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A comprehensive common weights data envelopment analysis model: Ideal and anti-ideal virtual decision making units approach

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Abstract

Data envelopment analysis (DEA) calculates the relative efficiency of homogenous decision-making units (DMUs) with multiple inputs and outputs. Classic DEA models usually suffer from several issues such as: discrimination power, variable weights of inputs/outputs, inaccurate efficiency estimation for small number of DMUs, incapability in working with zero and negative data, and not having exterior target. Ranking methods in DEA have been proposed to resolve such issues. In this paper an approach is proposed to overcome all of these issues, concurrently. This approach has five main properties: 1) using common weight methodology to reduce the chance of inefficient DMUs to be evaluated as efficient; 2) defining virtual ideal and anti-ideal DMUs in a unique model concurrently to improve the discrimination power and to make exterior target based on observed DMUs; 3) providing full ranking for even production possibility sets (PPS) with low number of DMUs; 4) handling for zero and non-positive data; 5) Ranking all DMUs in a single run which reduces the computational efforts effectively. The properties of the model of this study including convexity, feasibility, and optimality are discussed through several theorems. The validity of the model is illustrated through solving five benchmark numerical examples adopted from the literature of past works. The results of the model are compared with those of existing methods. The results illustrate the efficacy and comparability of proposed approach among the existing methods.

Keywords: Common weight DEA, virtual DMU, DEA ranking, efficiency evaluation

1-Introduction

Data Envelopment Analysis (DEA), as a nonparametric technique, has extensively been used to measure the efficiency of homogenous Decision-Making Units (DMUs) with the multiple inputs and outputs (Cooper et al., 2011; Zhu, 2009). Thirty five years after the seminal work by Charnes et al. (1978), a huge amount of research on different variations of DEA approaches has been reported in the literature (Liu et al., 2013). In traditional DEA models, each DMU can select the most preferred weights in order to maximize its efficiency score. Therefore, the efficiencies of DMUs are calculated by various weights and, consequently, inefficient DMUs may be evaluated as efficient and discrimination power of DEA may be reduced. As more than one efficient DMU usually exists, ranking will face some difficulties. To overcome these issues, many approaches have been proposed.

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Alder et al., (2002) have classified the rankings models into six groups as: 1) cross efficiency method; 2) super efficiency method; 3) benchmark ranking method; 4) multi-variable statistical methods; 5) non-efficient units ranking method by using relative values of non-efficiency dominance; 6) methods based on complementary data, including data related to the preferences of decision-makers and also the data related to the combination of multi-criteria decision-making approaches and DEA.

Sun et al., (2013) have classified ranking approaches into two main categories as cross efficiency methods, and the common weights methods. The first category provides a peer-evaluation score rather than a self-evaluation score. Then, by calculating the mean of optimal weights, a set of weights can be obtained. Kao and Hung (2005) claimed that cross efficiency methods did not consider all data in inputs and outputs and it was very likely that no DMU was recognized as efficient unit. The second category includes methods which are involved to calculate common weights.

Hosseinzadeh Lotfi et al. (2013) reviewed the literature of ranking models in DEA. Hossenzadeh Lotfi et al. (2013) divided the ranking models in DEA into seven groups as cross-efficiency matrix, optimal weights obtained from multiplier model of DEA, super-efficiency methods, benchmarking through setting a target for inefficient units, multivariate statistical techniques, ranking inefficient units through proportional measures of inefficiency, multiple-criteria decision methodologies combined with the DEA technique, and other methods of ranking units.

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In this paper, a ranking method is proposed. The ranking method of this study has four main properties:

1) using common weight methodology to reduce the chance of inefficient DMUs to be evaluated as efficient; 2) defining virtual ideal and anti-ideal DMUs in a unique model concurrently to improve the discrimination power and to make exterior target based on observed DMUs; 3) providing full ranking for even production possibility sets (PPS) with low number of DMUs; 4) handling for zero and non-positive data; 5) Ranking all DMUs in a single run which reduces the computational efforts effectively. The properties of the model of this study including convexity, feasibility, and optimality are discussed through several theorems. The applicability and efficacy of proposed approach is tested by several numerical examples. The results of proposed approach and those of existing methods are compared.

The rest of the paper is organized as follows. In section 2, the literature of relevant past research works is briefly reviewed. In section 3, the proposed ranking model and its theoretical properties are developed. In section 4, four numerical examples adopted from literature are solved using the proposed approach. The results of the model of this study are compared with those existing approaches in the literature to illustrate the applicability and efficacy of the proposed approach. In section 5, an algorithmic procedure is developed to handle the cases including negative and zero inputs/outputs. An illustrative example is also provided in Section 5 to demonstrate the applicability of proposed algorithm for zero and negative inputs/outputs. A statistical analysis is also described in section 5 in order to compare the performance of model of this study with existing methods in the literature. Finally, the paper will be concluded in section 6 with a brief summary, research findings, and contribution of the research, research limitations, managerial implications, and suggestions for future research.

2-Literature of Past Works

In this section literature of relevant past research works are reviewed and discussed.

