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A model of brand competition for durable goods supply chains in a dynamic framework

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Abstract

Game theory is an efficient tool to represent and conceptualize the problems concerning conflict and competition. In recent years and especially for durable products, competition between domestic and foreign brands for gaining market share has received a considerable attention. This paper study electronic commerce concepts by differential game theory and introduce a novel and comprehensive model for analyzing dynamic durable goods supply chains. Manufacturer of domestic brand as leader of the game announces his wholesale price to his retailer. Then the exclusive retailers of domestic and foreign brands play a Nash differential game in choosing their optimal retail prices and advertising efforts over time. Moreover, online pricing and advertising in a direct sales channel constitute other control variables of the manufacturer. Feedback equilibrium policies for the manufacturer and the retailers are obtained by assuming a linear demand function. A case study and sensitivity analysis are carried out to provide numerical results and managerial insights. We found that there is a reverse relationship between price sensitivity of demand and optimal levels of price and advertising efforts. Increase in advertising effectiveness parameter leads to enhancement of advertising efforts in relative marketing channel, but does not have a significant effect on pricing decisions.

Keywords: Stackelberg differential game, Nash differential game, Durable products, Sales–advertising dynamics, Feedback equilibrium, Electronic commerce.

1. Introduction

With global economic crisis, rising customers' expectations and increasing the number of brands supplied in the market, the problem of competition has become to an important issue for marketing managers and researchers. The typical approach for modeling and analyzing such

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systems is the game theory approaches. Among different types of product categories, durable products market has marked a considerable growth in recent years. These products are known to have useful physical function and relative long product life cycle (Jia & Zhang, 2013). However, by the current trend towards modern urban lifestyle, durable products and especially home appliances and household goods, play a significant role in business environment.

Nowadays electronic marketing is an inescapable necessity for national brands to survive in competitive business environment. Newly published data have suggested that online advertising is undergoing strong growth globally. The internet in 2013 passed newspapers to become the world's second-largest medium, behind TV, according to Zenith Optimedia. The internet now captures one in five ad dollars. Figure 1 provides comparative statistics indicating portion of worldwide major-media spending that went into each medium in 2005 and 2013.

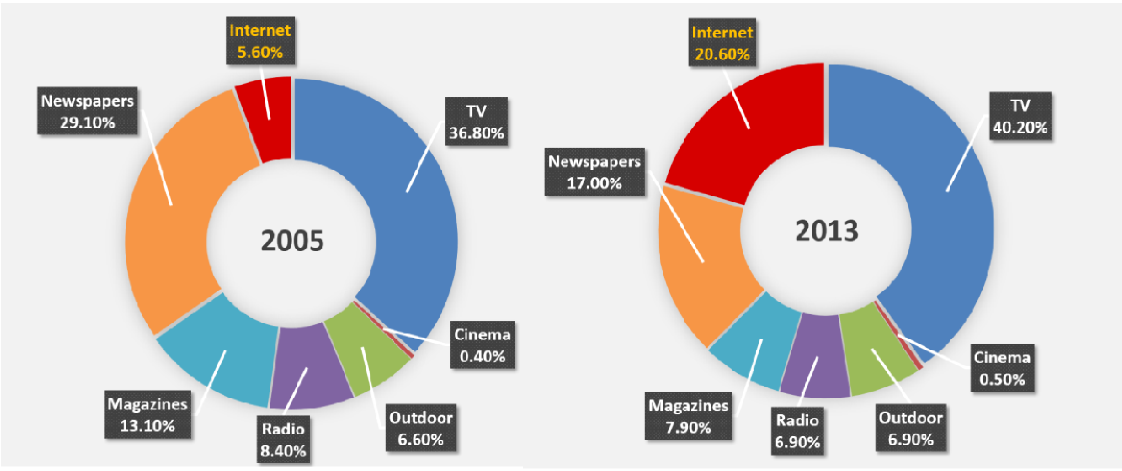

Figure 1. Portion of worldwide major-media spending that went into each medium in 2005 and 2013

Previous papers in differential games subject have rarely considered theelectronic commerce and direct marketing channel in their studies. To the best of our knowledge, there are only three papers addressing this issue. Fruchter and Tapiero(2005) considered a situation where a manufacturer is supplying his products through the Internet as well as selling them through an independent conventional retailer to customers. In fact they extended the setting in (Chiang et al., 2003) to a dynamic framework. Fruchter and Tapiero(2005) assumed consumers' heterogeneity on the acceptance of the online channel, with prices as strategies in a Stackelberg game. Rubel and Zaccour(2007) considered a manufacturer who offers online shopping to his customers as well as using a retail channel. They also assumed a shipment fee for online buyers. Recently Sayadi and Makui(2014) have studied a dual channel competition to investigate advertising strategies for promoting online and retail marketing channels. They modeled a channel formed by a manufacturer and a retailer and investigated the effect of some factors including compatibility factor of product with online marketing on the advertising decisions in the equilibrium. Thus there is no brand competition in (Fruchter & Tapiero, 2005), (Rubel & Zaccour, 2007) and (Sayadi & Makui, 2014) only one manufacturer is supposed. In addition, in none of these models, the product is assumed to be durable. The current research aims at filling this gap by considering both virtual and retail marketing channels, and presence of brand competition in a durable product market.

These are the main contribution of our proposed model besides considering both pricing and advertising as control variables, and both Stackelberg and Nash game settings.

The rest of this paper is organized as follows. In section 2, an overview of developed models with different features is presented. The main differences between cited works and our model are highlighted at the end of this section. Next the problem description and assumptions are addressed in section 3. Section 4 is dedicated to mathematical modeling. The Solving approach and preliminary results are provided in section 5. A case study and its numerical results are included in the sixth section. a sensitivity analysis of outcomes is also conducted to provide managerial insights in this section. Concluding remarks and directions for future research are suggested in section 7.

2.Literature Review

Since the presented model is a comprehensive one, the relative literature is reviewed from different aspects. Modeling durable goods market dynamics has long been paid attention by scholars in economics and management (e.g. (Mansfield, 1961) and (Bass, 1969)). Jørgensen and Zaccour(2004) Insisted that marketing problems must be considered as dynamic problems with strategic aspects. They suggest that chosen marketing instruments, can affect the competitors. Formerly Erickson (1992) had stated that competition in markets has a dynamic nature. Therefore a significant proportion of articles is devoted to differential games theory. This is why we construct our competitive model in dynamic framework.

In Chiang's (2012), “durability” means that each consumer adopts one unit of the product at most. Actually durable goods are those that once purchased by a customer, he does not require an immediate purchase during a lengthy period of time (Sethi et al., 2008). In contrast, consumables and perishable goods, such as grocery items, need to be repeatedly repurchased. The nature of durable goods entails that the market potential depletes with sales over time, and eventually, saturation is reached (Krishnamoorthy et al., 2010). Thus, the untapped market shrinks over time. Under such conditions the target of marketing efforts is to influence the remaining non-adopters.

A seminal model for characterizing the adoption process of a new durable product among a group of potential customers is the Bass diffusion model (Bass, 1969). Earlier proposed models were descriptive in nature, but soon researchers incorporated the effect of decision variables such as pricing and advertising into these models and attempted to issue normative guidelines, Reviewed by Mahajan et al. (1990). Robinson and Lakhani(1975) initiated a research stream in marketing by using the Bass (1969) model to obtain optimal pricing path for a new product. Years later Krishnan et al. (1999) criticized the extant literature and pointed out that in the real world, making pricing decisions based on the sales growth pattern is rarely found. They studied optimal pricing policies of a monopolist and concluded that either a monotonically declining or an increasing-decreasing pricing pattern is optimal. Krishnan et al. (2000) proposed a brand-level diffusion model and investigated the effect of a late entrant on the diffusion process of the existing brands and product category as whole.

Sethi and Bass (2003) extended the Bass (1969) model to capture pricing decisions effect on the cumulative sales of the products. On the other hand, Horsky and Simon (1983) extended the Bass (1969) model to include advertising policies using a nonlinear formulation. Thompson and Teng(1984) incorporated both price and advertising variables into a model of new product adoption in oligopoly setting. They found that the optimal pricing and advertising patterns are high initially and then fall over time. They assume that only the dominant firm determines the price in the market.

