk(1-g\alpha)e^{(a-b)n}-cke^{an}(1-g\alpha)-\frac{o}{T}-\frac{hke^{an}(1-g\alpha)T}{2}\\-C_p\alpha ke^{an}(1-g\alpha)(COD_R-COD_S)\end{aligned} \tag{26}$$

With appropriate proximity, we use this simplified profit function for solving the model, numerical examples, and sensitivity analysis.

6-Case study

In this section, a scheme is provided to obtain the optimal trade credit period and replenishment cycle time. The EOQ model is applied to find the optimum replenishment cycle and trade credit period. Behnoush is one of the first companies that started to produce beverage items in Iran. According to environmental regulations, beverage factories are forced to purify returning items before their disposal. Therefore, a purifying unit is established in the factory to purify large amounts of fluid per day.

We propose a numerical examples to illustrate the theoretical results as well as to provide some managerial insights. Let:

$a=5,\ b=3,\ k=1000,\ p=3\ \frac{\$}{unit},\ c=1\ \frac{\$}{unit},\ h=0.1\ \frac{\$}{unit/year},\ o=20\ \frac{\$}{order},\ m=1\ year,$ $C_p=0.01\ \frac{\$}{unit},\ \alpha=0.01,\ COD_R=500\ \frac{mg}{litre},\ COD_S=200\ \frac{mg}{litre}.$

Using Mathematica 11.2 in a personal computer with a 2.4 GHz processor and 6 GB RAM, the optimal solution will be:

$$n^* = 0.041$$
$$T^* = 0.584$$
$$\Pi(n^*, T^*) = 1824.12$$

The largest portion of the total cost is purchasing cost and the rest of the costs including ordering, holding, and waste treatment costs have smaller portions. Concavity of total profit function is illustrated in figure 1.

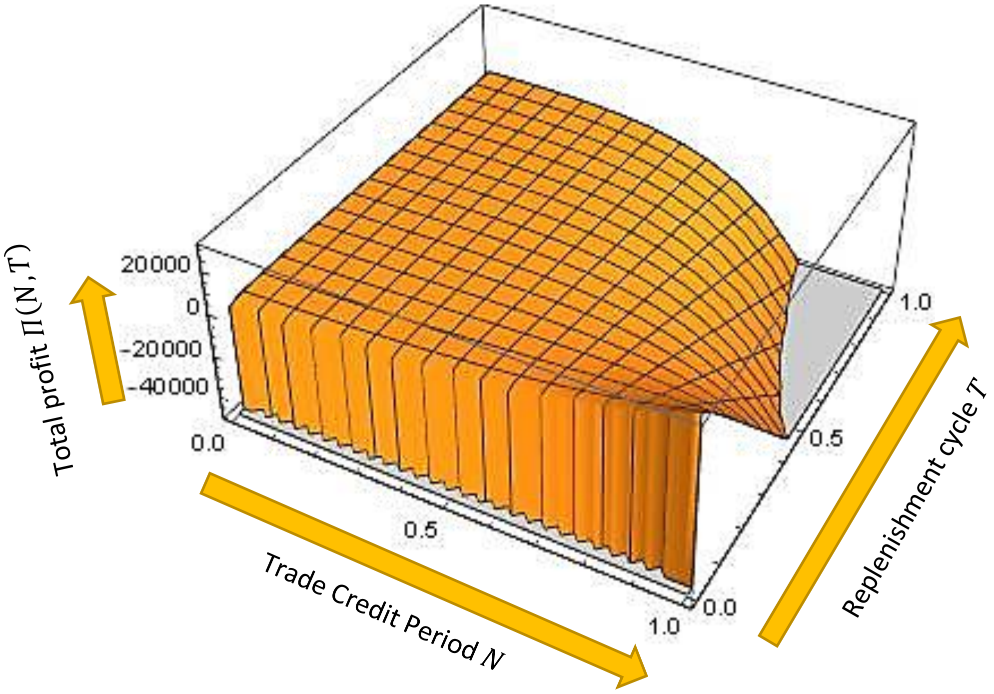


Fig 1. concavity of $\Pi(N^*, T^*)$ with respect to N and T

7-Sensitivity analysis

We have studied sensitivity analysis on the optimal solution with respect to each parameter in an appropriate unit using the same data as those in the numerical experiments in Table 3.

