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Solving linear equation system based on Z-numbers using big-M method

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Abstract

In real world, many decisions are made at any given moment, usually with uncertainty. Although there are many ways and tools to overcome these uncertainties, a powerful tool can be Z-numbers. In this study, inspiring Otadi-Mosleh researches and Big-M method, an extended model is proposed to solve the Z-number matrix equation. Also, numerical examples are provided to show the performance of this model.

Keywords: fuzzy concept, Z-number, Big-M method, matrix equation.

1-Introduction

Linear equation system appears in many various fields such as mathematics, physics, statistics, and engineering. Therefore, discovering an exact solution for this system of equations is so necessary. Since in many applications, data might be unrealistic and uncertain, fuzzy data is being used to describe the parameters. The system of linear equation $A\widetilde{X} = \tilde{b}$ for which the coefficients, a_{ij} of the matrix A are crisp and the elements $\widetilde{x}_i$ and $\tilde{b}_i$ of the vectors $\widetilde{X}$ and $\tilde{b}$ are fuzzy number-valued, is called Fuzzy System of Linear Equation (FSLE) and the linear system $\widetilde{A}\widetilde{X} = \tilde{b}$ for which all the coefficients a_{ij} of the matrix A and the elements $\widetilde{x}_i$ and $\tilde{b}_i$ of the vectors $\widetilde{X}$ and $\tilde{b}$ are considered to be fuzzy numbers, is called Fully Fuzzy Linear System (FFLS). Many authors studied these systems and presented various methods to solve them. For example, (Friedman et al., 1998) proposed a general method to find a solution for FSLE problems. LU decomposition method and the conjugate gradient were proposed (Abbasbandy et al., 2006; Abbasbandy et al., 2005; Asadi et al., 2005) to solve a general symmetric fuzzy linear system. For solving the dual linear system of the form $\widetilde{x} = A\widetilde{x} + \widetilde{u}$, where A is real $n \times n$ - matrix and $\widetilde{x}$ is the unknown vector and $\tilde{u}$ is the constant vector which both are fuzzy numbers, an iterative algorithm is presented (Wang et al., 2001). (Abbasbandy et al., 2008) studied the existence of a minimal solution for the system of the form $A\widetilde{x} + \widetilde{f} = B\widetilde{x} + \tilde{c}$ where A and B are real matrices, $\widetilde{x}$ is an unknown vector and $\widetilde{f}$, and $\tilde{c}$ are constant fuzzy numbers. (Ghanbari et al., 2010) obtained a solution by ranking function for the fuzzy linear system.

Also, (Muzzilo et al., 2006) considered FFLS of the form $\widetilde{A}_1\widetilde{x} + \tilde{b}_1 = \widetilde{A}_2\widetilde{x} + \tilde{b}_2$ where $\widetilde{A}_1$, $\widetilde{A}_2$ are square matrices of fuzzy coefficients and $\tilde{b}_1$, $\tilde{b}_2$ are fuzzy numbers. (Dehghan et al., 2006) presented FFLS of the

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form $\widetilde{A} \otimes \widetilde{x} = \widetilde{b}$ where $\widetilde{A}$ is a positive fuzzy matrix, $\widetilde{x}$ is unknown and $\widetilde{b}$ is a known positive fuzzy vector. (Kumar et al., 2011; Otadi and Mosleh, 2012) found an exact solution of FFLS by solving a Linear Programming (LP).

In most studies using LP method, it is essential to discover an exact solution for this system of equations. In addition, due to the action and teamwork in presenting data, it is often not possible to consider the obtained answers certainly. Therefore, in order to achieve more stable results against the comments of different people, it is necessary for the data to be done according to the uncertainty in the criteria. Also, in order to determine the validity of the results, the concept of reliability along with uncertainty for the data can be used. Z-numbers verify these aspirations over conventional fuzzy numbers.

(Otadi and Mosleh, 2012) found an exact solution of Fully Fuzzy Matrix Equation (FFME), however they did not consider the reliability of the data in their model. The advantage of using Z-numbers as parameters in the proposed model is considering the uncertainty in the opinion of experts and allocating credit in their notion. Therefore, the Z-numbers prioritize to the other generalization of fuzzy sets.

The main purpose of the proposed approach is to overcome some of the main deficiencies of the conventional fuzzy method, i.e., the FFME, which have been outlined by (Otadi and Mosleh, 2012) and other relevant studies. For this reason, in this paper, by motivating the Otadi-Mosleh method in solving FFME, an extended model is presented to solve the Z-number Matrix Equation (ZME).

