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Inventory control games with prepayments

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Abstract

The cost game arises when a group of retailers who observe demand for a common good decide to cooperate and make join orders following the EOQ policy. In this paper we present a new class of inventory games that is strategically equivalent to that class of inventory cost games: inventory games with advance payments. This model extends the traditional inventory model to the situation where advance payments of retailers are required. We propose a core distribution, which is based on a proportional allocation, as well as a population monotonic allocation scheme, for inventory games with advance payments. Then, we examine the stability of grand coalition from both a myopic and farsighted perspective, and conclude that it is always stable from both points of view. To complete our study, we develop a sensitivity analysis for the model and evaluate the changes produced on the proposed core distribution.

Keywords: Inventory games, advance payments, cooperative cost games, core distribution.

1- Introduction

Inventory cost games were introduced by Meca et al. (2004). This class of games arises when a group of retailers who observe demand for a common good decide to cooperate and make joint orders following the EOQ policy. By placing joint orders, these retailers can reduce their total cost of operations and get some benefits for the group. This kind of cooperation is becoming increasingly popular in the economic literature since the supply chain management has undergone radical changes in recent years with increasing emphasis on cooperation and information sharing. Recent surveys on cooperation among supply chain agents can be seen in Nagarajan and Sosic (2008), Meca and Timmer (2008), Dror and Hartman (2011), and Fiestras-Janeiro García-Jurado, Meca, and Mosquera (2011).

Multiple and various extensions of inventory cost games studied in Meca et al. (2004) can be found in the literature of game theory and inventory management. Meca et al. (2003) revisit inventory cost games but now allowing shortages. They see that n-person EPQ situations with shortages lead to exactly the same class of inventory cost games. In addition, they provide a non-cooperative approach to them. Necessary and sufficient conditions are given for the existence and uniqueness of the so-called constructive equilibrium in which all the players make joint orders.

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Meca, Guardiola and Toledo (2007) introduce the class p-additive games inspired by the class of inventory games that arises from inventory situations with temporary discounts (Toledo, 2002). It contains the class of inventory cost games.

Other variations of Meca et al. (2004) can be found in Anily and Haviv (2007), in Zhang (2009), and in Dror and Hartman (2007), but in all these papers authors consider problems that are different from all of the above. In the two first papers, it is considered that agents use POT (Power of Two) policies instead of EOQ policies, while in the third paper it is considered that, if a group of agents place a joint order, its fixed cost is the sum of the first component plus the sum of the second component of the agents in the group. Fiestras-Janeiro et al. (2012) deal with the cost allocation problem in an inventory transportation system with a single item and multiple agents that place joint orders using an EOQ policy. This model extends the one studied in Meca et al. (2004) by changing the structure of the fixed costs. They consider that a part of the fixed cost depends on each agent and that it is proportional to the distance between the agent and the supplier. It is assumed that agents are located on a line route. Then, they introduce and characterize a rule, with the same flavor as the Shapley value but less computational effort, which allows them to allocate the costs generated by the joint order. This has good properties from the point of view of stability. Fiestras-Janeiro et al. (2013) propose a new cost allocation rule, the so-called AMEF value, which is also inspired by the Shapley value. They prove that, under suitable conditions, the AMEF value provides core allocations.

Fiestras-Janeiro et al. (2014) deal with an inventory problem arising in a farming community in the Northwest of Spain. Each farm has its own silo (warehouse), with limited capacity, for keeping the feed. The only costs associated with the silos are their building costs since their maintenance costs are irrelevant; thus, the storage cost of each stockbreeder is in fact zero. They analyze then two models with *n* decision makers, all those facing continuous review inventory problems without holding costs, with limited capacity warehouses and without shortages. The fact that shortages are not allowed simplifies strongly the search for optimal policies. However, the case with shortages can be also used in this context Fiestras-Janeiro et al. (2015) shows. They consider that each firm uses its limited capacity warehouse for storing purposes and that it faces an economic order quantity model where storage costs are irrelevant (and assumed to be zero) and shortages are allowed. They show that firms can save costs by placing joint orders and obtain an optimal order policy for the firms. Some results that can be helpful for allocating the joint costs among the firms are finally provided.

A recent extension, very close to this paper, is Li et al. (2014). They present the class of inventory games with permissible delay in payments. The benefits retailers can obtain from permissible delay in payments by the supplier are obvious (i.e., a source of financing when they are short of cash). For suppliers, permissible delay in payments can promote their sales and reduce their on-hand stock. They prove that this class of games is balanced (the core is nonempty). Then, they propose a core distribution of the cost that can be reached through population monotonic allocation scheme (PMAS). Under this cost distribution, the grand coalition is shown to be stable from a farsighted point of view. In addition, we can mention other researches in the field of inventory cost games that done in the recent years. Lai et al. (2016) developed a distribution system, where multiple suppliers cooperate in supplying a product under two dispatching policies that called time-based policy and quantity-based policy. They proved this game is convex and monotone and also used PMAS scheme for cost allocation. Chen and Zhang (2016) proposed the inventory cost game with backlogging. They proved that class of games have a non-empty core but not necessarily the optimal dual solution defines a core allocation. Jouida et al. (2017) developed a horizontal cooperation game between a supplier and multiple retailers and studied the features of the stable coalitions. Hezarkhani et al. (2018) presented a two-stage cooperative inventory game for replenishment of multiple products. They showed that buyers can reduce their cost by cooperation.