Optimal infinite-horizon advertising decisions of a firm supplying a durable product in monopoly setting was analyzed by weber (2005). He proposed two important extension of well-known N-A (Nerlove & Arrow, 1962) and V-W (Vidale & Wolfe, 1957) models. Weber has modified the N-A (Nerlove & Arrow, 1962) model to emphasis the impact of decreasing returns on the advertising effort. The constant coefficient of advertising variable in the V-W (Vidale & Wolfe, 1957) model was also modified to capture evolution of the fraction of consumers who own the products. Chiang (2012) addressed the problem of pricing strategies in a supply chain which consists of a durable product manufacturer and an exclusive retailer. He presented a new dynamics for demand of durable products based on an exhaustible population of customers who differ in their reservation prices. Although Chiang (2012) did not incorporate advertising decisions, but obtained pricing policies in various scenarios such as decentralized or integrated supply chain and the players be myopic or far-sighted.

The majority of prior researches investigated pricing and advertising separately because joint pricing and advertising models lead to high levels of analytical complexity (Chutani & Sethi, 2012). But due to the interaction between pricing and advertising policies, thedecisions on these two variables must be made simultaneously (Chutani and Sethi,2012),therefore, in this research both of theabove variables are considered.Reviewing the extant literature reveals that most of developed models have been constructed to investigate pricing or advertising decisions in a single supply chain. For example Martín-Herrán et al. (2005) addressed a dynamic advertising problem involving two competing manufacturers and a single retailer. However, this paper aims at analyzing the brand competitions in retail and online supply chains.

Rubel and Zaccour(2007) considered a manufacturer who sells his products through a direct and a retail channel. They also assumed a logistics cost of selling online. The retailer tries to attract customer to buy from the offline channel. They finally identified a feedback Nash equilibrium and discussed the results. We pointed out in the former section that Fruchter and Tapiero(2005) and Sayadi and Makui(2014) have also considered the possibility of direct marketing using the Internet, but none of them assumed brand competition in their consideration. However, in real world systems customers face various brands with different features in the market and have the authority to select among them. Moreover, considering durable products is another distinction between our work and relative articles.

In 2008, Sethi et al. (2008) developed a dynamic equation based on two famous models; the Bass model (1969) and the Sethi model (1983). This model which is named the SPH model hereafter represents the rate of change in cumulative sales of a durable product by incorporating both pricing and advertising effects. Sethi et al. (2008) solved an optimal control problem explicitly for a monopolist seeking to maximize his discounted profit over an infinite horizon of time. They used both linear and isoelastic demand specifications and provided comparative statistics that examine the effects of parameter changes on the values of the total market, the optimal price, and the optimal advertising efforts.

The SPH model was extended to duopoly setting by Krishnamoorthy et al. (2010). They also presented two different versions of their model by employing linear and isoelastic demand formulations. Closed-form solutions demonstrated constant optimal price, while optimal advertising follows a decreasing pattern with cumulative sales. More recently Chutani and

Sethi(2012) developed the Competitive version of the SPH model in the context of cooperative advertising, channel coordination and brand level competition. In what follows we develop a novel model by incorporating electronic commerce into the model suggested by Chutani and Sethi(2012). The most relevant papers are summarized in table below to highlight our contribution to the literature.

Table 1. Summary of the most relevant papers and our research contribution to the literature

Paper	System Dynamics	Network Structure	Pricing Decisions	Advertising Decisions	durability of Product	Brand Competition	Online Marketing Channel	Game Type
Sethi et al. (2008)	Cumulative Sales	1 Channel	*	*	*			None
Krishnamoorthy et al. (2010)	Cumulative Sales	2 Channels	*	*	*			Nash
Chutani and Sethi(2012)	Cumulative Sales	2 Channels	*	*	*	*		Nash &Stackelberg
Fruchter and Tapiero(2005)	Probability of Online Purchase	2 Channels	*				*	Stackelberg
Rubel and Zaccour(2007)	Online Market Share	2 Channels		*			*	Nash
Sayadi and Makui(2014)	Market Shares	2 Channels	*	*			*	Nash
Present Research	**Cumulative Sales**	**3 Channels**	*	*	*	*	*	**Nash &Stackelberg**

3. Problem description and assumptions

The goal of this paper is to provide insights into the way optimal pricing and advertising policies change with respect to dynamics of durable goods supply chains including retail and online channels. We consider a domestic manufacturer with national brand who supplies his products to the market through an exclusive retailer that plays the follower role in this Stackelberg game. The manufacturer also employs an online marketing channel to sell his products directly and advertise his brand.

There is another retailer that buys the same durable product with a foreign brand from an importer company and releases them in the market. This means that the retailers are engaged in a Nash game to increase their market shares and improve their benefits. Figure 2 displays the system structure we consider in this research.

Figure 2. Supply chains structure of durable product market including domestic and foreign brands

We also consider the following assumptions for mathematical modeling of the system shown in Figure 2:

1. The market consists of potential customers of a durable product. The sales dynamics equation used in this paper is particularly valid for these types of products. Hence, generalization about the obtained results is not possible.
2. Durable products with different domestic and foreign brands are perfectly substitutable.
3. Demand in each marketing channel is a linear function of its own price.
4. The two retailers play a Nash game to compete in the market and determine their optimal retail prices and advertising efforts. However, in the Stackelberg game between manufacturer and domestic retailer, the manufacturer acts as the leader and the retailer plays the follower role.
5. In our model the wholesale price of the foreign brand and the unit cost of production for the manufacturer, are assumed to be exogenous variables. In addition, issues such as discount rate, advertising effectiveness on sales and price sensitivity of demand are constant parameters of the model.
6. All of three players seek to maximize their discounted profits over an infinite horizon of time.
7. In this model, there are seven control variables. The domestic manufacturer determines the wholesale price to charge his retailer. Moreover, he decides on online price and advertising efforts in the virtual channel. Retail price and level of advertising efforts constitute the control variable of the two retailers.

Next, given the above system structure and assumptions, mathematical modeling and solving procedure are presented.

4. Mathematical modeling

As shown in Figure 2, the network consists of three players using three marketing channels including two retail channels and one online to sell their products in the market. In the rest of the paper the following notations are used:

t: Time; $t \in [0, \infty)$.

i: Indices used to indicate Online (O), Domestic (D) and Foreign (F) marketing channels.$i = O, D, F$.

$X_i(t)$: Cumulative normalized sales related to Online (O), Domestic (D) and Foreign (F) marketing channels at time t; $X_i(t) \in [0, 1]$.

$A_i(t)$: Level of advertising efforts in Online (O), Domestic (D) and Foreign (F) marketing channels at time t.

$W_i(t)$: Wholesale price for retailers selling Domestic (D) and Foreign (F) brands at time t.

$P_i(t)$: Final price for customers in Online (O), Domestic (D) and Foreign (F) marketing channels at time t.

$D_i(P_i)$: Demand of goods sold by the players in Online (O), Domestic (D) and Foreign (F) marketing channels as a function of their own retail prices; $0 \leq D_i(P_i) \leq 1, \partial D_i(P_i)/\partial P_i < 0$.

ρ_i: Advertising effectiveness parameter related to Online (O), Domestic (D) and Foreign (F) marketing channels; $\rho_i > 0$.

r: Discount rate of the manufacturer and the retailers; $r > 0$.

V_j: Value functions related to the manufacturer (M), Domestic Retailer (RD) and Foreign Retailer (RF); $j = M, RD, RF$.

C: Unit production cost for domestic manufacturer.

Furthermore, we use the standard notations:

$Vj_{Xi} \coloneqq \partial Vj/\partial Xi\,,\ \ i = O, D, F\ \ \ j = M, RD, RF$

With respect to Figure 2, in the Stackelberg differential game between domestic manufacturer and his retailer, first the manufacturer as the leader announces his wholesale price policy; $W_D(t)$ at time$t$. Then the exclusive retailer of domestic brand –that acts as the follower in the Stackelberg game- plays a Nash differential game with the exclusive retailer of foreign brand. The retailers determine their optimal pricing and advertising levels to maximize their discounted profits. On the other hand the manufacturer decides on online price and internet advertising efforts in order to attract customers to buy from the electronic shop.

In order to characterize the equilibrium of two differential games mentioned above, first the optimization problems of retailers must be investigated. The formulation of these optimal control problems is somehow the same, so that the retailers seek to maximize the present value of his profit stream over the infinite horizon. Given the wholesale prices of the manufacturer and the importer company, the optimal control problem of retailer$j$has the following formulation:

$$V_j(X_O, X_D, X_F) \coloneqq \max_{P_i(t), A_i(t), t\geq 0} \int_0^{\infty} e^{-rt}\left(\left(P_i(t) - W_i(t)\right)\dot{X}_i(t) - A_i^2(t)\right) dt\,, i = D, F;\ j = RD, RF \tag{1}$$

Subject to

$$\dot{X}_i(t) = {}^{dX_i(t)}/_{dt} = \rho_i A_i(t) D_i\left(P_i(t)\right) * \sqrt{1 - X_O(t) - X_D(t) - X_F(t)}, X_i(0) = X_i \in [0, 1], i = O, D, F \tag{2}$$

And

$$D_i(P_i(t)) = 1 - \eta_i P_i(t), i = O, D, F \tag{3}$$

In fact $P_i(t) - W_i(t), i = D, F$ is the margin of retailers and $V_j(X_O, X_D, X_F)$ can be defined as the value function of retailer $j = RD, RF$. The discount rate r can be considered equal to inflation rate. Assuming the cost of advertising effort to be a convex function implies increasing marginal costs in converting Ainto X. Using a second order function is common in the literature (e.g. (Sethi, 1983) and (Sorger, 1989)).