The rest of the paper is organized as follows: In section 2, some necessary and useful results of fuzzy set theory are reviewed, in section 3, standard form of FFME is presented and different cases that $\widetilde{x}$ might be the solution to this equation are investigated. The concept of Z-number is proposed in Section 4. a brief description of Kang's model is expressed for converting Z-numbers to classical fuzzy numbers. Finally, in Section 5, a general form of ZME is proposed and then an extended model is stated to solve the ZME. In the way that, first using the Kang's method, Z-number matrices transform to fuzzy number matrices, then motivating Otadi-Mosleh and Big-M method, an extended model is proposed to solve the ZME. In the proposed method, to reduce the calculations and increase the computational speed, the FFME is converted to ZME; however this approach loses some information which can be considered as disadvantages of this method.

2-Prelimininaries

Fuzzy set and number theory were first introduced by (Zadeh, 1965). Since then, many researchers studied the properties and applications of fuzzy numbers (Celikyilmaz and Turksen, 2009; Coppi et al., 2006; Jain and Martin, 1998; Nguyen and Sugeno, 2012; Zhang and Lio, 2006). It was undeniable that most of the phenomena in the real world deal with uncertainty. Fuzzy set theory as a beneficial tool manages uncertainty and vagueness. In this section, essential concepts of fuzzy set theory are introduced. To analyze the data by fuzzy logic, the fuzzy membership function is needed:

Definition 2.1. (Kaufmann and Gupta, 1985). The characteristic function μ_A of a crisp set $A \subseteq X$ assigns a value either 0 or 1 to each member in X. This function can be generalized to a function $\mu_{\widetilde{A}}$ such that the value assigned to the element of the universal set X falls within a specified range i.e. $\mu_{\widetilde{A}} : X \to [0,1]$. The assigned value indicates the membership grade of the element in the set A.

The function $\mu_{\widetilde{A}}$ is called the membership function and the set $\widetilde{A} = \{(x, \mu_{\widetilde{A}}(x); x \in X\}$ defined by $\mu_{\widetilde{A}}(x)$ for each $x \in X$ is called a fuzzy set.

Definition 2.2 (Kaufmann and Gupta, 1985). A favorite fuzzy number $\widetilde{A} = (a,b,c)$ is said to be a triangular fuzzy number if its membership function is given by

$$\mu_{\widetilde{A}}(x)=\begin{cases}\dfrac{x-a}{b-a}, & a\le x\le b,\\ \dfrac{c-x}{c-b}, & b\le x\le c,\ .\\ 0, & \text{otherwise,}\end{cases}$$

Definition 2.3 (Najafi et al., 2016). An unrestricted fuzzy number is of the form $\widetilde{A}=(a,b,c)$ where $a,b,c\in\mathbb{R}$. The set of unrestricted fuzzy numbers can be represented by $F(\mathbb{R})$.

Definition 2.4 (Kaufmann and Gupta, 1985). A nonnegative triangular fuzzy number is of the form $\widetilde{A}=(a,b,c)$ if and only if $a\ge 0$. The set of all these triangular fuzzy numbers are denoted by $F(\mathbb{R}^{+})$.

Definition 2.5 (Kaufmann and Gupta, 1985). Two triangular fuzzy numbers $\widetilde{A}=(a,b,c)$, $\widetilde{B}=(e,f,g)$ are said to be equal, $\widetilde{A}=\widetilde{B}$, if and only if $a=e,b=f,c=g$.

Definition 2.6 (Kaufmann and Gupta, 1985). The arithmetic operations between two triangular fuzzy numbers are presented as follows: Let $\widetilde{A}=(a,b,c)$ and $\widetilde{B}=(e,f,g)$ be two triangular fuzzy numbers and $k\in\mathbb{R}$.

(i) $k\ge 0, k\widetilde{A}=(ka,kb,kc)$,

(ii) $k\le 0, k\widetilde{A}=(kc,kb,ka)$,

(iii) $\widetilde{A}\oplus\widetilde{B}=(a,b,c)\oplus(e,f,g)=(a+e,b+f,c+g)$,

(iv) $\widetilde{A}\ominus\widetilde{B}=(a,b,c)\ominus(e,f,g)=(a-g,b-f,c-e)$,

(v) Let $\widetilde{A}=(a,b,c)$ be any triangular fuzzy number and $\widetilde{B}=(e,f,g)$ be a nonnegative one. If the fuzzy multiplication is denoted by $\hat{*}$ (Feuring and Lippe, 1995), then

$$\widetilde{A}\hat{*}\widetilde{B}=\begin{cases}(ae,bf,cg), & a\ge 0,\\ (ag,bf,cg), & a<0,c\ge 0,\\ (ag,bf,ce), & c<0.\end{cases}$$

Definition 2.7 (Dubois and Prade, 1980). A matrix $\widetilde{A}=(\tilde{a}_{ij})$ is called a fuzzy number matrix, if each element of $\widetilde{A}$ is a fuzzy number. $\widetilde{A}$ will be a positive (negative) fuzzy matrix and denoted by $\widetilde{A}>0$ $(\widetilde{A}<0)$ if each element of $\widetilde{A}$ is positive (negative). Similarly, non-negative and non-positive fuzzy matrices are defined.