In this paper, we propose a new extension of inventory cost games (Meca et al., 2004), to the situation with payments in advance. Advance payment purchasing systems are very common in supply chain transactions. In the real world, especially in monopolistic or oligopolistic market, when a supplier is powerful, (s)he wants to reduce the risk of the cash flow, and would like the buyer to pay in a fixed period before delivering the product. This scheme called advance payment or prepayment in the literature. The advance payment scheme is very common in Europe and U.S. utility markets, real estate etc. (Schulz et al., 2015). In such systems the capital cost of the retailers will increase, because they have incurred interest costs on the purchasing cost of products that have not been received yet but on the other side, the supplier earn interested capital. Each retailer seeks for determining its optimal inventory costs by developing an economic order quantity model with advance payment for multiple retailers. If the retailers decide to cooperate and place their orders jointly, they can reduce the total average cost generated by the cooperation. By using cooperative game theory we study such cooperation and show that cooperation of all the retailers in ordering is always beneficial; i.e., the grand coalition can achieve a lower cost than the added cost if they act individually.

Our paper is also related to the literature on inventory models with payments in advance. Maiti et al. (2009) consider inventory model under advance payment and stochastic lead time and solved their model with genetic algorithm. Gupta et al. (2009) also solved the inventory model under advance payment with genetic algorithm but they considered interval valued inventory costs. Chen et al. (2012) examined the three payment scheme on the newsvendor problem, this schemes including payment in the time of delivery, delayed payment, advance payment. Mateuta and Zanchettin (2013) studied the interaction between supplier credit sales and customer advance payment. Taleizadeh et al. (2013) developed the economic order quantity (EOQ) model with considering advance payment and partial backordering. Zhang et al. (2014) developed the inventory model under partial advance payment and partial delay in payment. Taleizadeh (2014) developed EOQ model for deteriorating item under multiple prepayments. Beullens and Janssens (2014) developed an inventory model with considering the timing of the cash flow and used the net present value concept to analyze the model under three payment structures including symmetric time, delay in payment and advance payment. Zhang et al. (2016) investigated the supply risks on capacity in advance payment scheme. Teng et al. (2016) studied an EOQ model with advance payment and considering expiration date for deteriorating goods. Taleizadeh (2017) presented a lot-sizing model with considering advance payment, partial backordering and product rejection as a stochastic event.

The contribution of our paper is threefold. First, we present the class of inventory games with advance payments and show that this class of games is strategically equivalent to the class of inventory cost games introduced by Meca et al. (2004). This fact guarantees the existence of a core distribution as well as a population monotonic allocation scheme, and allows us to conclude that inventory games with advance payments are totally balanced, and so the grand coalition is a myopic stable outcome. Second, we examine the stability of grand coalition from a farsighted perspective. We show that grand coalition belongs to the largest consistent set, i.e., it is a farsighted stable outcome under the proposed cost distribution. Third, we develop a sensitivity analysis for the model and evaluate the changes produced on the proposed cost distribution.

The rest of this paper is organized as follows. We start by introducing preliminaries on cooperative game theory in section 2. In section 3 we introduce the basic inventory model with advance payments under cooperation in order. Section 4 presents the class of inventory games with advance payments. Section 5 presents a numerical example to illustrate those games and a sensitivity analysis for the model. Concluding remarks in section 6 close the paper.

2- Preliminaries

To begin with we will introduce some basic concepts of cooperative game theory that we will use throughout the paper and will enable a self-reading of it. A cost game is a pair of (N,c) with $N=\{1,2,\ldots,n\}$ being the set of players (finite), and c is the characteristic function, which measures the cost generated by each of the possible coalitions that can be formed between players of N. Formally, $c:2^n \rightarrow R, c(S), \forall S \in 2^n, c(\varnothing)=0$. We will consider some properties of cost games. A cost game (N,c) is concave if for all $i \in N$ and for all $S \subset T \subset N\{i\}$, we have that $c(S\cup\{i\})-c(s) \geq c(T\cup\{i\})-c(T)$ and it is monotone if for all $S \subset T \subset N$ it holds that

$c(S) \le c(T)$. The concavity property provides us with additional information about the game: the marginal contribution of an agent diminishes as a coalition grows, and so it is profitable for the agents in *N* to form the grand coalition.

One of the main issues treated in cooperative game theory is how to divide the benefits from cooperation if the grand coalition has formed. In the case of cost games, we are interested in providing a distribution of the total cost, *c(N)*, so that no coalition has an incentive to leave the grand coalition and pay less. One way to share these benefits is according to an allocation in the core. The core of a cost game (N,c) is the set

$$C(c) = \{x \in R^n \left| \sum_{i \in N} x_i = c(N) and \sum_{i \in S} x_i \le c(S) for\ all\ S\ N, S \ne \emptyset \right.\} \tag{1}$$

When an element of the core $x \in C(c)$ is proposed (henceforth, a core distribution), where player *i* has to pay x_i, every coalition *S* of players should pay at most his own cost since $\sum_{i \in S} x_i \le c(S)$. A game (N,c) is balanced if it has a non-empty core (see Bondareva, 1963, and Shapley, 1967), and it is called totally balanced if each sub game (S,c_S) is balanced, where $c_S(T) = c(T), \forall T \subseteq S$. In addition, Shapley (1971) shows that concave games are always balanced.

A refinement of the core is the set of distributions of the total cost *x* that can be reached through a population monotonic allocation scheme, in short, PMAS. These schemes were introduced in Sprumont (1990) and defined as follows. A vector $y = \left(y_i^S\right)_{i \in S \subseteq N, S \ne \emptyset}$ is a PMAS of the cost game (N,c) if and only if satisfies the following two conditions. Firstly, it should hold that $\sum_{i \in S} y_i^S = c(S)$, for all non-empty coalition *S* of *N*. Secondly, for all non-empty coalition *S* and *R* in *N* and for all player *i* in *S* it should be hold that $S \subset R \subset N$ implies $y_i^S \ge y_i^R, \forall i \in S$. Also from Sprumont (1990) it follows that a core distribution, $x \in C(c)$, is reached through a PMAS of the game (N,c) if there exists $y = \left(y_i^S\right)_{i \in S \subseteq N, S \ne \emptyset}$ a PMAS such that $y_i^N = x_i$. Moreover, every cost game with PMAS is totally balanced. However, it is not possible to get a PMAS with any random selection of cost-distributions and there are totally balanced cost games without PMAS.