Equation (2) which is named the state equation in optimal control literature is an extension of the model proposed by Sethi et al. (2008) for durable products. Formerly Krishnamoorthy et al. (2010) had addressed a differential-game extension of SPH model with duopoly setting. Here we considered three channels competing for market share by controlling pricing and advertising variables to build the equation (2). Actually we consider a total market potential of one with the cumulative normalized sales of marketing channel i at time $t$as $X_i(t), i = O, D, F$. So the rate of change of cumulative units sold, which is the instantaneous sales, is denoted by $\dot{X}_i(t)$, and is given by (2). The term $X_O(t) + X_D(t) + X_F(t)$ shows the total cumulative sales of the market at time t. The other terms in (2) are introduced earlier.

Demand formulation may have different forms. However, the linear demand function is very common in the relative literature (e.g., (Sethi et al., 2008), (Krishnamoorthy et al., 2010) and (Krishnan et al., 1999)). Thus the demand specification (3) is considered in which η_i is sensitivity of demand to the price $P_i(t)$ and is a positive constant.

The solution to the Nash differential game defined by (1)-(3) would give the optimal policies for control variables of the retailers including retail prices $P_i^*, i = D, F$ and advertising efforts $A_i^*, i = D, F$ in feedback form. These solutions can be written as below:

$$P_i^*\big(X_D(t), X_F(t) \big| W_D(t), W_F(t)\big), i = D, F \tag{4}$$
$$A_i^*\big(X_D(t), X_F(t) \big| W_D(t), W_F(t)\big), i = D, F$$

The domestic manufacturer anticipates his retailer's optimal responses and incorporates them into his optimization problem, which is a stationary infinite horizon optimal control problem. The manufacturer's problem is formulated as follows:

$$V_M(X_O, X_D, X_F) := \max_{W_D(t), P_O(t), A_O(t), t\geq 0} \int_0^\infty e^{-rt}\Big((P_O(t) - C)\dot{X}_O(t) + (W_D(t) - C)\dot{X}_D(t) - A_O^2(t)\Big)dt \tag{5}$$

Subject to

$$\dot{X}_i(t) = {dX_i(t)}/{dt} = \rho_i A_i(t) D_i\big(P_i(t)\big) * \sqrt{1 - X_O(t) - X_D(t) - X_F(t)}, X_i(0) = X_i \in [0,1], i = O, D, F \tag{6}$$

And

$$D_i(P_i(t)) = 1 - \eta_i P_i(t), i = O, D, F \tag{7}$$

Since C is the unit cost of production for domestic manufacturer and $W_D(t)$ is the wholesale price of domestic brand, the terms $P_O(t) - C$ and $W_D(t) - C$ are margins of the manufacturer in online and retail marketing channels respectively. Equation (6) is the state equation and (7) shows the demand specification. Solving the optimal control problem (5)-(7) determines the optimal policies for online pricing and advertising efforts in the electronic shop as well as the whole price to charge the exclusive retailer. These solutions which are obtained in feedback form are denoted by P_O^*, A_O^* and W_D^* respectively.

The set of optimal feedback policies of the domestic manufacturer and the exclusive retailers of domestic and foreign brands constitute a time-consistent feedback equilibrium of Stackelberg and Nash Games defined by (1)-(7). Then cumulative sales vector and decisions of the players can be obtained by substituting the equilibrium values into the state equation.

5.Solving procedure and preliminary results

In this part since the game type between domestic manufacturer and his retailer has been assumed to be Stackelberg, the optimization problems of the retailers are first analyzed to find their Nash equilibrium decisions on pricing and advertising efforts, given the wholesale prices announced by the manufacturer and the importer company. It should be noted that all of computations in this article have been done using mathematical software.

According to principles of optimal control theory for characterizing feedback solutions, Hamilton-Jacobi-Bellman (HJB) equations for the value functions of the exclusive retailers of domestic and foreign brand i.e., $V_j(X_O, X_D, X_F), j = RD, RF$ are as follows:

$$HJB_j := rV_j(X_O, X_D, X_F) = \max_{P_i(t), A_i(t)} \left(\left(P_i(t) - W_i(t)\right)\dot{X}_i(t) - A_i^2(t) \right) + \left(Vj_{XO}\dot{X}_O(t) \right) + \left(Vj_{XD}\dot{X}_D(t) \right) + \left(Vj_{XF}\dot{X}_F(t) \right), \quad (8)$$
$$i = D, F \quad j = RD, RF$$

Where the term Vj_{Xi} represents a marginal increase in the total discounted profit of retailer j, $j = RD, RF$, with respect to increase in the cumulative sale of channel i, $i = O, D, F$.

Proposition 5.1. According to principles of optimal control theory (Sethi & Thompson, 2000) and differential games theory (Jørgensen & Zaccour, 2004), for given wholesale prices of domestic and foreign brands; W_D and W_F respectively, the optimal feedback pricing and advertising policies of retailer $j, j = RD, RF$ is:

$$P_i^* = \frac{1 + W_i\eta_i - Vj_{Xi}\eta_i}{2\eta_i}, i = D, F \quad j = RD, RF \quad (9)$$

$$A_i^* = \frac{(1 - (W_i - Vj_{Xi})\eta_i)^2 \rho_i \sqrt{1 - X_O - X_D - X_F}}{8\eta_i}, i = D, F \quad j = RD, RF \quad (10)$$

Proof: See Appendix A.

The optimal feedback pricing policies in (9) shows that increase in wholesale prices; $W_i, i = D, F$ leads to retail price increment for both retailers proportional to their price sensitivity of demand; $\eta_i, i = D, F$. On the other hand the equation (10) indicates a reverse relationship between optimal level of advertising efforts of the retailers and the wholesale prices charged by the manufacturer of domestic brand and the importer company of foreign brand. It can be found from (9) that although increase in retailers' marginal benefit with respect to their own cumulative sales makes the retailer prices go down, but motivates the retailers to spend more on advertising.

Generally, equations (9)-(10) demonstrate that decrease in wholesale prices; $W_i, i = D, F$ causes a decrease in prices and advertising efforts in retail level competition which both lead to their market shares improvement. Moreover, when the untapped market, i.e., $1 - X_O - X_D - X_F$ is higher, the advertising efforts are greater to capture the remaining market. This result is consistent with the observation that companies begin to drastically reduce their advertising efforts as the potential customers shrink in the decline stage of the product life cycle.

Now by substituting the optimal policies characterized in (9)-(10) into the equations in (8), the HJB equations for the retailers are rewritten as below:

$$
\begin{aligned}
HJB_{RD} &\coloneqq rV_{RD}(X_O,X_D,X_F) = \frac{81}{16384}\frac{1}{\eta_D^2\eta_O\eta_F}(1-X_O-X_D-X_F) * \\
&\Big[\rho_D^2\eta_O\eta_F(C-VRD_{XD}-VM_{XD})^4\eta_D^4 - 4\rho_D^2\eta_O\eta_F(C-VRD_{XD}-VM_{XD})^3\eta_D^3 \\
&\quad + \Big(\frac{1024}{81}\rho_F^2\eta_O VRD_{XF}(VRF_{XF}-W_F)^3\eta_F^3 \\
&\quad + \frac{1024}{27}\rho_F^2\eta_O VRD_{XF}(VRF_{XF}-W_F)^2\eta_F^2 \\
&\quad + \Big(\frac{-1024}{81}\rho_O^2 VRD_{XO}(C-VM_{XO})^3\eta_O^3 + \frac{1024}{27}\rho_O^2 VRD_{XO}(C-VM_{XO})^2\eta_O^2 \\
&\quad + \Big(6(C-VRD_{XD}-VM_{XD})^2\rho_D^2 - \frac{1024}{27}C\rho_O^2 VRD_{XO} \\
&\quad + \frac{1024}{27}\rho_F^2(VRF_{XF}-W_F)VRD_{XF} + \frac{1024}{27}\rho_O^2 VRD_{XO}VM_{XO}\Big)\eta_O \\
&\quad + \frac{1024}{81}\rho_O^2 VRD_{XO}\Big)\eta_F + \frac{1024}{81}\rho_F^2\eta_O VRD_{XF}\Big)\eta_D^2 \\
&\quad - 4\rho_D^2\eta_O\eta_F(C-VRD_{XD}-VM_{XD})\eta_D + \rho_D^2\eta_O\eta_F\Big]
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
HJB_{RF} &\coloneqq rV_{RF}(X_O,X_D,X_F) = \frac{-1}{16}\frac{1}{\eta_F^2\eta_O\eta_D}(1-X_O-X_D-X_F) * \\
&\Big[\frac{27}{64}\rho_D^2\eta_F^2\eta_O VRF_{XD}(C-VRD_{XD}-VM_{XD})^3\eta_D^3 \\
&\quad - \frac{81}{64}\rho_D^2\eta_F^2\eta_O VRF_{XD}(C-VRD_{XD}-VM_{XD})^2\eta_D^2 \\
&\quad + \Big(\frac{-1}{4}\rho_F^2\eta_O(VRF_{XF}-W_F)^4\eta_F^4 - \rho_F^2\eta_O(VRF_{XF}-W_F)^3\eta_F^3 \\
&\quad + \Big(\rho_O^2 VRF_{XO}(C-VM_{XO})^3\eta_O^3 - 3\rho_O^2 VRF_{XO}(C-VM_{XO})^2\eta_O^2 \\
&\quad + \Big(\Big(\frac{81}{64}(C-VRD_{XD}-VM_{XD})\rho_D^2 VRF_{XD} + 3C\rho_O^2 VRF_{XO} \\
&\quad - \frac{3}{2}\rho_F^2(VRF_{XF}-W_F) - 3\rho_O^2 VRF_{XO}VM_{XO}\Big)\eta_O\Big) - \rho_O^2 VRF_{XO}\Big)\eta_F^2 \\
&\quad - \rho_F^2\eta_O(VRF_{XF}-W_F)\eta_F - \frac{1}{4}\rho_F^2\eta_O\Big)\eta_D - \frac{27}{64}\rho_D^2\eta_F^2\eta_O VRF_{XD}\Big]
\end{aligned}
\tag{12}
$$

According to principles of differential games theory, here by obtaining the optimal solutions of domestic retailer as the follower of the Stackelberg competition, we can go to the leader optimization problem. The Manufacturer of domestic brand takes into account his exclusive retailer's optimal response to his wholesale price, and solves his optimal control problem to determine the optimal wholesale price for the retailer. The HJB equation for the manufacturer's value function presented in (5) is:

$$HJB_M \coloneqq rV_M(X_O, X_D, X_F) = \max_{W_D(t),P_O(t),A_O(t)} \left((P_O(t) - C)\dot{X}_O(t) + (W_D(t) - C)\dot{X}_D(t) - A_O^2(t)\right) + \left(VM_{XO}\dot{X}_O(t)\right) + \left(VM_{XD}\dot{X}_D(t)\right) + \left(VM_{XF}\dot{X}_F(t)\right) \tag{13}$$

Where the term VM_{Xi} represents a marginal increase in the total discounted profit of the manufacturer with respect to increase in the cumulative sale of channel i, $i = O, D, F$. However, since the equilibrium solutions of the retailers are computed in (9)-(10), the HJB equation of the manufacturer in (13) can be rewritten as follows:

$$\begin{aligned} HJB_M \coloneqq rV_M(X_O, X_D, X_F) = \\ \max_{W_D(t),P_O(t),A_O(t)} \Bigg(\frac{1}{16}\frac{1}{\eta_D\eta_F}\Big(& 16\rho_O\eta_D\eta_F A_O(P_O\eta_O - 1)(C - VM_{XO} \\ & - P_O)\sqrt{1 - X_O - X_D - X_F} \\ & - VM_{XF}\rho_F^2\eta_D(VRF_{XF} - W_F)^3(-1 + X_O + X_D + X_F)\eta_F^3 \\ & - 3VM_{XF}\rho_F^2\eta_D(VRF_{XF} - W_F)^2(-1 + X_O + X_D + X_F)\eta_F^2 \\ & + \big(\rho_D^2(VRD_{XD} - W_D)^3(-1 + X_O + X_D + X_F)(C - W_D - VM_{XD})\eta_D^3 \\ & + 3\rho_D^2(VRD_{XD} - W_D)^2(-1 + X_O + X_D + X_F)(C - W_D - VM_{XD})\eta_D^2 \\ & + (3(-1 + X_O + X_D + X_F)(VRD_{XD} - W_D)(C - W_D - VM_{XD})\rho_D^2 \\ & - 3\rho_F^2(-1 + X_O + X_D + X_F)(VRF_{XF} - W_F)VM_{XF} - 16A_O^2)\eta_D \\ & + \rho_D^2(-1 + X_O + X_D + X_F)(C - W_D - VM_{XD})\big)\eta_F \\ & - VM_{XF}\rho_F^2\eta_D(-1 + X_O + X_D + X_F)\Big)\Bigg) \end{aligned} \tag{14}$$

We can now obtain the manufacturer's optimal policies as shown in the following results.

Proposition 5.2. According to principles of optimal control theory (Sethi & Thompson, 2000) and differential games theory (Jørgensen & Zaccour, 2004), the optimal feedback marketing policies of the manufacturer including online pricing; P_O, online advertising efforts; A_O and the wholesale price for the retailer of domestic brand; W_D are:

$$P_O^* = \frac{C\eta_O - VM_{XO}\eta_O + 1}{2\eta_O} \tag{15}$$

$$A_O^* = \frac{(1 - (C - VM_{XO})\eta_O)^2\rho_O\sqrt{1 - X_O - X_D - X_F}}{8\eta_O} \tag{16}$$

$$W_D^* = \frac{3C\eta_D - 3VM_{XD}\eta_D + VRD_{XD}\eta_D + 1}{4\eta_D} \tag{17}$$

Proof: See Appendix B.

Equations (15)-(16) show that if the manufacturer's marginal benefit with respect to cumulative sales from online channel goes up, the optimal price in internet shop decreases and online advertising efforts increases. A rise in production costs makes the online advertising budget limited and leads to increase in online price and especially wholesale price for the exclusive retailer of domestic brand. Moreover, it is obvious that if the effectiveness of online advertising is high, the manufacturer has a stronger desire to spend on online advertising to attract the potential customers to the virtual channel. Similar to the retailers, the manufacturer also makes greater advertising efforts in the early stage of product life cycle which the number of potential customers, i.e., $1 - X_O - X_D - X_F$ is high.

It is evident from the equation (17) that there is a direct relationship between the wholesale price of domestic brand and the marginal benefit of domestic retailer with respect to his cumulative sales. This means, if the benefit of retailer from his sales is high, the manufacturer increases the wholesale price, because he knows that the retailer has his own incentive to improve his sales by reducing P_D^* and increasing A_D^* which is again evident from (9) and (10). In contrast, if VM_{XD}, i.e., the marginal benefit of manufacturer from the retail channel is high, the manufacturer incentivizes his exclusive retailer to increase sales by reducing the wholesale price for him.

Now by revealing the optimal decisions of the Stackelberg game leader in (15)-(17), we can get the final equilibrium policies of domestic retailer by substituting W_D^* in (17) into equations (9)-(10) as below:

$$P_D^* = \frac{5 + 3C\eta_D - 3VM_{XD}\eta_D - 3VRD_{XD}\eta_D}{8\eta_D} \tag{18}$$

$$A_D^* = \frac{3\rho_D}{32\eta_D}\sqrt{1 - X_O - X_D - X_F} * (1 - C\eta_D + VM_{XD}\eta_D + VRD_{XD}\eta_D)^2 \tag{19}$$

Since the wholesale price of foreign brand, i.e., W_F^* has been considered as an exogenous variable, the equations (9)-(10) for the foreign retailer do not change. It can be found from (18)-(19) that if the marginal benefit of domestic manufacturer and retailer with respect to cumulative sale in the retail channel, i.e., VM_{XD} and VRD_{XD} respectively increase, the optimal retail price decreases and the advertising efforts increase which together act in improving the sales of retailer of domestic brand.