3-Fully fuzzy matrix equation

A matrix system such as

$$\begin{pmatrix}\tilde{a}_{11} & \dots & \tilde{a}_{1n}\\ \vdots & \ddots & \vdots\\ \tilde{a}_{n1} & \cdots & \tilde{a}_{nn}\end{pmatrix}\begin{pmatrix}\tilde{x}_{11} & \dots & \tilde{x}_{1n}\\ \vdots & \ddots & \vdots\\ \tilde{x}_{n1} & \cdots & \tilde{x}_{nn}\end{pmatrix}=\begin{pmatrix}\tilde{b}_{11} & \dots & \tilde{b}_{1n}\\ \vdots & \ddots & \vdots\\ \tilde{b}_{n1} & \cdots & \tilde{b}_{nn}\end{pmatrix},$$

where $\widetilde{a}_{ij}$, $1 \le i, j \le n$ are arbitrary triangular fuzzy numbers, the elements $\widetilde{b}_{ij}$, are fuzzy numbers and the unknown elements $\widetilde{x}_{ij}$, are non-negative ones, is called a General Fuzzy Matrix Equation (GFME) (Otadi and Mosleh, 2012). A fuzzy number matrix $\widetilde{x} = (\widetilde{x}_1, \widetilde{x}_2, ..., \widetilde{x}_n)$ is a solution of FFME

$$\widetilde{A} \hat{*} \widetilde{X} = \widetilde{B}. \tag{3.1}$$

If $\widetilde{A} \hat{*} \widetilde{x}_j = \widetilde{b}_j$; $1 \le j \le n$,

where $\widetilde{x}_j = ((y_{1j}, x_{1j}, z_{1j}), (y_{2j}, x_{2j}, z_{2j}), ..., (y_{nj}, x_{nj}, z_{nj}))^T$; $1 \le j \le n$,

and $\widetilde{b}_j = ((g_{1j}, b_{1j}, h_{1j}), (g_{2j}, b_{2j}, h_{2j}), ..., (g_{nj}, b_{nj}, h_{nj}))^T$; $1 \le j \le n$, are the j[th] columns of the fuzzy matrices $\widetilde{X}$ and $\widetilde{B}$, respectively. If in the $n \times n$ FFME (3.1), each element of $\widetilde{A}$, $\widetilde{X}$ and $\widetilde{B}$ is a non-negative fuzzy number, then the system (3.1) is called a non-negative FFME. Considering $\widetilde{A} = (M, A, N)$ where M, A, and N are three crisp matrices with the same size of $\widetilde{A}$, $\widetilde{X} = (Y, X, Z)$ where Y, X, Z are three crisp matrices with the same size of $\widetilde{X}$ and $\widetilde{B} = (G, B, H)$ where G, B, H are also three crisp matrices with the same size of $\widetilde{B}$, then $\widetilde{X}$ is called a solution of (3.1) if:

$$\begin{cases} MY = G, \\ AX = B, \\ NZ = H. \end{cases}$$

If $Y \ge 0, X - Y \ge 0$ and $Z - X \ge 0$, then $\widetilde{X}$ is said a consistent solution of the non-negative FFME (3.1).

Theorem 2.1 (Otadi and Mosleh, 2012). Let $\widetilde{A} = (M, A, N) \ge 0$, $\widetilde{B} = (G, B, H) \ge 0$, and each of the matrices M, A, N be a product of a permutation matrix by a diagonal one. Also, let $M^{-1}G \le A^{-1}B \le N^{-1}H$. Then, the non-negative FFME (3.1) has a non-negative consistent fuzzy solution.

4-Z-number theory

In real life, most of the information is uncertain and indefinite. To overcome this uncertainty, (Zadeh, 2011) defined a new theory based on uncertainty and named it the theory of Z-numbers. A Z-number denoted with "Z" consists of an ordered pair $(\widetilde{A}, \widetilde{B})$, of fuzzy numbers, where" $\widetilde{A}$ " is a restriction on a real variable such as X and" $\widetilde{B}$ " shows the reliability of the first component. Examples are presented below to comprehend the concept:

The price of a house: (Approximately 2 million dollars, very likely).

The temperature in autumn: (Medium, usually).