Let (N,c) and (N,c') cooperative cost games. We say that *c* and *c'* are strategically equivalent if there exist $k > 0$ and $a = (a_1, a_2, ..., a_n)$ such that $c(S) = k.c'(S) + \sum_{i \in S} a_i$, for all coalition *S* in *N*. Note that *c* and *c'* play symmetric roles and we can also write

$$c'(S) = \frac{c(S)}{k} - \sum_{i \in S} \frac{a_i}{k} \tag{2}$$

There exists a relationship between "equivalence" and "core-distributions". The following well known theorem tells us that if we are studying a game in characteristic function form, then we are simultaneously studying all games which are strategically equivalent to it.

Theorem1. Let (N,c) and (N,c') be strategically equivalent cooperative cost games such that $c(S) = k.c'(S) + \sum_{i \in S} a_i$. Then, the following statements hold:

1. *If* (N,c') is concave, (N,c) is concave.
2. *If* (N,c') is monotone, (N,c) is monotone.
3. A distribution *x* is in the core of *c'* if and only if *(k·x +a)* is in the core of *c*.

As Chwe (1994) points out, core-distributions provide a kind of stability from myopic point of view. As we already mentioned, the idea behind the core is that no subset of players can benefit by defecting from the grand coalition with one step, if so the grand coalition is considered to be unstable. However, this idea avoids the possibility that an initial defection may trigger a sequence of further moves, which eventually can lead to an outcome wherein the players who initiated the deviations would receive higher cost than that they would obtain in the grand coalition. Therefore, farsighted players may not choose to defect in the first place, and thus the grand coalition, which appeared unstable from a myopic view, may actually be stable from a farsighted point of view. A new solution concept, the largest consistent set (in short, LCS), which allows players to look ahead and consider possible further deviations, was introduced by Chwe (1994). Basically, the LCS approaches stability analysis from a farsighted perspective, i.e., considers the effect of externalities and allows players to consider multiple possible further deviations, while the core approaches stability analysis from a myopic perspective, i.e., considers only one step deviation.

Formally we define the LCS as follows. By L we denote coalition structures where L is a partition of the player set N, i.e. $L=\{L_1,L_2,\ldots,L_m\}$. For two coalition structures L_1, L_2, we say that player i strongly prefers coalition structure L_2 to L_1, i.e., $L_1<_L L_2$ if the cost given to him/her under L_2 is strictly lower than under L_1. In other words, $L_1<_L L_2$ if and only if $x_i^{L2}<x_i^{L1}$, where x_i^{Li} denotes player i's cost under coalition structure L. For a coalition S in N, $L_1<_S L_2$, if $L_1<_L L_2$ for all $i\in S$. By $\rightarrow_S$ we denote the following relation: $L_1<_S L_2$ if the coalition structure L_2 is obtained when S deviates from coalition structure L_1. We say that L_1 is directly dominated by L_2, i.e., $L_1<L_2$ if there exists a coalition S such that $L_1 \rightarrow_S L_2$ and $L_1<_S L_2$. We say that L_1 is indirectly dominated by L_m, i.e., $L_1<<L_m$ if there exist $L_1,L_2,\ldots,L_m$ and $S_1,S_2,\ldots,S_m$ such that $L_i\rightarrow_{Si}L_{i-1}$ and $L_i<_{Si}L_m$ for $i=1,2,\ldots,m-1$.

A set Y is called consistent if the following condition holds: $L\in Y$ if and only if for all L', C such that $L\rightarrow_C L'$ there exists $B\in Y$ where $L'=B$ or $L'<<B$ such that $L<\neq_C B$. Chwe (1994) shows that although there can be many consistent sets, there uniquely exists a LCS, which contains all other consistent sets. The LCS has the following merit that if one outcome is not in the LCS, it cannot possibly be stable. The LCS is the set of all outcomes that can possibly be stable.

We conclude this preliminary section by describing inventory games introduced by Meca et al. (2004) as models for cooperation in inventory situations. The player set N consists of a group of retailers dealing with the ordering of a certain commodity (every individual agent's problem being an EOQ problem). In an inventory cost game, a group of players minimize their total cost by placing their orders together as one big order (paying a fix ordering cost A). To coordinate the ordering policy of the retailers, some public information is needed: the demand and holding cost for each retailer, i.e., d_i and h_i for all $i\in N$. Then an inventory cost situation is given by the following 3-tuple $<N,A,\{\mathrm{d_i,h_i}\}_{\mathrm{i\in N}}>$ with $\mathrm{A}>0,\mathrm{d_i}\geq 0,\mathrm{h_i}>0,\forall i\in N$. The corresponding inventory cost game (N, c_I) is defined as follows,

$$c_I(S):=\sqrt{\sum_{j\in S}2Ad_jh_j},\forall S\subseteq N,S\neq\varnothing. \tag{3}$$

Meca et al. (2004) show that inventory cost games are concave and monotone. Moreover, the c^2-proportional rule with $c^2=(c_I(i)^2/c_I(N))_{i\in N}$, or SOC-rule, on inventory cost games is a core-distribution which can be reached through a PMAS for (N, c_I). Meca et al. (2003) revisit inventory cost games and the SOC-rule. They prove that the wider class of n-person EPQ inventory situations with shortages leads to exactly the same class of cost games.

3- The basic inventory model with advance payments

We consider a supply chain with one supplier and a finite number of retailers. The retailers purchase one common good from the supplier and they are asked to pay all of the purchasing cost before the date of delivery. There is a single good and each retailer has its own private warehouse. The demand for the good for all retailers is assumed to be known, constant. No retailer is allowed to run out of stock i.e., shortages are not allowed. The replenishment lead time is assumed to be deterministic and constant, and without loss of generality equal to zero. The supplier offers a discount retailer prices if all the payment are paid in advance. During the time between that the purchasing cost has been paid and the inventory has been settled, the payment generates interest cost. This supply chain can be seen as an advance payment purchasing system. In such a system the capital cost of the retailers will increase because they has incurred interest costs on the purchasing cost of products which have not yet been received. Each retailer seeks for determining its optimal inventory costs. Then, an economic order quantity model with advance payment for multiple retailers is developed.