Table 3. Sensitivity analysis

Parameter	Value	n^*	T^*	$\Pi(n^*, T^*)$
a	5	0.041	0.584	1824.12
	6	0.118	0.455	1986.44
	8	0.194	0.297	2764.85
b	1	0.842	0.079	16293.9
	2	0.271	0.329	2511
	3	0.041	0.584	1824.12
k	1000	0.041	0.584	1824.12
	2000	0.044	0.41	3688.45
	3000	0.045	0.334	5559.52
p	3	0.041	0.584	1824.12
	4	0.139	0.457	2969.96
	5	0.215	0.378	4329.46
	0.4	0.331	0.283	3246.47

Table 3. Continued

Parameter	Value	n^*	T^*	$\Pi(n^*,T^*)$
c	0.7	0.155	0.439	2289.64
	1	0.041	0.584	1824.12
o	15	0.042	0.504	1833.3
	20	0.041	0.584	1824.12
	25	0.040	0.655	1816.06
h	0.1	0.041	0.584	1824.12
	0.2	0.037	0.417	1795.93
	0.3	0.034	0.343	1774.48
α	0.01	0.041	0.584	1824.12
	0.02	0.031	0.615	1693.92
	0.03	0.022	0.648	1568.87
C_p	0.001	0.041	0.584	1824.12
	0.002	0.032	0.598	1789.86
	0.003	0.022	0.612	1757.17
COD_R	500	0.041	0.584	1824.12
	600	0.038	0.589	1812.52
	700	0.035	0.594	1801.1

Some critical analyses have been explained as follows:
According to figure 2 and figure 3, by increasing in c and k, the amount of T^* rises and n^* decreases. The parameter a can affect the demand positively and its growth causes an increase in demand. Rising in c increases the purchasing cost and as the demand decreases, net revenue falls and total profit will be decreased as well.

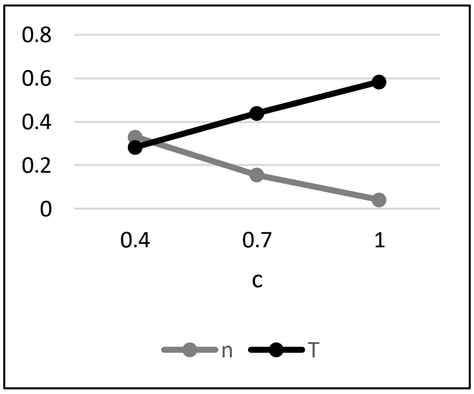


Fig 2. Effect of c on decision variables

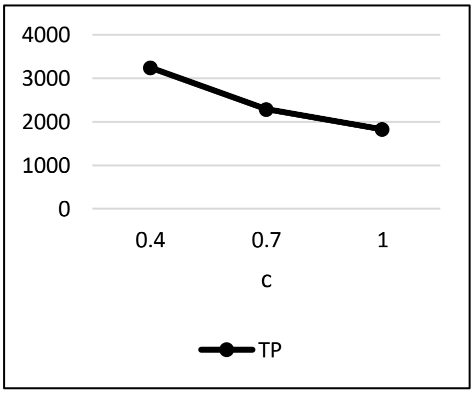
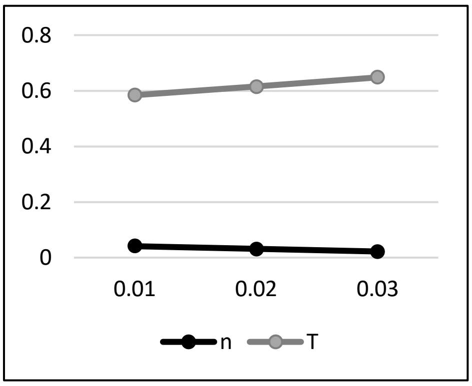


Fig 3. Effect of c on objective function

According to figure 4 and figure 5, as α increases, n^* and $\Pi(n^*,T^*)$ will be decreased and T^* will be increased. A growth in α increases the purchasing cost and as the demand decreases, net revenue falls and total profit will be decreased as well. As the fraction of returning items increases, the demand will be decreased and this causes falling in the total profit function.