Since, most of the phenomena can be explained by Z-numbers, expert's preferred Z numbers over fuzzy numbers and they quickly applied their achievements to various sciences (Abbasi et al., 2020; Daryakenari et al., 2020). (Akbarian Sarvari et al., 2019) presented a new approach based on Z-number Data Envelopment Analysis (DEA) model to control uncertainty. (Azadeh et al., 2013; Bobar et al., 2020) used Z-numbers in Analytical Hierarchy Process (AHP) and introduced the Z-AHP concept. (Sadi-Nezhad and Sotoudeh-Anvari, 2016) proposed a new DEA model in indefinite cases called Z-DEA by using Z-numbers. (Aliev and Zeinalova, 2014) obtained some direct calculations based on Z-numbers. (Aliev et al., 2015) used Z-numbers in LP problems. (Jafari et al., 2017) solved fuzzy equations based on Z-numbers using neural networks. (Jafari et al., 2020) modeled fuzzy nonlinear system with Z-number coefficients. (Kang et al., 2012) suggested a method to convert Z-numbers to regular fuzzy numbers. This method was also used in many other types of research. In this research, Kang's method is also used to convert the ZME into FFME.

The following is a brief explanation of Kang's method for converting Z-numbers to classical fuzzy numbers:

4-1- Suggested method for converting Z-numbers to fuzzy numbers

Since working with regular fuzzy numbers is easier than Z-numbers, (Kang et al., 2012) suggested a model for converting Z-numbers to regular fuzzy numbers. Although this approach loses some information but it reduces calculations.

Consider a Z-number $Z=(\widetilde{A},\widetilde{B})$, where $\widetilde{A}=\{(x,\mu_{\widetilde{A}}(x))\mid x\in[0,1]\}$ and $\widetilde{B}=\{(x,\mu_{\widetilde{B}}(x))\mid x\in[0,1]\}$, $\mu_{\widetilde{A}}(x)$ and $\mu_{\widetilde{B}}(x)$ are triangular membership functions. To convert a Z-number to a regular fuzzy number, the following three steps are suggested:

Step 1. First, convert the second part (reliability) into a crisp number.

$$\alpha=\frac{\int x\,\mu_{\widetilde{B}}(x)dx}{\int \mu_{\widetilde{B}}(x)dx}, \tag{4.1}$$

where $\int$ indicates an algebraic integration.

Step 2. Second, add the weight of the second part (reliability) to the first part (restriction). The weighted Z-number can be written as

$$\widetilde{Z}^{\alpha}=\{(x,\mu_{\widetilde{A}^{\alpha}}(x))\mid \mu_{\widetilde{A}^{\alpha}}(x)=\alpha\mu_{\widetilde{A}}(x),\ x\in[0,1]\}. \tag{4.2}$$

Step 3. Lastly, convert the irregular fuzzy number to a regular fuzzy number. The regular fuzzy set is given by

$$\widetilde{Z}^{'}=\{(x,\mu_{\widetilde{Z}^{'}}(x))\mid \mu_{\widetilde{Z}^{'}}(x)=\mu_{\widetilde{A}}(\frac{x}{\sqrt{\alpha}}),\ x\in[0,1]\}. \tag{4.3}$$

Table 1. Transformation rules of linguistic variables of reliabilities

Linguistic terms	Membership function
Very Low (VL)	(0,0,0.3)
Low(L)	(0.1,0.3,0.5)
Medium(M)	(0.3,0.5,0.7)
High(H)	(0.5,0.7,0.9)
Very High (VH)	(0.7,1.0,1.0)

Example 4.1. Suppose that one capital market expert believes that the efficiency of one particular stock in the next year can be expressed by a triangular fuzzy number $\widetilde{A}=(0.14,0.25,0.35;1)$. This number indicates that the next year's efficiency of the stock will not be less than 0.14. On the other hand, the efficiency of this stock does not exceed 0.35. Also, the membership number of the efficiency of the stock in 0.25 is equals to 1. Now according to the experience of the expert as well as the predictability of the stock over recent years, one can show the confidence level of his prediction by another triangular fuzzy number such as $\widetilde{B}=(0.5,0.7,0.9)$ (table 1). In other words, the probability of the correctness of the expert's prediction can be expressed by a fuzzy number $\widetilde{B}$. So the Z-number represents the future efficiency of this stock as:

$Z=(\widetilde{A},\widetilde{B})=((0.14,0.25,0.35),(0.5,0.7,0.9))$

Now, this number can be converted to a fuzzy number by the method described in section 3. To do this the following steps are taken.

Step 1. First, according to (4.1) the second part (reliability) is converted into a crisp number

$$\alpha = \frac{\int_{0.5}^{0.7} x\,(\frac{x-0.5}{0.7-0.5})dx + \int_{0.7}^{0.9} x\,(\frac{0.9-x}{0.9-0.7})dx}{\int_{0.5}^{0.7} (\frac{x-0.5}{0.7-0.5})dx + \int_{0.7}^{0.9} (\frac{0.9-x}{0.9-0.7})dx} = 0.7.$$