We denote the demand and holding cost per time unit of retailer $i \in N$ by $D_i \geq 0$ and $h_i > 0$, respectively. The rest of the parameters, that are common to all retailers, are the unit purchasing cost with advance payment (C>0), the fixed ordering cost per order ($A>0$), the length of advance payment ($t_0 \geq 0$), the interest charges per euro investment in stocks per year, ($Ic \geq 0$) and the replenishment cycle (T).

If the retailers place orders separately, each retailer $i \in N$ has to pay a purchasing cost of $D_i T_i C$, an ordering cost A at time t_0, and a capital cost $D_i T_i C I_c t_0$ from time t_0 to (t_0+T). After receiving inventory (s) he incurs a holding cost of $D_i T_i h_i / 2$, and the cost of the interest when the goods are kept in stock during one cycle is $D_i T_i^2 C I_c / 2$. Hence, the average total cost per time unit, as a function of the replenishment cycle, is given by

$$TC_i(T_i) = \frac{A}{T_i} + \frac{D_i T_i h_i}{2} + D_i C I_c t_0 + \frac{D_i C T_i I_c}{2} \tag{4}$$

The minimal cost is obtained in T_i^* with $TC_i^{'}\left(T_i^*\right) = 0$ and $TC_i^{"}\left(T_i^*\right) > 0$. It follows that

$$T_i^* = \sqrt{\frac{2A}{D_i\,(h_i + CI_c)}} \tag{5}$$

and the minimal average cost per time unit is

$$TC_i^*\left(T_i^*\right) = \sqrt{2AD_i\left(h_i + CI_C\right)} + D_i C I_C t_0 \tag{6}$$

If a group of retailers decide to cooperate and place orders jointly, the ordering cost can be shared among them but each retailer has to pay its own holding cost. Consider $S \subseteq N$ the group of retailers decides to cooperate. Cycle length should be the same for all retailers (by a similar argument as the one given by Meca et al. 2004), let sayT_S, they have a common fixed cost to order jointly, but each retailer has its own holding cost. Once the cost of interest charges before receiving products and holding cost including capital cost have been calculated, the average total cost per time unit is now given by the following:

$$TC_s(T_s) = \frac{A}{T_s} + \sum_{i \in S} \frac{D_i h_i T_s}{2} + \sum_{i \in S} D_i C I_c t_0 + \sum_{i \in S} \frac{D_i C I_c T_s}{2} \tag{7}$$

It is easy to show that the optimal replenishment cycle length for coalition $S \subseteq N$ is

$$T_s^* = \sqrt{\frac{2A}{\sum_{i \in s} D_i\left(h_i + CI_c\right)}} \tag{8}$$

and the minimal average total cost is

$$TC_s\left(T_s^*\right)=\sqrt{\sum_{i\in s}2AD_i\left(h_i+CI_c\right)}+\sum_{i\in s}D_iCI_ct_0 \tag{9}$$

Next Proposition shows that optimal replenishment cycle length satisfies a monotonicity property: it is decreasing with respect to coalitions; i.e., the larger a coalition is, the shorter is the optimal replenishment cycle length.

Proposition1. If $\varnothing \neq S \subset R \subseteq N$, then $(T_s^* \geq T_R^*)$.

Proof. Since $\sum_{i\in s}D_i\left(h_i+CI_c\right)$ is increasing in the number of elements and A is fixed, it follows immediately that when number of firms in coalition increase the cycle length decreasing. Hence $T_s^* \geq T_R^*$.

The following Proposition proves that the cooperation of all the retailers is always beneficial in a basic inventory model with advance payments; i.e., the grand coalition can achieve a lower cost than the added cost if all retailers act individually.

Proposition2. Given an inventory model with advance payments, it holds that $TC_N\left(T_N^*\right)<\sum_{i\in N}TC_i\left(T_i^*\right)$.

Proof. Using the Proposition 1, we prove that

$$\begin{aligned}TC_N\left(T_N^*\right)&=\sum_{i\in N}\left(h_i+CI_c\right)D_iT_N^*+\sum_{i\in N}D_iCI_ct_0=\sum_{i\in N}[(h_i+CI_c)D_iT_N^*+D_iCI_ct_0]\\&\leq\sum_{i\in N}[(h_i+CI_c)D_iT_i^*+D_iCI_ct_0]=\sum_{i\in N}TC_i\left(T_i^*\right)\end{aligned} \tag{10}$$

We define an *inventory situation with advance payments* as the 6-tuple $<N,A,C,\mathrm{t}_0,\mathrm{I}_{\mathrm{c},}\left\{\mathrm{D}_{\mathrm{i},}\mathrm{h}_{\mathrm{i},}\right\}_{\mathrm{i}\in\mathrm{N}}>$ with $A,C,t_0,I_{c,}>0,D_i\geq 0,h_i>0,$ for all $\mathrm{i}\in\mathrm{N}$.

Now we are ready to introduce the class of inventory games with advance payments that is based on the inventory situation described just above.

4- Inventory cost games with advance payments

In this section we focus on the study of interactions among possible coalitions of retailers. We are interested in finding a stable distribution of the total cost generated by the grand coalition that allow us to conclude that this coalition is a stable outcome, from both a myopic and farsighted perspective, for inventory cost games with advance payments. From the myopic perspective, every coalition can freely form and the rest of the retailers may regroup. But if retailers are farsighted, they need to consider a set of ultimate outcomes instead of their individual outcomes. The farsighted coalition will be stable when the LCS forms as we will describe in this section. The LCS defines possible stable outcomes of all retailers and a coalition may chose not to deviate.