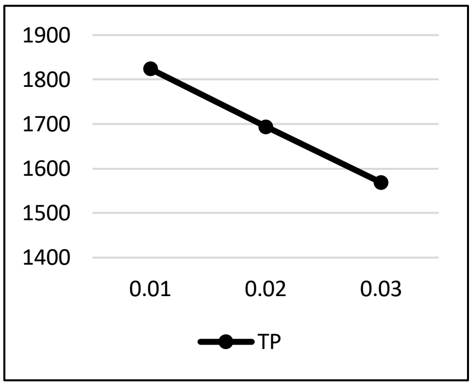


Fig 4. Effect of α on decision variables

Fig 5. Effect of α on objective function

8-Conclusion

Green inventory and the possible purification operation of returning items under the trade credit period has been a vital issue in many industries. In this paper, three concepts including deterioration, delay in payment, and green inventory related to refining of returning items were considered concurrently; such that (1) Items not only deteriorate but also have expiration dates. (2) Offering a trade credit period increases demand and default risk. (3) Items that their expiration date has arrived, will be taken back to the factories and there are purification stations established in the factories for depleting them. In this case, we were seeking for optimal trade credit period and optimal replenishment cycle time. To simplify the model, the model was developed in a special case and solved by Mathematica 11.2, by assigning values to parameters, and then running sensitivity analysis. The results showed that with the long expiration date, the trade credit period and replenishment cycle time will be longer and the total profit will be increased. Moreover, as the trade credit period increases, the total profit will be increased and as the replenishment cycle increases, the total profit will fall. It shows that the higher trade credit period stimulates demand and the higher demand results in more profit. Also, using results from numerical examples and by altering parameters in sensitivity analysis, we observed the effect of each parameter on the total profit function. Rising in parameters which scale up demand (a and k), will also increase the total profit. Unit selling price (p) which is a parameter related to net revenue, increases the total profit. But the growth in the relevant unit cost parameters, decrease the total profit.

For future research, the mathematical model can be extended in several ways such as considering allowable shortages, cash discount, inflation rate, and the time value of money, etc. Also, the demand rate and deterioration rate can be considered stochastic. Dividing the buyers into good-credit and bad-credit buyers can be another recommendation. The trade-credit period should be offered to only the good-credit buyers and the bad-credit buyers should pay immediately. This policy decreases the default risk and causes an increase in demand and total profit.

In conclusion, it should be noted that a practical model by insisting on considering the real-world conditions was developed in this paper. Companies can use the model developed in this paper to obtain the optimal replenishment cycle and trade credit period. Therefore, the number of returned items will be decreased and the amount of purification cost will fall down and as result, the total profit will rise.

Corollary 1

For any given value of m and T, we have

$$\frac{(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) \geq 1 \qquad \text{for all} \qquad T \geq 0 \tag{27}$$

Proof

Let us define

$$f(T) = (1+m)\ln\left(\frac{1+m}{1+m-T}\right) - T \tag{28}$$

Then we have $f(0) = 0$ and

$$\frac{df(T)}{dT} = \frac{1+m}{1+m-T} - 1 > 0 \qquad \text{for all} \qquad T \geq 0 \tag{29}$$

Therefore,

$$f(T) > 0 \qquad \text{for all} \qquad T \geq 0 \tag{30}$$

As a result, we know that

$$\frac{(1+m)}{T}\ln\left(\frac{1+m}{1+m-T}\right) \geq 1 \qquad \text{for all} \qquad T \geq 0 \tag{31}$$

This completes the proof of corollary 1.

Appendix A

Proof of theorem 1.