Step 2. Second, the weight of reliability is added to the constraint

$$\widetilde{Z}^{\alpha} = (0.14, 0.25, 0.35; 0.7).$$

Step 3. Lastly, the weighted Z-number is converted to a regular fuzzy number as proposed in (4.3)

$$\widetilde{Z}^{'} = (\sqrt{0.7}\times 0.14, \sqrt{0.7}\times 0.25, \sqrt{0.7}\times 0.35) = (0.12, 0.21, 0.29).$$

5-Proposed method to find the fuzzy optimal solution for the ZME

Definition 5.1. A matrix $A = [(\tilde{u}_{ij}, \tilde{r}_{ij})]$ is called a Z-number matrix if each element of A is a Z -number. A matrix equation system such as

$$A \hat{*} \widetilde{X} = B\ , \tag{5.1}$$

where $A = [(\tilde{u}_{ij}, \tilde{r}_{ij})]_{n\times n}$, $B = [(\tilde{e}_{ij}, \tilde{f}_{ij})]_{n\times n}$ are Z-number matrices and $\widetilde{X} = [\tilde{x}_{ij}]_{n\times n}$ is a non-negative fuzzy number matrix, is called a ZME.

In this section, an extended method is proposed to find a fuzzy optimal solution for the ZME (5.1). In the way that, first using the Kang's method, Z-number matrices are transformed to fuzzy number matrices, then motivating Otadi-Mosleh and Big-M method, an extended model is proposed to solve the ZME (5.1), in the following steps:

Step 1. Using equations (4.1), (4.2) and (4.3), first the Z-numbers $(\tilde{u}_{ij}, \tilde{r}_{ij})$ and $(\tilde{e}_{ij}, \tilde{f}_{ij})$ are converted to fuzzy numbers $\tilde{a}_{ij}$ and $\tilde{b}_{ij}$, respectively. So the model (5.1) can be transformed as follows:

$$\widetilde{A} \hat{*} \widetilde{X} = \widetilde{B}, \tag{5.2}$$

where $\widetilde{X} = [\tilde{x}_{ij}]_{n\times n}$, $\widetilde{A} = [\tilde{a}_{ij}]_{n\times n}$, $\widetilde{B} = [\tilde{b}_{ij}]_{n\times n}$, $\tilde{a}_{ij}, \tilde{b}_{ij} \in F(\mathbb{R})$, and $\tilde{x}_{ij} \in F(\mathbb{R}^{+})$.

Step 2. Considering (5.2), if $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_n)$ where

$$\tilde{x}_j = ((y_{1j}, x_{1j}, z_{1j}), (y_{2j}, x_{2j}, z_{2j}), ..., (y_{nj}, x_{nj}, z_{nj}))^T, \quad 1 \le j \le n,$$

and

$$\tilde{b}_j = ((g_{1j}, b_{1j}, h_{1j}), (g_{2j}, b_{2j}, h_{2j}), ..., (g_{nj}, b_{nj}, h_{nj}))^T, \quad 1 \le j \le n,$$

the model (5.2) can be written as follows:

$$\widetilde{A} \hat{*} \tilde{x}_j = \tilde{b}_j, \quad 1 \le j \le n. \tag{5.3}$$

Step 3. Similar to the method of Otadi-Mosleh, consider $\tilde{a}_{ij} = (m_{ij}, a_{ij}, n_{ij})$, $\tilde{x}_{ij} = (y_{ij}, x_{ij}, z_{ij})$ and $\tilde{b}_{ij} = (g_{ij}, b_{ij}, h_{ij})$, also the crisp matrices $M = \{m_{ij}\}$, $A = \{a_{ij}\}$, $N = \{n_{ij}\}$, $X = \{x_{ij}\}, Y = \{y_{ij}\}, Z = \{z_{ij}\}$, $G = \{g_{ij}\}, B = \{b_{ij}\}$ and $H = \{h_{ij}\}$. The model (5.3) can be translated to $(M, A, N)(Y, X, Z) = (G, B, H)$.

Step 4. Assuming $(m_{ik}, a_{ik}, n_{ik})\hat{*}(y_{kj}, x_{kj}, z_{kj}) = (w_{ik}^{(j)}, q_{ik}^{(j)}, u_{ik}^{(j)})$, that $1 \le i, j, k \le n$ where each $(y_{kj}, x_{kj}, z_{kj}) \in F(\mathbb{R}^+)$, the model (5.3) can be written as follows:

$$\sum_{k=1}^{n} (w_{ik}^{(j)}, q_{ik}^{(j)}, u_{ik}^{(j)}) = (g_{ij}, b_{ij}, h_{ij}), \quad 1 \le i \le n. \tag{5.4}$$

Step 5. Using arithmetic operations and the Big-M method, the following LP is proposed, where the artificial variables r_i, $i = 1, 2, ..., 3n^2$ are added.