Given an inventory situation with advance payments $<N,A,C,\mathrm{t}_0,\mathrm{I}_{\mathrm{c},}\left\{\mathrm{D}_{\mathrm{i},}\mathrm{h}_{\mathrm{i},}\right\}_{\mathrm{i}\in\mathrm{N}}>$, we define the corresponding *inventory cost game with advanced payments* (N, c_A) as follows. For all coalitions $S\subseteq N$,

$$\begin{aligned}&c_A(S)=\sqrt{\sum_{i\in s}2AD_i\left(h_i+CI_c\right)}+\sum_{i\in s}D_iCI_ct_0\\&c_A(\varnothing)=0\end{aligned} \tag{11}$$

The reader may notice that the inventory cost with advance payments generated by coalition $S\subseteq N$ consists of two parts: (1) inventory cost $\sqrt{\sum_{i\in s}2AD_i\left(h_i+CI_c\right)}$, (2) capital cost due to the interest charged $\sum_{i\in s}D_iCI_ct_0$.

Next Proposition shows that the study of class of inventory cost games with advanced payments can be done simultaneously to the class of inventory cost games (see Meca et al. 2004), since both classes are strategically equivalent.

Proposition3. Inventory cost games with advanced payments are strategically equivalent to inventory cost games.

Proof. Denote by $H_i = h_i + CI_c$. Then, for all $S \in N$, there exists *k*=1 and $a_i = D_i CI_c t_0, i \in N$, such that,

$$c_A(S) = \sqrt{\sum_{i \in s} 2AD_i \left(h_i + CI_c\right)} + \sum_{i \in s} D_i CI_c t_0 = \sqrt{\sum_{i \in s} 2AD_i H_i} + \sum_{i \in s} a_i = c_I(S) + \sum_{i \in s} a_i \qquad (12)$$

Hence (N, c_A) and (N, c_I) are strategically equivalent.

We consider now the following issue. Is it profitable for the agents in N to form the grand coalition to place joint orders? The following Proposition proves that the answer to this question is positive because inventory cost games with advanced payments are concave.

Proposition4. Inventory cost games with advanced payments are concave and monotone.

Proof. It is a direct consequence of Theorem1 and Proposition 3.

Based on the relationship between "equivalence" and "core-distributions", we can define a cost distribution for retailer $i \in N$ as follows:

$$\beta_i(c_A) = \frac{D_i(h_i + CI_c)}{\sum_{i \in N} D_i(h_i + CI_c)} \sqrt{\sum_{i \in N} 2AD_i(h_i + CI_c)} + D_i CI_c t_0 \qquad (13)$$

$$\beta_i(c_A) = (\beta_i(c_A))_{i \in N}$$

The first part of this cost distribution β allocates the inventory cost in proportion to $D_i\left(h_i + CI_c\right)$, and the second part is the individual capital cost charged. $D_i CI_c t_0$.

Next proposition shows that β is always a core distribution and can be reached through a PMAS. Hence, inventory cost games with advanced payments are totally balanced. It means that the grand coalition is always a stable outcome from a myopic perspective.

Theorem2. Let $< N, A, C, \mathrm{t}_0, \mathrm{I}_{\mathrm{c}}, \{\mathrm{D}_{\mathrm{i}}, \mathrm{h}_{\mathrm{i}}\}_{\mathrm{i} \in \mathrm{N}} >$ be an inventory situation with advance payments and (N, c_A) the corresponding inventory game. The cost distribution β always belongs to the core and it can be reached through a PMAS.

Proof. First we prove that β belong to the core of (N, c_A). Indeed, denote by $c_I(S) = \sqrt{\sum_{i \in S} 2AD_i H_i}$, where $H_i := h_i + CI_C, a_i := D_i CI_C t_o, \forall i \in N$.

Then, $\beta_i(c_A) := \dfrac{D_i H_i}{\sum_{i \in S} D_i H_i} + a_i = \dfrac{c_I(i)^2}{c_I(N)} + a_i$.

We know by Meca et al. (2004) that $\left(\frac{c_I(i)^2}{c_I(N)}\right)_{i\in N} \in C(c_I)$. Hence, $\beta(c_A)\in C(c_A)$.

Second, we prove that β can be reached through a PMAS. We define

$$y_i^s = \frac{D_i(h_i + CI_c)}{\sum_{i\in N} D_i(h_i + CI_c)} \sqrt{\sum_{i\in N} 2AD_i(h_i + CI_c)} + D_i CI_c t_0, \forall i \in S, \varnothing \neq S \subseteq N \quad (14)$$

Obviously $\sum_{i\in S} y_i^S = c_A(S)$. In addition, by Proposition 1, we can see that $\forall i \in S, \varnothing \neq S \subset R \subset N$,

$$y_i^S = D_i T_s^*(h_i + CI_C) + D_i CI_C t_o \geq D_i T_R^*(h_i + CI_C) + D_i CI_C t_o = y_i^R. \quad (15)$$

Hence, $y = (y_i^S)_{i\in S\leq N, S\neq\emptyset}$ is a PMAS such that $y_i^N = \beta_i(c_A)$.

To complete this section, we prove that *the grand coalition is also a stable outcome from a farsighted perspective.* As we already announced, we adopt the concept of LCS to analyze stability from a farsighted view. A Similar application of LCS can be seen in Li et al. (2014). The following Theorem states that under the core distribution *β,* the grand coalition is also farsighted stable.

Theorem3. Let $< N, A, C, \mathrm{t}_0, \mathrm{I}_{\mathrm{c},} \{\mathrm{D}_{\mathrm{i},}\mathrm{h}_{\mathrm{i},}\}_{\mathrm{i}\in\mathrm{N}} >$ be an inventory situation with advance payments and (*N*, c_A) the corresponding inventory game. The grand coalition is a farsighted stable outcome under the core distribution *β*.