The HJB equation of the manufacturer can be rewritten as follows by substituting the optimal policies obtained in (15)-(17) into (14):

$$\begin{aligned}
HJB_M &\coloneqq rV_M(X_O, X_D, X_F) = \frac{1}{64}\frac{1}{\eta_O^2\eta_D^2\eta_F}(1 - X_O - X_D - X_F) * \\
&\bigg[\frac{27}{64}\rho_D^2\eta_O^2\eta_F(C - VRD_{XD} - VM_{XD})^4\eta_D^4 - \frac{27}{16}\rho_D^2\eta_O^2\eta_F(C - VRD_{XD} - VM_{XD})^3\eta_D^3 \\
&\quad + \bigg(4\rho_F^2\eta_O^2VM_{XF}(VRF_{XF} - W_F)^3\eta_F^3 + 12\rho_F^2\eta_O^2VM_{XF}(VRF_{XF} - W_F)^2\eta_F^2 \\
&\quad + \bigg(\rho_O^2(C - VM_{XO})^4\eta_O^4 - 4\rho_O^2(C - VM_{XO})^3\eta_O^3 \\
&\quad + \bigg(\frac{81}{32}(C - VRD_{XD} - VM_{XD})^2\rho_D^2 + 6C^2\rho_O^2 - 12C\rho_O^2VM_{XO} + 6\rho_O^2VM_{XO}^2 \\
&\quad + 12\rho_F^2VM_{XF}(VRF_{XF} - W_F)\bigg)\eta_O^2 - 4\rho_O^2(C - VM_{XO})\eta_O + \rho_O^2\bigg)\eta_F \\
&\quad + 4\rho_F^2\eta_O^2VM_{XF}\bigg)\eta_D^2 - \frac{27}{16}\rho_D^2\eta_O^2\eta_F(C - VRD_{XD} - VM_{XD})\eta_D + \frac{27}{64}\rho_D^2\eta_O^2\eta_F\bigg]
\end{aligned} \tag{20}$$

In the dynamic programing (11)-(12)-(20), we see that the value functions $V_M(X_O, X_D, X_F)$ and $V_i(X_O, X_D, X_F), i = RD, RF$ are linear in X_O, X_D and X_F and are a multiple of $1 - X_O - X_D - X_F$. We therefore, conjecture the following form of value functions:

$$V_j(X_O, X_D, X_F) = \beta_j(1 - X_O - X_D - X_F), j = RD, RF \tag{21}$$

$$V_M(X_O, X_D, X_F) = \alpha_M(1 - X_O - X_D - X_F) \tag{22}$$

Now the coefficients α_M, β_{RD} and β_{RF} must be solved to obtain the optimal strategies in feedback form. With this form of value functions, the first-order derivatives with respect to the cumulative sales are:

$$Vj_{Xi} = -\beta_j, \ \ i = O, D, F \ \ \ j = RD, RF \tag{23}$$

$$VM_{Xi} = -\alpha_M, \ \ i = O, D, F \tag{24}$$

Considering the equations in (23)-(24) and comparing the coefficients of X_O, X_D and X_F and the constant term of the value functions $V_M(X_O, X_D, X_F)$ and $V_i(X_O, X_D, X_F), i = RD, RF$ in (11)-(12)-(20), with those in (21)-(22), we obtain the following system of equations to be solved for the coefficients α_M, β_{RD} and β_{RF}:

$$\begin{aligned} r\beta_{RD} = {} & \frac{1}{16384}\frac{1}{\eta_D^2\eta_O\eta_F} * \\ & \Big[81\rho_D^2\eta_O\eta_F(C + \beta_{RD} + \alpha_M)^4\eta_D^4 - 324\rho_D^2\eta_O\eta_F(C + \beta_{RD} + \alpha_M)^3\eta_D^3 \\ & + \big(1024\rho_F^2\eta_O\beta_{RD}(\beta_{RF} + W_F)^3\eta_F^3 - 3072\rho_F^2\eta_O\beta_{RD}(\beta_{RF} + W_F)^2\eta_F^2 \\ & + \big(1024\rho_O^2\beta_{RD}(C + \alpha_M)^3\eta_O^3 - 3072\rho_O^2\beta_{RD}(C + \alpha_M)^2\eta_O^2 \\ & + \big(486\rho_D^2\beta_{RD}^2 \\ & + \big((972C + 972\alpha_M)\rho_D^2 + 3072\rho_O^2\alpha_M + 3072C\rho_O^2 \\ & + 3072\rho_F^2(\beta_{RF} + W_F)\big)\beta_{RD} + 486\rho_D^2(C + \alpha_M)^2\big)\eta_O - 1024\rho_O^2\beta_{RD}\big)\eta_F \\ & - 1024\rho_F^2\eta_O\beta_{RD}\big)\eta_D^2 - 324\rho_D^2\eta_O\eta_F(C + \beta_{RD} + \alpha_M)\eta_D + 81\rho_D^2\eta_O\eta_F\Big] \end{aligned} \tag{25}$$

$$\begin{aligned} r\beta_{RF} = {} & \frac{1}{1024}\frac{1}{\eta_F^2\eta_O\eta_D} * \\ & \Big[27\rho_D^2\eta_F^2\eta_O\beta_{RF}(C + \beta_{RD} + \alpha_M)^3\eta_D^3 - 81\rho_D^2\eta_F^2\eta_O\beta_{RF}(C + \beta_{RD} + \alpha_M)^2\eta_D^2 \\ & + \big(16\rho_F^2\eta_O(\beta_{RF} + W_F)^4\eta_F^4 - 64\rho_F^2\eta_O(\beta_{RF} + W_F)^3\eta_F^3 \\ & + \big(64\rho_O^2\beta_{RF}(C + \alpha_M)^3\eta_O^3 - 192\rho_O^2\beta_{RF}(C + \alpha_M)^2\eta_O^2 \\ & + \big(96\rho_F^2\beta_{RF}^2 \\ & + \big((81C + 81\beta_{RD} + 81\alpha_M)\rho_D^2 + 192C\rho_O^2 + 192\rho_F^2W_F + 192\rho_O^2\alpha_M\big)\beta_{RF} \\ & + 96\rho_F^2W_F^2\big)\eta_O - 64\rho_O^2\beta_{RF}\big)\eta_F^2 - 64\rho_F^2\eta_O(\beta_{RF} + W_F)\eta_F + 16\rho_F^2\eta_O\big)\eta_D \\ & - 27\rho_D^2\eta_F^2\eta_O\beta_{RF}\Big] \end{aligned} \tag{26}$$

$$r\alpha_M = \frac{1}{4096}\frac{1}{\eta_O^2\eta_D^2\eta_F} * \tag{27}$$

$$\Big[27\rho_D^2\eta_O^2\eta_F(C+\beta_{RD}+\alpha_M)^4\eta_D^4 - 108\rho_D^2\eta_O^2\eta_F(C+\beta_{RD}+\alpha_M)^3\eta_D^3 + \Big(256\rho_F^2\eta_O^2\alpha_M(\beta_{RF}+W_F)^3\eta_F^3 - 768\rho_F^2\eta_O^2\alpha_M(\beta_{RF}+W_F)^2\eta_F^2 + \Big(64\rho_O^2(C+\alpha_M)^4\eta_O^4 - 256\rho_O^2(C+\alpha_M)^3\eta_O^3 + \Big((162\rho_D^2+384\rho_O^2)\alpha_M^2 + \big((324C+324\beta_{RD})\rho_D^2 + 768C\rho_O^2 + 768\rho_F^2(\beta_{RF}+W_F)\big)\alpha_M + 162(C+\beta_{RD})^2\rho_D^2 + 384C^2\rho_O^2\Big)\eta_O^2 - 256\rho_O^2(C+\alpha_M)\eta_O + 64\rho_O^2\Big)\eta_F - 256\rho_F^2\eta_O^2\alpha_M\Big)\eta_D^2 - 108\rho_D^2\eta_O^2\eta_F(C+\beta_{RD}+\alpha_M)\eta_D + 27\rho_D^2\eta_O^2\eta_F\Big]$$

The system of equations (25)-(27) is heavily nonlinear and it is almost impossible to obtain an explicit solution for that. Nevertheless, it is easy to solve these equations numerically and study the dependence of control variables, i.e., $P_i^*, A_i^*, W_D^*, i = O, D, F$ on model parameters, i.e., $\eta_i, \rho_i, i = O, D, F$.

6. Case study and numerical results

In this section, we perform a case study and numerical analysis using a mathematical software. The proposed model can be adapted to each durable goods producer who is engaged in brand competition and uses online marketing channel as well as retail channel to supply his products. An Iranian company who produces 20 different categories of home appliances as durable goods is considered. This manufacturer sells his products through an online marketing channel as well as a retail channel. Moreover, there is a brand competition in the market between this domestic brand and a substitute product with a foreign brand. Therefore, the market structure is similar to our system introduced in figure 2. Thus by collecting data from this manufacturer and his exclusive retailer, we investigated the numerical results of the proposed model.