$$\text{Minimize } \mathrm{M}_1 r_1 + M_2 r_2 + ... + M_{3n^2} r_{3n^2},$$

$$\text{subject to} \begin{cases} \sum_{k=1}^{n} w_{1k}^{(1)} + r_1 = g_{11}, \\ \sum_{k=1}^{n} w_{2k}^{(1)} + r_2 = g_{21}, \\ \vdots \\ \sum_{k=1}^{n} w_{nk}^{(1)} + r_n = g_{n1}, \\ \sum_{k=1}^{n} w_{1k}^{(2)} + r_{n+1} = g_{12}, \\ \vdots \\ \sum_{k=1}^{n} u_{nk}^{(n)} + r_{3n^2} = h_{nn}, \\ y_{kj}, z_{kj}, x_{kj} \ge 0, \; x_{kj} - y_{kj} \ge 0, \\ r_p \ge 0, 1 \le i, j, k \le n, \; 1 \le p \le 3n^2. \end{cases}$$

There are different methods for obtaining these artificial variables. One of these methods consists of minimizing their sum, with respect to the constraints (5.4) and $r_i \ge 0$, $i = 1, 2, ..., 3n^2$. If the FFME (5.2) has a solution, then the optimal value of this problem is zero, where all the artificial variables approach to zero (Bazaraa et al., 1990; Murty, 1984).

Example 5.1. Consider the following ZME as the following:

$$\begin{cases} ((1,\ 2,\ 3),\ (0.7,\ 1,\ 1))\hat{*}\tilde{x}_1 \oplus ((2,\ 3,\ 5),\ (0.7,\ 1,\ 1))\hat{*}\,\tilde{x}_2 = ((4,\ 19,\ 46),\ (0.5,\ 0.7,\ 0.9)), \\ \big((2,\ -1,\ 2),(0.5,\ 0.7,\ 0.9)\big)\hat{*}\tilde{x}_1 \oplus \big((1,\ 2,\ 3),(0.7,\ 1,\ 1)\big)\hat{*}\tilde{x}_2 = \big((13,\ 1,\ 29),(0.7,\ 1,\ 1)\big), \end{cases}$$

where $\tilde{x}_1, \tilde{x}_2 \in F(\mathbb{R}^+)$.

Solution: First, according to (4.1) and (4.3) the parameters of ZME are converted to regular fuzzy numbers. So the given model can be converted to:

$$\begin{cases}(0.95,\ 1.90,\ 2.85)\hat{*}\tilde{x}_1 \oplus (1.90,\ 2.85,\ 4.74)\hat{*}\tilde{x}_2 = (3.35,\ 15.90,\ 38.49),\\(1.67,\ -0.84,\ 1.67)\hat{*}\tilde{x}_1 \oplus (0.95,\ 1.90,\ 2.85)\hat{*}\tilde{x}_2 = (12.33,\ 0.95,\ 27.51),\end{cases}$$

where $\tilde{x}_1, \tilde{x}_2 \in F(\mathbb{R}^+)$. Let $\tilde{x}_1 = (y_1,\ x_1,\ z_1)$ and $\tilde{x}_2 = (y_2,\ x_2,\ z_2)$ then the above model can be written as.

$$\begin{cases}(0.95,\ 1.90,\ 2.85)\hat{*}(y_1,\ x_1,\ z_1)\oplus (1.90,\ 2.85,\ 4.74)\hat{*}(y_2,\ x_2,\ z_2) = (3.35,\ 15.90,\ 38.49),\\(1.67,\ -0.84,\ 1.67)\hat{*}(y_1,\ x_1,\ z_1)\oplus (0.95,\ 1.90,\ 2.85)\hat{*}(y_2,\ x_2,\ z_2) = (12.33,\ 0.95,\ 27.51).\end{cases}$$

Now using the arithmetic operations, the above model can be proposed as:

$$\begin{cases}(0.95y_1 + 1.90y_2,\ 1.90x_1 + 2.85x_2,\ 2.85z_1 + 4.74z_2) = (3.35,\ 15.90,\ 38.49),\\(1.67y_1 + 0.95y_2,\ -0.84x_1 + 1.90x_2,\ 1.67z_1 + 2.85z_2) = (12.33,\ 0.95,\ 27.51).\end{cases}$$

Therefore, the following crisp system is obtained:

$$\begin{cases}0.95y_1 + 1.90y_2 = 3.35,\\1.90x_1 + 2.85x_2 = 15.90,\\2.85z_1 + 4.74z_2 = 38.49,\\1.67y_1 + 0.95y_2 = 12.33,\\-0.84x_1 + 1.90x_2 = 0.95,\\1.67z_1 + 2.85z_2 = 27.51.\end{cases}$$

Now, using the Big-M method the following LP is given:

Minimize

$$M_1r_1 + M_2r_2 + M_3r_3 + M_4r_4 + M_5r_5 + M_6r_6$$

$$\text{subject to}\begin{cases}0.95y_1 + 1.90y_2 + r_1 = 3.35,\\1.90x_1 + 2.85x_2 + r_2 = 15.90,\\2.85z_1 + 4.74z_2 + r_3 = 38.49,\\1.67y_1 + 0.95y_2 + r_4 = 12.33,\\-0.84x_1 + 1.90x_2 + r_5 = 0.95,\\1.67z_1 + 2.85z_2 + r_6 = 27.51,\end{cases}$$

where $r_1, r_2, r_3, ..., r_6, y_1, y_2, x_1 - y_1, x_2 - y_2, z_1, x_1, z_2, x_2 \geq 0$.