Proof. Suppose we have *n* players in game from the set of {1,2,...,*n*}, we show that any deviation from the grand coalition is deterred and return in the grand coalition. Hence as the results the grand coalition is farsighted stable. In following has shown if *k* retailers want to deviate from the grand coalition is deterred by sequence:

$$\{1,2,...,n\} \rightarrow c\{(1,...,k),(k+1,..,n)\} \rightarrow c_1\{1,(2,...,k),(k+1,...,n)\} \rightarrow .... \rightarrow c_n\{1,..,n\} \quad (16)$$

Consider,

$$C = \{1,2,...,k\}\ ,\ C_1 = \{1\}\ ,\ ....\ ,\ C_n = \{n\}\ ,\ C_{n+1} = \{1,2,...,n\} \quad (17)$$

Let,

$$\omega_1 = \{(1,...,k),(k+1,..,n)\}\ ,\ \omega_2 = \{1,(2,...,k),(k+1,...,n)\}\ ,\ \omega_{n+1} = \{1,..,n\}\ ,\ \omega_{n+2} = \omega = \{(1,...,n)\} \quad (18)$$

According to lemma (2), we have:

$$\omega_1 < c_1\omega\ ,\ \omega_2 < c_2\omega\ ,\ ...\ ,\ \omega_{n+1} < c_{n+1}\omega \quad (19)$$

We can see $\omega_1 \subset \omega$ and $\omega_1 \neq c\omega$, hence the deviation by (1,..,*k*) is deterred.

Our last result shows that under the core distribution *β* the grand coalition is in the LCS, that is, the grand coalition is also farsighted stable. Therefore, from Theorems 2 and 3, we can conclude that the grand coalition is stable both from a myopic and farsighted point of view.

In the following we illustrate our cooperative model by means of a numerical example and develop sensitivity analysis that evaluates the changes produced on the core distribution proposed.

5- Numerical example and sensitivity analysis

We consider a two echelons supply chain with one supplier and four retailers who purchase a single good. The common parameters are A=900, C=500, I_c=0.15, t_0=0.1, and the individual parameters for each retailers shown in table 2.

Table 2. Data for each retailer

i	D_i	h_i
1	1000	400
2	1400	250
3	900	300

First we study the optimal replenishment cycle and minimal average cost per time unit for each retailer when they place their orders separately. All of them can be obtained from (2) and (3), respectively, and they are shown in table 3.

Table 3. Optimal replenishment cycle and minimal average cost when the retailers place order separately

I	T_i^*	$TC_i(T_i^*)$
1	0.061559	36740.38
2	0.062897	39118.18
3	0.07303	31397.52

When the retailers cooperate, the inventory cost with advance payments for each coalition can be obtained from (6), and the optimum replenishment cycle time for each coalition is calculated according (5). By using Theorem2 we can obtain a core distribution β that is reached through a PMAS (β_i^S). In table 4 we can see the optimal replenishment cycle, the cost game, and the PMAS. Notice that the distribution for the grand coalition is given by $\beta_1=25400.12$, $\beta_2=27646.43$, $\beta_3=19468.5$.

Table 4 shows that (1) larger coalitions have smaller replenishment cycle time ($T_N^*<T_S^*$, *for all S*), (2) the cost of gran coalition (72515.05) is lower than the added individual costs (74186.08), (3) the PMAS, which the core distribution β is reached through, reduces considerably the costs of the retailers in the grand coalition ($\beta_i^N << \beta_i^S$, *for all S*).

Now we analyze the changes produced in the core distribution β and the optimal replenishment cycle for the grand coalition (T_N^*) under small changes in the parameters of the model. The results of this sensitivity analysis for β and T_N^* are shown in tables (5) and (6), respectively.

Table 4. Cost game and cost distributions

Coalition	T_S*	$c_A(S)$	(β_i^S) PMAS
{1}	0.061559	36740.38	$\beta_1^{\{1\}}$=36740.38
{2}	0.062897	39118.18	$\beta_2^{\{2\}}$=39118.18
{3}	0.073030	31397.52	$\beta_3^{\{3\}}$=31397.52
{1,2}	0.043900	58914.54	$\beta_1^{\{1,2\}}$=28397.21 , $\beta_2^{\{1,2\}}$=30517.33
{1,3}	0.047000	52492.65	$\beta_1^{\{1,3\}}$=29857.24 , $\beta_2^{\{1,3\}}$=22635.41
{2,3}	0.047600	55019.03	$\beta_2^{\{2,3\}}$ =32184.43 , $\beta_3^{\{2,3\}}$=22834.6
{1,2,3}	0.037600	72515.05	β_1^N=25400.12 , β_2^N =27646.43 , β_3^N =19468.5

Table 5. Sensitivity analysis for β

Variable β	Changes in parameter A (%)					
	-75	-50	-25	25	50	75
$\beta 1$	16450.1	20157.3	23002	27512.9	29423.1	31179.6
$\beta 2$	19073.2	22624.4	25349.2	29670.3	31500	33182.6
$\beta 3$	13109.3	15743.3	17764.5	20969.7	22326.9	23575
	Changes in parameter C (%)					
	-75	-50	-25	25	50	75
$\beta 1$	18955.6	21106	23254.2	27543.5	29684.3	31822.5
$\beta 2$	17972	21217.5	24441.6	30833.6	34004.5	37160.5
$\beta 3$	13389	15425	17451.3	21477.4	23478.7	25472.7
	Changes in parameter D_i (%)					
	-75	-50	-25	25	50	75
$\beta 1$	10825.1	16407.3	21127	29387.9	33173.1	36804.6
$\beta 2$	11198.2	17374.4	22724.2	32295.3	36750	41057.6
$\beta 3$	8046.75	12368.3	16077	22657.2	25701.9	28637.5
	Changes in parameter h_i (%)					
	-75	-50	-25	25	50	75
$\beta 1$	17973.8	20908.3	23311.4	27270.9	28980	30563.4
$\beta 2$	22021.2	24149.1	25995.2	29153.4	30548	31852.2
$\beta 3$	14829.8	16621.2	18134.2	20675.6	21786	22820
	Changes in parameter I_c (%)					
	-75	-50	-25	25	50	75
$\beta 1$	18955.6	21106	23254.2	27543.5	29684.3	31822.5
$\beta 2$	17972	21217.5	24441.6	30833.6	34004.5	37160.5
$\beta 3$	13389	15425	17451.3	21477.4	23478.7	25472.7
	Changes in parameter t_0 (%)					
	-75	-50	-25	25	50	75
$\beta 1$	19775.1	21650.1	23525.1	27275.1	29150.1	31025.1
$\beta 2$	19771.4	22396.4	25021.4	30271.4	32896.4	35521.4
$\beta 3$	14406	16093.5	17781	21156	22843.5	24531