We selected one of durable products of the mentioned manufacturer which it's average production cost is 108800 units of currency. The wholesale price of the substitute product with foreign brand is also estimated at 141440 units of currency. For ease of computations, we divide the prices by 1000000 to reduce the scale of numbers. To estimate the level of discount rate, we calculated the mean of inflation rate in the last 10 years which is %19/12 according to official statistics. Therefore, the exogenous parameters of the models are as follows:

$C = 0.101800, W_F = 0.141440, r = 0.1912$

We consider the interval $[3, 7]$ for variations in values of price sensitivity of demands, i.e., $\eta_i, i = O, D, F$. The interval for variations in values of effectiveness of advertising, i.e., $\rho_i, i = O, D, F$ is assumed to be $[5, 25]$. These intervals are estimated base on the papers in relative literature.

6.1. Effect of price sensitivity of demand in online channel (η_O)

Figures 3 and 4 show thatan increase in η_O leads to a sharp drop in level of optimal online advertising efforts and optimal price in online shop. In addition, the wholesale price of domestic brand also decreases but not as much as the price in the virtual channel. As η_O increases, the exclusive retailer of domestic brand enhances his advertising efforts to attract the market share of the online channel.

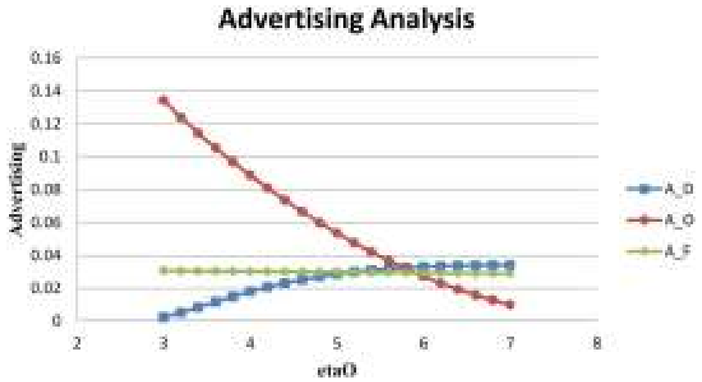

Figure 3. Impact of η_O on advertising efforts of manufacturer and retailers of domestic and foreign brands

According to figures 3 and 4 a slight increase in retail price and a slight decrease in advertising efforts in foreign brand marketing channel are the other result of high price sensitivity of demand in the online channel.

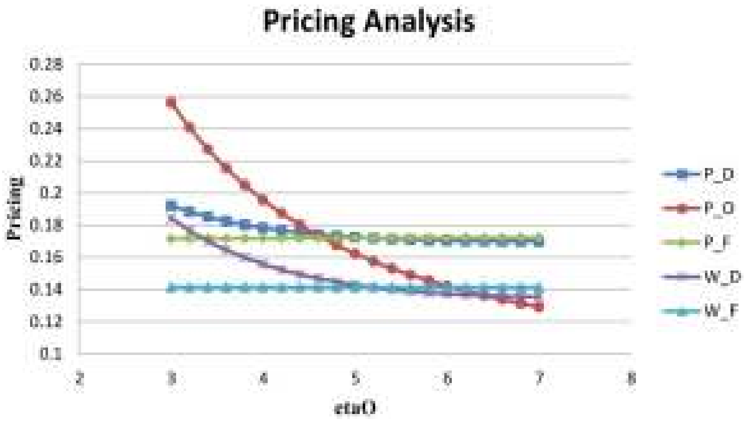

Figure 4. Impact of η_O on optimal wholesale and final prices in online and retail marketing channels

Figure 5 indicates that increase in η_O causes a rapid decline in profit of the manufacturer. In fact for high values of η_O the benefit gained by the three players are almost the same. This might happen if the customers in the market do not have any desire to buy online. If η_O is low, the domestic retailer gains a little more benefit than the retailer selling the foreign brand. However, this relation is vice versa for high values of η_O.

Figure 5. Impact of η_O on benefit of manufacturer and retailers of domestic and foreign brands

6.2. Effect of the advertising effectiveness parameter in online channel (ρ_O)

From Figures 6 and 7 it is obvious that if ρ_O rises, the optimal online advertising efforts increase drastically. The optimal wholesale and retail prices also go up but this rise is not sharp. In addition, the exclusive retailer of domestic brand gradually loses his incentive to spend on advertising efforts. On the other hand, he makes his products a little expensive to compensate his loss which is also due to increase in wholesale price announced by the manufacturer.

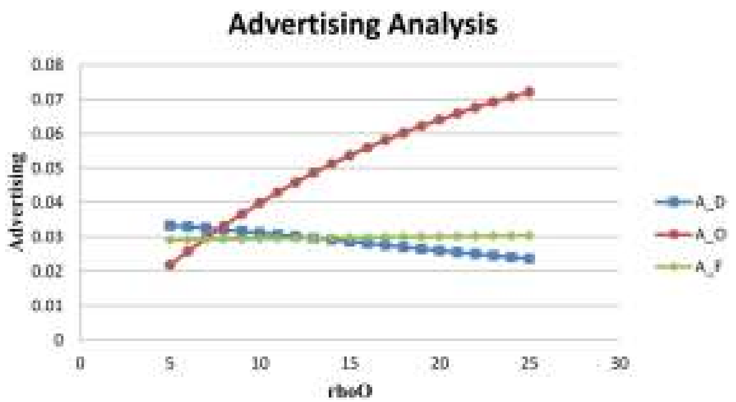

Figure 6. Impact of ρ_O on advertising efforts of manufacturer and retailers of domestic and foreign brands

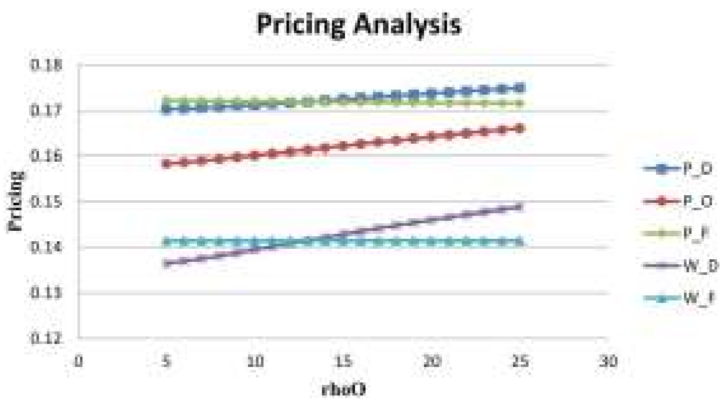

Figure 7. Impact of ρ_O on optimal wholesale and final prices in online and retail marketing channels

The extreme increase in manufacturer's benefit shown in figure 8, lies in more advertising efforts for high values of advertising effectiveness in the online channel which improves its market share. Figure 8 also indicates that as customers have stronger desire to buy online, the conventional retailer of domestic brand incurs a loss. The customers' attitude to buy domestic brand especially form the internet shop, has also bad effects on the benefit of exclusive retailer of foreign brand. However, under this condition selling the foreign brand in retail channels is a little more economical than domestic brand.

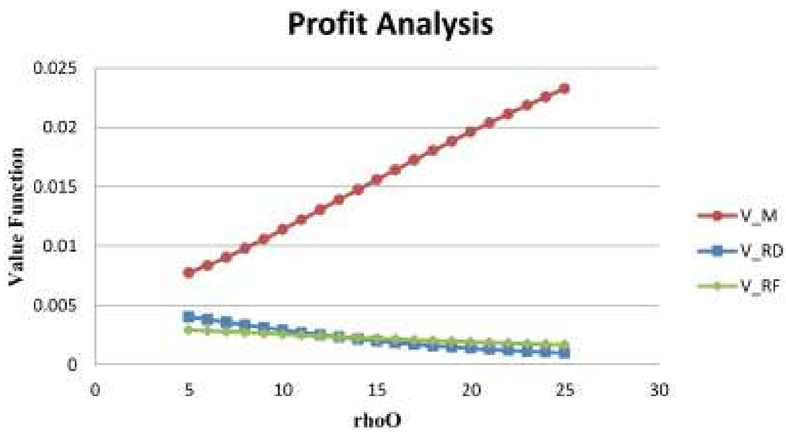

Figure 8. Impact of ρ_O on benefit of manufacturer and retailers of domestic and foreign brands

6.3. Effect of price sensitivity of demand in domestic retail channel (η_D):

It is evident from figure 9 that a sharp drop happens in optimal advertising expenditures of the domestic retailer as a result of increase in his price sensitivity of demand. On the other hand, the manufacturer tries to attract the potential customers to the online channel by his advertising efforts. As expected, the advertising policy of the retailer of foreign brand does not show any sensitivity to η_D.