For simplicity, let's use (13) to define

$$F(n)=\frac{d\Pi(n,T)}{dn} = pk(a-b)(1-g\alpha)e^{(a-b)n} - \frac{k}{T}ae^{an}(1-g\alpha)C_p\alpha(COD_R-COD_S)(1+m)\ln\left(\frac{1+m}{1+m-T}\right) - \frac{c}{T}ake^{an}(1-g\alpha)(1+m)\ln\left(\frac{1+m}{1+m-T}\right) - \frac{hake^{an}(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right] \quad \text{(A1)}$$

We then have

$$F(0)=pk(a-b)(1-g\alpha)-\frac{k}{T}a(1-g\alpha)C_p\alpha(COD_R-COD_S)(1+m)\ln\left(\frac{1+m}{1+m-T}\right)-\frac{c}{T}ak(1-g\alpha)(1+m)\ln\left(\frac{1+m}{1+m-T}\right)-\frac{hak(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]\neq 0 \quad \text{(A2)}$$

and

$$F(\infty)=\lim_{n\to\infty} ke^{an}\left\{pk(a-b)e^{-bn}(1-g\alpha)-\frac{a(1-g\alpha)\alpha}{T}C_p\alpha(COD_R-COD_S)(1+m)\ln\left(\frac{1+m}{1+m-T}\right)-\frac{ca(1-g\alpha)}{T}(1+m)\ln\left(\frac{1+m}{1+m-T}\right)-\frac{ha(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]\right\}=-\infty \quad \text{(A3)}$$

Simplifying (13) and using corollary 1, we will obtain

$$\frac{dF(n)}{dn}\leq pk(a-b)^2(1-g\alpha)e^{(a-b)n}-\frac{k}{T}a^2e^{an}(1-g\alpha)C_p\alpha(COD_R-COD_S)(1+m)\ln\left(\frac{1+m}{1+m-T}\right)-\frac{c}{T}a^2ke^{an}(1-g\alpha)(1+m)\ln\left(\frac{1+m}{1+m-T}\right)-\frac{ha^2ke^{an}(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2}\ln\left(\frac{1+m}{1+m-T}\right)+\frac{T^2}{4}-\frac{(1+m)T}{2}\right]$$
$$\leq pk(a-b)^2(1-g\alpha)e^{(a-b)n}-a^2ke^{an}(1-g\alpha)C_p\alpha(COD_R-COD_S)-ca^2ke^{an}(1-g\alpha)\leq 0 \quad \text{(A4)}$$
$$If \quad (a-b)^2p-a^2C_p(COD_R-COD_S)-ca^2\leq 0$$

Consequently, if $\Delta(n)\leq 0$ then $F(0)\leq 0$. From (A2) - (A4), we know that $F(n)\leq 0$ for all $n\geq 0$. Therefore, $\Pi(n,T)$ is decreasing in n, and maximized at $n^*=0$. Otherwise if $\Delta(n)>0$ (i.e., $(0)>0$), we use the Mean Value theorem into $F(0)>0$ and $F(\infty)>-\infty$, there exists a unique $n^*>0$ such that $F(n^*)>0$ and hence $\Pi(n,T)$ is maximized at $n^*>0$. This completes the proof of theorem 1.

Appendix B

Proof of theorem 2.

For simplicity, let's use (15) to define

$$G(T)=\frac{d\Pi(n,T)}{dT}$$
$$=\frac{o}{T^2}-\frac{cke^{an}(1-g\alpha)(1+m)}{(1+m-T)T}+\frac{cke^{an}(1-g\alpha)(1+m)}{T^2}\ln\left(\frac{1+m}{1+m-T}\right)-\frac{ke^{an}(1-g\alpha)C_p\alpha(COD_R-COD_S)(1+m)}{(1+m-T)T}+\frac{ke^{an}(1-g\alpha)C_p\alpha(COD_R-COD_S)(1+m)}{T^2}\ln\left(\frac{1+m}{1+m-T}\right)-\frac{hke^{an}(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2(1+m-T)\mathrm{T}}+\frac{T}{4}-\frac{(1+m)^2}{2T}\ln\left(\frac{1+m}{1+m-T}\right)\right] \quad \text{(B1)}$$