Each parameter changes from -25% to +75% and the effect of these changes on distribution β is shown in figures (2) - (7). We can conclude that the cost distribution β is slightly sensitive to increases in each parameter. It is highly sensitive to variations in demand but it is almost insensitive to variations in holding cost per time unit.

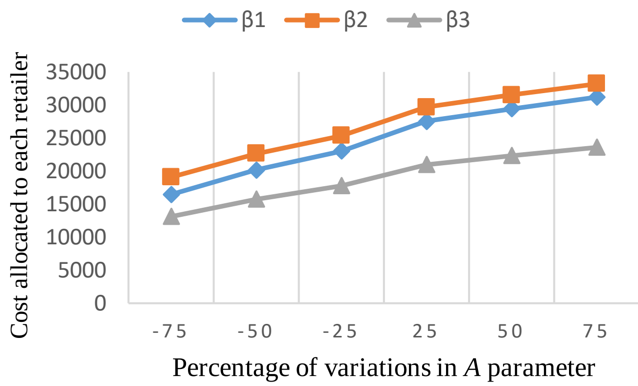

Fig 2. Effect of changes in A on β

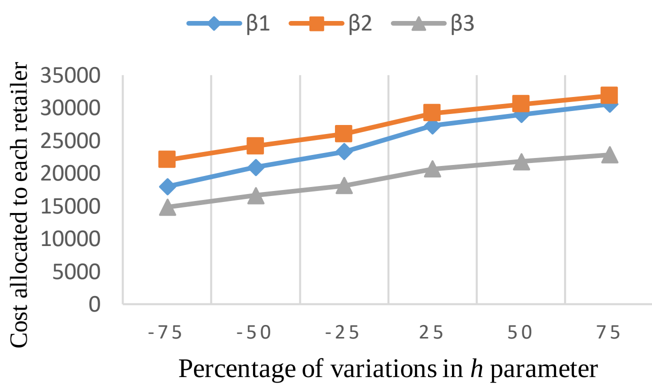

Fig 3. Effect of changes in h_i on β

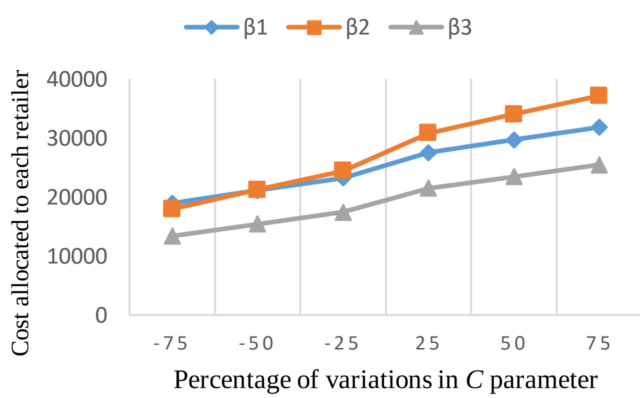

Fig 4. Effect of changes in C on β

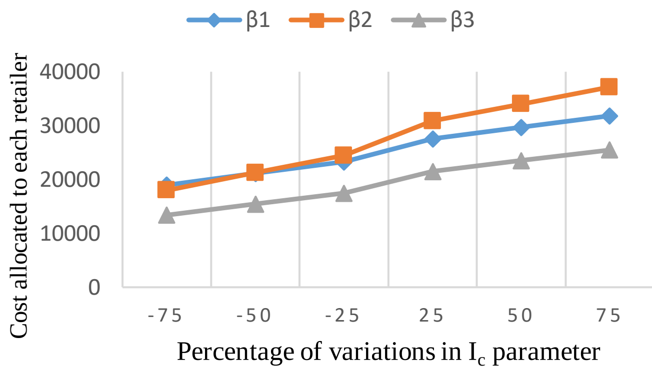

Fig 5. Effect of changes in I_c on β

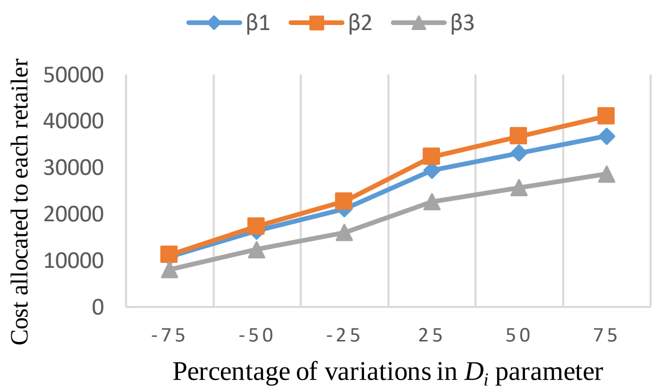

Fig 6. Effect of changes in D_i on β

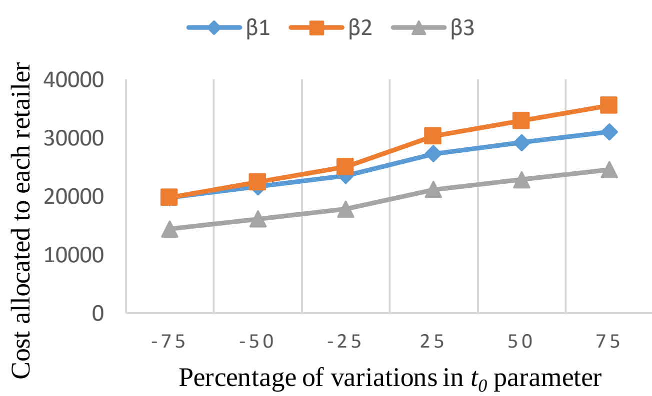

Fig 7. Effect of changes in t_0 on β

Sensitivity analysis for T_N^* in return of parameters changes shown in table (6), and graphically can be seen in figure (8). It shows that the optimal replenishment cycle for the grand coalition is also highly sensitive to variations in demand as well as to variations in fixed ordering cost per order and holding cost per time unit.

Table 6. Sensitivity analysis for T_N^*

Variable T_s^*						
	Changes in parameter A (%)					
	-75	-50	-25	25	50	75
T_s^*	0.01884	0.02665	0.03264	0.04213	0.04615	0.04985
	Changes in parameter C (%)					
	-75	-50	-25	25	50	75
T_s^*	0.04079	0.03967	0.03864	0.0368	0.03597	0.0352
	Changes in parameter D_i (%)					
	-75	-50	-25	25	50	75
T_s^*	0.07537	0.05329	0.04351	0.03371	0.03077	0.02849
	Changes in parameter h_i (%)					
	-75	-50	-25	25	50	75
T_s^*	0.05985	0.04875	0.04216	0.03438	0.03182	0.02976
	Changes in parameter I_c (%)					
	-75	-50	-25	25	50	75
T_s^*	0.04079	0.03967	0.03864	0.0368	0.03597	0.0352
	Changes in parameter t_0 (%)					
	-75	-50	-25	25	50	75
T_s^*	0.03768	0.03768	0.03768	0.03768	0.03768	0.03768

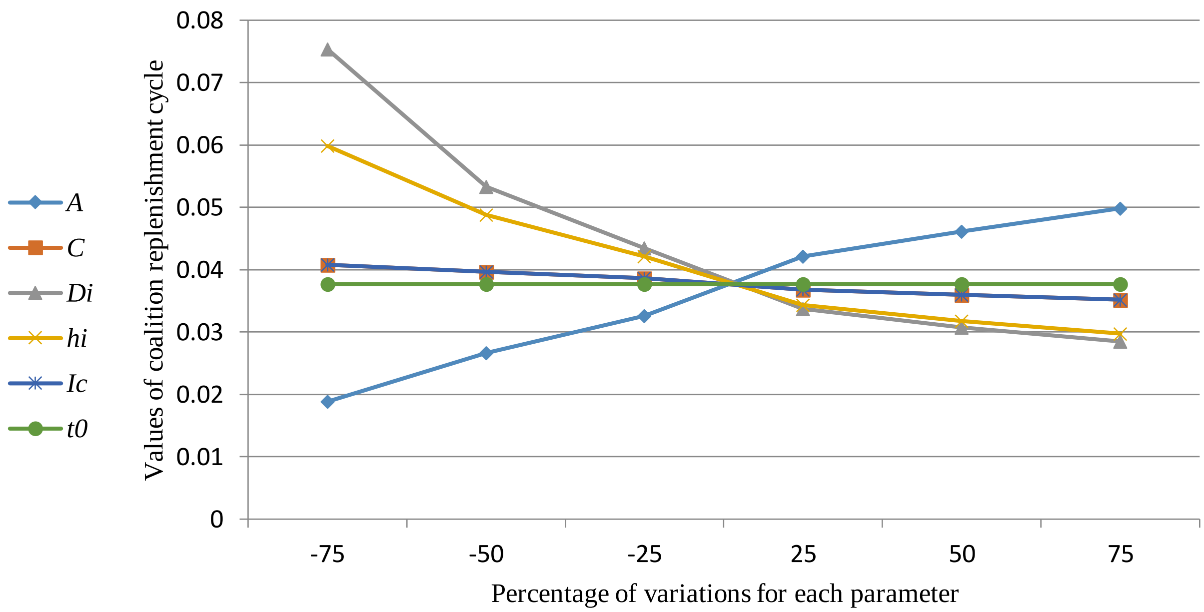