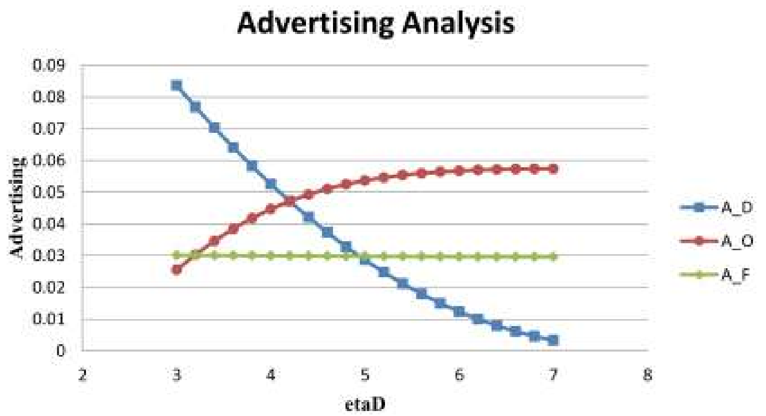

Figure 9. Impact of η_D on advertising efforts of manufacturer and retailers of domestic and foreign brands

Increase in η_D leads to a fall in both retail price of domestic brand and the marginal benefit of domestic retailer as shown in figure 10. In fact the manufacturer decreases the wholesale price to help his exclusive retailer improve his conditions, while keeping the internet price roughly constant.

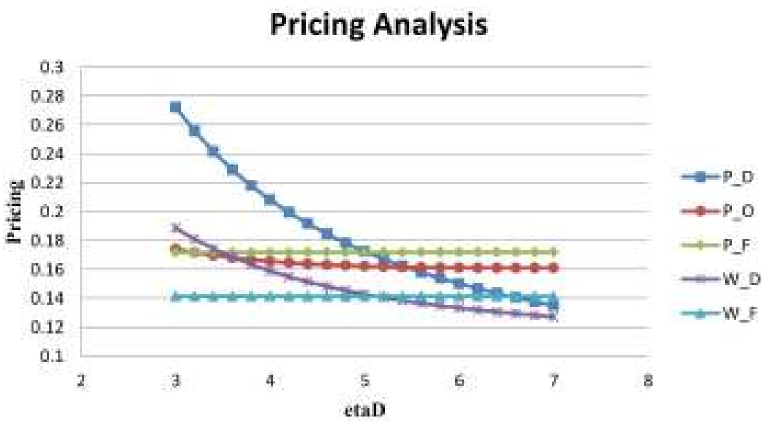

Figure 10. Impact of η_D on optimal wholesale and final prices in online and retail marketing channels

Figure 11 confirms the bad effect of increase in price sensitivity of demand in domestic retail channel on the benefits of both manufacturer and his exclusive retailer. It can be found that when $\eta_D$is more than 6, the benefit of the retailer of domestic brand is completely ruined. Although for low values of η_D, selling the products with domestic brand in retail channel is much more beneficial than foreign brand.

Figure 11. Impact of η_D on benefit of manufacturer and retailers of domestic and foreign brands

6.4. Effect of the advertising effectiveness parameter in domestic retail channel (ρ_D)

Figure 12 demonstrates a significant increase in the optimal advertising expenditures of domestic retailer as a result of rise in his advertising effectiveness. When the customers are more impressed by advertising in retail channel, the manufacturer reduces his efforts to spend on internet advertising.

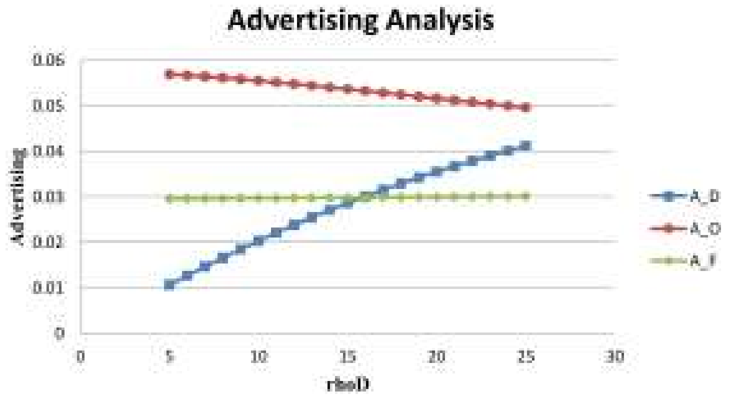

Figure 12. Impact of ρ_D on advertising efforts of manufacturer and retailers of domestic and foreign brands

Figure 13 indicates that none of pricing policies is that sensitive to changes in advertising effectiveness parameter in domestic retail channel.

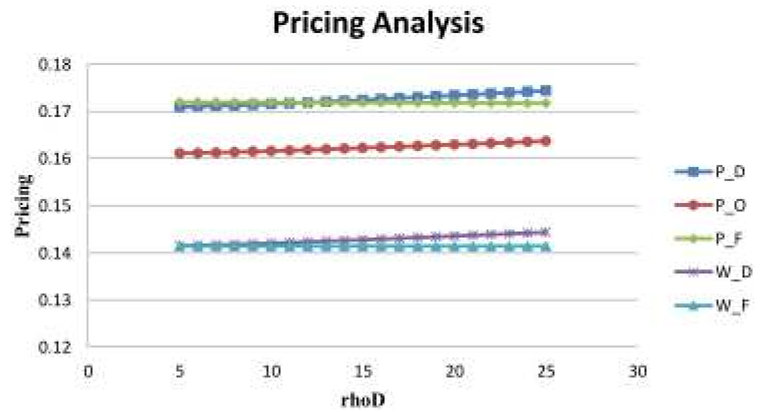

Figure 13. Impact of ρ_D on optimal wholesale and final prices in online and retail marketing channels

It can be found from figure 14 that increase in ρ_D improves the benefit of domestic brand's supply chain members, while having no effect on the gain of exclusive retailer of foreign brand.

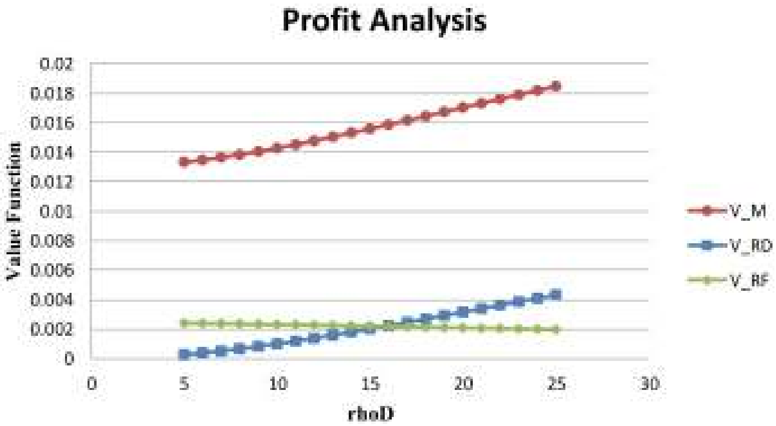

Figure 14. Impact of ρ_D on benefit of manufacturer and retailers of domestic and foreign brands

6.5. Managerial insights

The sensitivity analysis conducted above provides some remarkable insights into the way pricing and advertising decisions should be made. We discuss about them as below:

1. Marketing and sales managers of domestic product producer should keep in mind that with increase in price sensitivity of demand, the online price must be set in a lower level. As well as seasonal effects on product demand, some other issues like compatibility factor of product with online marketing and public confidence in organization website can be the determinants of consumer buying patterns. Managers should also take into account the price sensitivity of demand in internet shop when they budget for online advertising efforts.

2. One of the most important factors which affects the price sensitivity of demand is to provide various services to the customers while and after selling the products. This will leads to lower price sensitivity of demand and provide the opportunity for managers to set higher prices and make much more benefits.

3. Availability of the internet in different areas plays a major role in developing online advertising strategies based on the level of advertising effectiveness parameter in the online

channel. Although this is out of control of the company managers but they should pay attention to such cultural and infrastructural conditions of the target markets.

4. Business managers in retail marketing channels must schedule their promotion programs wisely and choose crowded places for outdoor advertising. Using mass media is another way for administrators to make their advertising plans more influential.
5. Generally, both features of the online channel including price sensitivity of demand and advertising effectiveness have a significant effect of domestic company profit level. However, about the retail channel; the impact of price sensitivity of demand factor is much more than advertising effectiveness parameter on profit of the both members of domestic goods supply chain.