The optimal solution of the LP problem is achieved with the usage of Lingo software as $r_1, r_2, r_3, r_5 = 0$, $r_4 = 6.441053$, $r_6 = 4.367278, y_1 = 3.526316$, $y_2 = 0, x_1 = 4.580696$, $x_2 = 2.525150$, $z_1 = 0$, *and* $z_2 = 8.120253$.

Therefore, the fuzzy optimal solution is obtained by
$\tilde{x}_1 = (3.526316,\ 4.580696,\ 0)$ and $\tilde{x}_2 = (0,\ 2.525150,\ 8.120253)$.

Example 5.2. Consider the following ZME as

$$\begin{pmatrix} ((1,\ 2,\ 3),(0.5,\ 0.7,\ 0.9)) & ((1,\ 2,\ 4),(0.7,\ 1,\ 1)) \\ ((2,\ 3,\ 4),(0.7,\ 1,\ 1)) & ((3,\ 4,\ 5),(0.7,\ 1,\ 1)) \end{pmatrix}\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} = \begin{pmatrix} ((3,\ 10,\ 29),(0.7,\ 1,\ 1)) & ((5,\ 16,\ 39),(0.7,\ 1,\ 1)) \\ ((8,\ 18,\ 37),(0.5,\ 0.7,\ 0.9)) & ((13,\ 28,\ 50),(0.7,\ 1,\ 1)) \end{pmatrix},$$

where $\tilde{x}_{11}, \tilde{x}_{12}, \tilde{x}_{21}, \tilde{x}_{22} \in F(\mathbb{R}^+)$.

Solution: Similar to the previous example, first the Z-numbers are converted to regular fuzzy numbers:

$$\begin{pmatrix} (0.84,\ 1.67,\ 2.51) & (0.95,\ 1.90,\ 3.79) \\ (1.90,\ 2.85,\ 3.79) & (2.85,\ 3.79,\ 4.74) \end{pmatrix}\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} \\ \tilde{x}_{21} & \tilde{x}_{22} \end{pmatrix} = \begin{pmatrix} (2.85,\ 9.49,\ 27.51) & (4.74,\ 15.18,\ 37) \\ (6.69,\ 15.06,\ 30.96) & (12.33,\ 26.56,\ 47.43) \end{pmatrix}.$$

Let $\tilde{x}_{11} = (y_{11},\ x_{11},\ z_{11})$, $\tilde{x}_{12} = (y_{12},\ x_{12},\ z_{12})$, $\tilde{x}_{21} = (y_{21},\ x_{21},\ z_{21})$, $\tilde{x}_{22} = (y_{22},\ x_{22},\ z_{22})$, the given ZME can be written as:

$$\begin{cases} (0.84,\ 1.67,\ 2.51)\hat{*}(y_{11},\ x_{11},\ z_{11}) \oplus (0.95,\ 1.90,\ 3.79)\hat{*}(y_{21},\ x_{21},\ z_{21}) = (2.85,\ 9.49,\ 27.51), \\ (0.84,\ 1.67,\ 2.51)\hat{*}(y_{12},\ x_{12},\ z_{12}) \oplus (0.95,\ 1.90,\ 3.79)\hat{*}(y_{22},\ x_{22},\ z_{22}) = (4.74,\ 15.18,\ 37), \\ (1.90,\ 2.85,\ 3.79)\hat{*}(y_{11},\ x_{11},\ z_{11}) \oplus (2.85,\ 3.79,\ 4.74)\hat{*}(y_{21},\ x_{21},\ z_{21}) = (6.69,\ 15.06,\ 30.96), \\ (1.90,\ 2.85,\ 3.79)\hat{*}(y_{12},\ x_{12},\ z_{12}) \oplus (2.85,\ 3.79,\ 4.74)\hat{*}(y_{22},\ x_{22},\ z_{22}) = (12.33,\ 26.56,\ 47.43). \end{cases}$$