Then we have

$$G(m)=\frac{1}{m^2}\left\{o-cke^{an}(1-g\alpha)(1+m)[m-\ln(1+m)]-ke^{an}(1-g\alpha)C_p\alpha(COD_R-COD_S)(1+m)[m-\ln(1+m)]-\frac{hke^{an}}{4}[m^2-2(1+m)^2\ln(1+m)+2(1+m)^2]\right\} \quad \text{(B2)}$$

and

$$G(0)=\lim_{T\to 0}G(T)$$
$$=\lim_{T\to 0}\{\frac{o}{T^2}-\frac{cke^{an}(1-g\alpha)(1+m)}{(1+m-T)T}+\frac{cke^{an}(1-g\alpha)(1+m)}{T^2}\ln\left(\frac{1+m}{1+m-T}\right)-\frac{ke^{an}(1-g\alpha)C_p\alpha(COD_R-COD_S)(1+m)}{(1+m-T)T}+\frac{ke^{an}(1-g\alpha)C_p\alpha(COD_R-COD_S)(1+m)}{T^2}\ln\left(\frac{1+m}{1+m-T}\right)-\frac{hke^{an}(1-g\alpha)}{T}\left[\frac{(1+m)^2}{2(1+m-T)\mathrm{T}}+\frac{T}{4}-\frac{(1+m)^2}{2T}\ln\left(\frac{1+m}{1+m-T}\right)\right]\} \quad \text{(B3)}$$
$$=\infty$$

Simplifying (15) and using corollary 1, we will obtain

$$\frac{dG(T)}{dT} = \frac{d^2\Pi(n,T)}{dT^2} = $$
$$= -\frac{2o}{T^3} - cke^{an}(1-g\alpha)\left[-\frac{2(1+m)}{(1+m-T)T^2} + \frac{(1+m)}{T(1+m-T)^2} + \frac{2(1+m)\ln\left(\frac{1+m}{1+m-T}\right)}{T^3}\right]$$
$$- ke^{an}(1-g\alpha)C_p\alpha(COD_R - COD_S)\left[-\frac{2(1+m)}{(1+m-T)T^2} + \frac{(1+m)}{T(1+m-T)^2} + \frac{2(1+m)\ln\left(\frac{1+m}{1+m-T}\right)}{T^3}\right]$$
$$- hke^{an}(1-g\alpha)\left[\frac{2(1+m)^2}{T^3}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{2(1+m)^2}{(1+m-T)T^3} + \frac{(1+m)^2}{(1+m-T)^2T^2} - \frac{(1+m)^2}{(1+m-T)T^2}\right] \quad \text{B4)}$$

In order to prove theorem 2, we assume that the term in the brackets is positive, then, for any given n and T, this Theorem is proved. Therefore, we have

$$-\frac{2(1+m)}{(1+m-T)T^2} + \frac{(1+m)}{T(1+m-T)^2} + \frac{2(1+m)\ln\left(\frac{1+m}{1+m-T}\right)}{T^3} \geq 0 \quad \text{(B5)}$$

And

$$\frac{2(1+m)^2}{T^3}\ln\left(\frac{1+m}{1+m-T}\right) - \frac{2(1+m)^2}{(1+m-T)T^3} + \frac{(1+m)^2}{(1+m-T)^2T^2} - \frac{(1+m)^2}{(1+m-T)T^2} \geq 0 \quad \text{(B6)}$$

Simplifying B5 and B6, we get

$$\ln\left(\frac{1+m}{1+m-T}\right) \geq \frac{(2+2m-3T)T}{2(1+m-T)^2} \quad \text{(B7)}$$
$$\ln\left(\frac{1+m}{1+m-T}\right) \geq \frac{(2+2m-3T)+(1+m-T)T}{2(1+m-T)^2} \quad \text{(B8)}$$

Consequently, if $\Delta(T) \geq 0$ then $G(m) \geq 0$. From (B1)-(B8) we know that $G(T) \geq 0$ for all $T \leq m$. Therefore, $\Pi(n,T)$ is increasing in T, and maximized at $T^* = m$. Otherwise if $\Delta(T) \leq 0$ (i.e., $G(m) \geq 0$), we use the Mean Value theorem into $G(m) = \infty$ and $G(m) \leq 0$, there exists a unique $T^* \in (0,m)$ such that $G(T^*) = 0$ and hence $\Pi(n,T)$ is maximized at unique $\Pi(n,T)$. This completes the proof of theorem 2.

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