Fig 8. Optimal replenishment cycle values related to changes in each parameter

6- Conclusions

In this paper we had considered an advance payment purchasing system. In such a system the capital cost of the retailers will increase because they have incurred interest costs on the purchasing cost of products which have not been received yet. Each retailer seeks for determining its optimal inventory costs by developing an economic order quantity model with advance payment for multiple retailers. If the retailers decide to cooperate and place their orders jointly, they can reduce the total average cost generated by the cooperation; i.e. cooperation of all the retailers is more beneficial than individual actions. By using cooperative game theory we have studied such cooperation and have presented the class of inventory games with advance payments. We have proved that this class of games is strategically equivalent to the class of inventory cost games introduced by Meca et al. (2004). A core distribution, as well as a population monotonic allocation scheme, for those games has been proposed. Then we have examined the stability of grand coalition from both a myopic and farsighted perspective, and have come to the conclusion that the grand coalition is always stable from both points of view. Finally, we have developed a sensitivity analysis for the core distribution proposed and for the optimal replenishment cycle of the grand coalition. This analysis has shown that the core distribution is highly sensitive to variations in demand but it is almost insensitive to variations in holding cost per time unit. Likewise, the optimal replenishment cycle for the grand coalition is highly sensitive to variations in demand as well as to variations in fixed ordering cost per order and holding cost per time unit.

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