Using arithmetic operations, the above model can be transformed to the following model:

$$\begin{cases} (0.84y_{11} + 0.95y_{21},\ 1.67x_{11} + 1.90x_{21},\ 2.51z_{11} + 3.79z_{21}) = (2.85,\ 9.49,\ 27.51), \\ (0.84y_{12} + 0.95y_{22},\ 1.67x_{12} + 1.90x_{22},\ 2.51z_{12} + 3.79z_{22}) = (4.74,\ 15.18,\ 37), \\ (1.90y_{11} + 2.85y_{21},\ 2.85x_{11} + 3.79x_{21},\ 3.79z_{11} + 4.74z_{21}) = (6.69,\ 15.06,\ 30.96), \\ (1.90y_{12} + 2.85y_{22},\ 2.85x_{12} + 3.79x_{22},\ 3.79z_{12} + 4.74z_{22}) = (12.33,\ 26.56,\ 47.43). \end{cases}$$

Therefore, the following crisp model is achieved:

$$\begin{cases} 0.84y_{11} + 0.95y_{21} = 2.85, \\ 1.67x_{11} + 1.90x_{21} = 9.49, \\ 2.51z_{11} + 3.79z_{21} = 27.51, \\ 0.84y_{12} + 0.95y_{22} = 4.74, \\ 1.67x_{12} + 1.90x_{22} = 15.18, \\ 2.51z_{12} + 3.79z_{22} = 37, \\ 1.90y_{11} + 2.85y_{21} = 6.69, \\ 2.85x_{11} + 3.79x_{21} = 15.06, \\ 3.79z_{11} + 4.74z_{21} = 30.96, \\ 1.90y_{12} + 2.85y_{22} = 12.33, \\ 2.85x_{12} + 3.79x_{22} = 26.56, \\ 3.79z_{12} + 4.74z_{22} = 47.43. \end{cases}$$

Now using the Big-M method, the following the LP model is obtained:

Minimize $M_1r_1 + M_2r_2 + ... + M_{12}r_{12}$

$$\text{subject to}\begin{cases} 0.84y_{11} + 0.95y_{21} + r_1 = 2.85, \\ 1.67x_{11} + 1.90x_{21} + r_2 = 9.49, \\ 2.51z_{11} + 3.79z_{21} + r_3 = 27.51, \\ 0.84y_{12} + 0.95y_{22} + r_4 = 4.74, \\ 1.67x_{12} + 1.90x_{22} + r_5 = 15.18, \\ 2.51z_{12} + 3.79z_{22} + r_6 = 37, \\ 1.90y_{11} + 2.85y_{21} + r_7 = 6.69, \\ 2.85x_{11} + 3.79x_{21} + r_8 = 15.06, \\ 3.79z_{11} + 4.74z_{21} + r_9 = 30.96, \\ 1.90y_{12} + 2.85y_{22} + r_{10} = 12.33, \\ 2.85x_{12} + 3.79x_{22} + r_{11} = 26.56, \\ 3.79z_{12} + 4.74z_{22} + r_{12} = 47.43, \end{cases}$$

$$\text{where}\begin{cases} r_1,\cdots, r_{12} \ge 0 \\ y_{1i}\ , y_{2i} \ \ge 0 \\ x_{1i} - y_{1i}\ , x_{2i} - y_{2i} \ \ge 0 \\ z_{1i}\ , x_{1i}\ , z_{2i}\ , x_{2i} \ \ge 0 \end{cases} \text{for } i = 1, 2.$$

The optimal solution of LP problem is obtained as $r_1, r_4, \ldots, r_{12} = 0,\ r_2 = 0.6653684, r_3 = 2.755063, y_{11} = 3,\ y_{21} = 0.3473684, y_{12} = 3.048387,\ y_{22} = 2.294058,$ $x_{11} = 5.284211,\ x_{21} = 0, x_{12} = 7.730723,\ x_{22} = 1.194575,\ z_{11} = 0, z_{21} = 6.531646,\ z_{12} = 1.775530,$ *and* $z_{22} = 8.586654.$

Therefore, the requested fuzzy optimal solution is as

$\tilde{x}_{11} = (3,\ 5.284211,\ 0), \tilde{x}_{12} = (3.048387,\ 7.730723,\ 1.775530), \tilde{x}_{21} = (0.3473684,\ 0,\ 6.531646),$

and $\tilde{x}_{22} = (2.294058,\ 1.194575,\ 8.586654).$

6-Conclusion

In most studies using LP method, it is essential to discover an exact solution for this system of equations. Since the concept of reliability along with uncertainty of the data have not been considered in the conventional models, the authors proposed an extended model of matrix equations based on Z-numbers (ZME) to consider this reliability. To reduce the calculations and increase the computational speed, first the ZME system is transformed to FFME system by Kang' method, then inspiring the method of Otadi and Mosleh and Big-M method, the FFME system solved. The extended model is validated by some benchmark examples. Since this approach loses some information, solving ZME system without converting the Z-number parameters to fuzzy parameters is one of the future goals. Also, the limitation of this study includes the lack of consideration of the causal relationships between the variables, which can be addressed using the fuzzy cognitive map based on the Z-number theory in the future studies.

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