7.Conclusion

Electronic commerce and online marketing are new concepts in analyzing pricing and advertising decisions in the markets including brand competition. Actually using an online channel to supply the products and promote the sales is a vital issue in modern lifestyle. Reviewing the articles in the research area of optimal control theory and differential games reveals that there is no paper considering both online and retail marketing channels in the presence of brand competition in durable goods market. Therefore, in this research we addressed a model including a manufacturer who uses internet to supply and advertise his durable products as well as selling them through a conventional retail channel.

We extended the SPH (Sethi et al., 2008) model to incorporate online marketing channel as well as brand competition between retailers into the state equation of the system. The players peruse their own objectives to maximize their discounted profits over an infinite horizon of time. The feedback equilibrium policies of both Stackelberg and Nash differential games were obtained to determine the optimal decisions on final price and advertising efforts in each of three channels. The optimal feedback policy of wholesale price of domestic brand was also characterized. We finally carried out a case study and provided managerial insights based on a sensitivity analysis of the obtained results to show the applicability aspects of the proposed model.

Generally, the results show that as price sensitivity of demand increases, the optimal price and advertising efforts decrease. This might be the result of seasonal changes or fluctuations in product quality. Furthermore, the cultural compatibility of people with online shopping in various countries is different. The widespread availability of the internet leads to greater values of advertising effectiveness parameter of the online channel. Investigating on branding and using expensive and influential advertising mediums like TV can be the other reasons of increase in advertising effectiveness parameter. This increase leads to enhancement of advertising efforts in relative marketing channel, however, does not have a significant effect on pricing decisions.

As suggestions for future research, we recommend the following directions:

1. Developing a reverse logistic model especially because the products are durable and it is possible for them to be returned. Then different pricing and advertising policies for remanufactured and refurnished products could be analyzed.
2. Considering different kinds of system structures, for example assuming hyper markets which supply substitute products with various brands.

3. Investigating market segmentation policy and its effects on pricing and advertising decisions.
4. Assuming the products to be perishable or consumable to see how the optimal pricing and advertising decisions change.
5. Considering more complex and complete forms of demand specifications.

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Appendix A. Proof of Proposition 5.1

Writing the first-order conditions with respect to P_i and A_i, $i = D, F$ in the HJB equations (8) gives the following:

$$\partial HJB_j \big/ \partial P_i = 0 \Rightarrow -2\rho_i A_i \sqrt{1 - X_O - X_D - X_F} * \left(-\frac{1}{2} + \left(P_i - \frac{W_i}{2} + \frac{Vj_{Xi}}{2}\right)\eta_i\right) = 0, \quad (A.1)$$

$i = D, F \quad j = RD, RF$

$$\partial HJB_j \big/ \partial A_i = 0 \Rightarrow \frac{(1 - (W_i - Vj_{Xi})\eta_i)^2 \rho_i \sqrt{1 - X_O - X_D - X_F}}{4\eta_i} - 2A_i = 0, i = D, F \quad j = RD, RF \quad (A.2)$$

We obtain the optimal feedback decisions on pricing and advertising efforts of the retailers as (9) and (10) by solving the equations (A.1) and (A.2).

Furthermore to verify the second-order conditions, we compute $\partial^2 V_j / \partial P_i^2$, $\partial^2 V_j / \partial A_i^2$ and $\partial^2 V_j / \partial P_i \partial A_i$, $i = D, F$; $j = RD, RF$ for $P_i = P_i^*$ and $A_i = A_i^*$, $i = D, F$ which gives the following for each of retailers:

$$\begin{aligned}
\partial^2 V_j / \partial P_i^2 &= -2\rho_i \eta_i A_i * \sqrt{1 - X_O - X_D - X_F} \xrightarrow{A_i = A_i^*} \left(-1/4\right) * \rho_i^2 \left(1 - \eta_i (W_i - Vj_{Xi})\right)^2 \\
&\quad * (1 - X_O - X_D - X_F) < 0, i = D, F \quad j = RD, RF \\
\partial^2 V_j / \partial A_i^2 &= -2 < 0, i = D, F \quad j = RD, RF \\
\partial^2 V_j / \partial P_i \partial A_i &= \rho_i \sqrt{1 - X_O - X_D - X_F} * (1 - (2P_i - W_i + Vj_{Xi})\eta_i) \xrightarrow{P_i = P_i^*} 0, \\
i = D, F \quad j = RD, RF
\end{aligned} \quad (A.3)$$

Thus the Hessian matrix for the two retailers is:

$$\begin{bmatrix} \left(-1/4\right) * \rho_i^2 \left(1 - \eta_i (W_i - Vj_{Xi})\right)^2 * (1 - X_O - X_D - X_F) < 0 & 0 \\ 0 & -2 \end{bmatrix}$$

As determinant of the first-order sub-matrix is negative and that of the second-order sub-matrix is positive, we found that the Hessian matrix for the value functions of retailer j, $j = RD, RF$ is negative definite for the control variables $P_i = P_i^*$ and $A_i = A_i^*$, $i = D, F$ obtained in (9)-(10), thus indicating the global maximum.

Appendix B. Proof of Proposition 5.2

By writing the first-order conditions with respect to P_O, A_O and W_D in the HJB equation (14), we have:

$$\partial HJB_M \big/ \partial P_O = 0 \Rightarrow \rho_O A_O \sqrt{1 - X_O - X_D - X_F} * (1 + (C - 2P_O - VM_{XO})\eta_O) = 0 \quad (B.1)$$

$$\partial HJB_M/_{\partial A_O} = 0 \Rightarrow \frac{(1-(C-VM_{XO})\eta_O)^2\rho_O\sqrt{1-X_O-X_D-X_F}}{4\eta_O} - 2A_O = 0 \tag{B.2}$$

$$\partial HJB_M/_{\partial W_D} = 0 \Rightarrow {}^3/_{16\eta_D} * (1-(W_D-VRD_{XD})\eta_D)^2 * \rho_D^2(1-X_O-X_D-X_F)$$
$$* \left({}^1/_3 + \left(C - {}^{4W_D}/_3 - VM_{XD} + {}^{VRD_{XD}}/_3\right)\eta_D\right) = 0 \tag{B.3}$$

Solving the equations (B.1)-(B.3) yields the optimal feedback decisions on online pricing, online advertising and Wholesale price for domestic retailer as mentioned in (15)-(17).

Furthermore to verify the second-order conditions, we consider hessian matrix for the value function of domestic manufacturer by computing second-order derivatives with respect to $P_O = P_O^*$, $A_i = A_O^*$ and $W_O = W_O^*$ as shown below:

$$\begin{aligned}
&\partial^2 V_M/\partial P_O^2 = -2\rho_O\eta_O A_O * \sqrt{1-X_O-X_D-X_F} \xrightarrow{A_O=A_O^*} \left(-{}^1/_4\right) \\
&\qquad\qquad * \rho_O^2\big(1-\eta_O(C-VM_{XO})\big)^2 * (1-X_O-X_D-X_F) < 0 \\
&\partial^2 V_M/\partial A_O^2 = -2 < 0 \\
&\partial^2 V_M/\partial W_D^2 = 0 \\
&\partial^2 V_M/\partial P_O\partial A_O = \rho_O\sqrt{1-X_O-X_D-X_F} * (1-(2P_O-C+VM_{XO})\eta_O) \xrightarrow{P_O=P_O^*} 0 \\
&\partial^2 V_M/\partial P_O\partial W_D = 0 \\
&\partial^2 V_M/\partial W_D\partial A_O = 0
\end{aligned} \tag{B.4}$$

Thus the Hessian matrix for the manufacturer of domestic brand is:

$$\begin{bmatrix} \left(-{}^1/_4\right) * \rho_O^2\big(1-\eta_O(C-VM_{XO})\big)^2 * (1-X_O-X_D-X_F) < 0 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

As determinant of the first-order sub-matrix is negative and that of the second-order sub-matrix is positive and determinant of the whole matrix equals to zero, we found that the Hessian matrix for the value function of manufacturer is negative semi-definite for the control variables $P_O = P_O^*$, $A_i = A_O^*$ and $W_D = W_D^*$ obtained in (14)-(16), thus indicating a maximum.

It should be noted that the equation (B.3) has three roots, out of which two are identical. Thus solving the first-order condition gives two distinct solutions. We can show that one of them, which gives $W_D = VRD_{XD} + {}^1/_{\eta_D}$, yields $P_i^* = {}^1/_{\eta_i}, i = D, F$ from (8) with the corresponding demand $D_i(P_i^*) = 0, i = D, F$ in (3), and can therefore be ruled out from further consideration. Thus the remaining solution for W_D which was introduced in (16) is indeed the global maximum solution.