2-1-Ranking Approaches in DEA

Unfortunately, classic DEA models may suffer from a critical problem called weak discrimination power. For production possibility sets (PPSs) which include low number of DMUs in comparison with number of inputs and outputs, several DMUs may be assessed as efficient with efficiency score equal to one. Experimentally, as a rule of thumb, the relation $n \geq 3\times(m+s)$ should be satisfied for an acceptable discrimination power in a DEA models, where *n* is number of DMUs, *m* is number of inputs, and *s* is number of outputs. Several ranking procedures have been proposed to overcome the above mentioned issue. The most notable ones are Andersen and Petersen's (1993) super-efficiency model, and the slack-based measure introduced by Tone (2002). Zhu (2001) discussed and reviewed the use of super-efficiency approach in data envelopment analysis (DEA). Amirteimoori (2007) proposed an alternative efficiency measure by using efficient and anti-efficient frontiers. Jahanshahloo et al. (2010) proposed two ranking methods. In the first method, an ideal line was defined and common set of weights was determined for efficient DMUs. In the second method, a special line was defined. Then all efficient DMUs were compared with it.

Lee et al. (2011) developed a two-stage process for calculating super-efficiency scores. The proposed approach examined whether the standard VRS super-efficiency DEA model was infeasible. When the model was feasible, the proposed approach by Lee et al. (2011) yielded super-efficiency scores that were identical to those arising from the original model. Under the super-efficiency model for efficient infeasible DMUs, the proposed approach by Lee et al. (2011) yielded super-efficiency scores that characterized input savings and/or output surpluses. Chen and Liang (2011) decreased the computation burden of model proposed by Lee et al. (2011) and proposed a single DEA-based model in order to check the infeasibility of a standard VRS super-efficiency model.

Noura et al. (2011) presented a super-efficiency method by which units that were more effective and useful in society had better ranks. In fact, in order to determine super-efficiency using the method by Noura et al. (2011), the effectiveness of each unit in society was considered rather than the cross-comparison of the units. Noura et al. (2011) divided the inputs and outputs into two groups, desirable and undesirable, at the discretion of the manager, and assigned weights to each input and output. Noura et al. (2011) determined the rank of each DMU according to the weights and the desirability of inputs and outputs. Khalili-Dmaghani et al. (2011) proposed a hybrid approach based on fuzzy DEA and simulation to measure the efficiency of agility in supply chain. Khalili-Dmaghani et al. (2011) used a simulation based approach in order to rank the DMUs in presence of uncertain data. Lee and Zhu (2011) proposed a super-efficiency ranking approach for variable return to scale (VRS) conditions.

Foroughi (2013) proposed a generalized model to find most BCC-efficient DMU. The proposed approach by Foroughi (2013) was applicable for all assumptions of returns-to-scale. Chen (2013) proposed a super-efficiency approach to rank efficient DMUs using slack based measure (SBM). Based on proposed approach by Chen (2013) simultaneous computation of SBM scores for inefficient DMUs and super-efficiency for efficient DMUs were achieved.

Chen et al. (2013) addressed the problem of infeasibility in conventional radial super-efficiency DEA models under VRS. Chen et al. (2013) developed a measure of super-efficiency based on a directional distance function to resolve this problem. Chen et al. (2013) proposed a model to modify the directional distance function by selecting proper feasible reference bundles. As a result, the modified VRS super-efficiency model proposed by Chen et al. (2013) successfully addressed the infeasibility issues occurring either in conventional VRS models or in the super-efficiency model. Recently, Oukil and Amin (2015) proposed a ranking method based on cross-evaluation, preference voting and ordered weighted averaging in order to improve the discrimination among DMUs while offering more flexibility to the decision process. They applied the proposed approach on an example involving 15 baseball players. Khodabakhshi and Aryavash (2015) proposed an equitable method for ranking DMUs based on minimum and maximum efficiency values in DEA.

Tavassoli et al. (2015) integrated slacks-based measure (SBM), strong complementary slackness condition (SCSC), and data envelopment analysis–discriminant analysis DEA–DA to improve the discrimination power of classic DEA models. They applied the approach in a case study of electricity distribution units.

Oral et al. (2015) used the benefit of multiple optimal solutions in DEA in order to develop a cross-efficiency method that was most appreciative for all DMUs being cross-evaluated by all others. They proved the theoretical superiority of proposed model and illustrated its applicability using benchmark instances adopted from literature.

Wu el al. (2015) proposed a cross-efficiency evaluation approach based on Pareto improvement, which contained two models (Pareto optimality estimation model and cross-efficiency Pareto improvement model) and an algorithm. The Pareto optimality estimation model was used to estimate whether the given set of cross-efficiency scores were Pareto-optimal solutions. If these cross-efficiency scores were not Pareto optimal, the Pareto improvement model was used to make cross-efficiency Pareto improvement for all the DMUs.

Hinojosa et al. (2017) proposed a new approach based on the Shapley value, for ranking efficient DMUs. They formulated the problem for a radial, input-oriented with constant returns to scale (CRS) DEA model. Liu (2018) proposed a new ranking method based on cross-efficiency and signal-to-noise (SN). In order to overcome the difficulties caused by the possible existence of multiple optimal weights for the DEA, he considered cross-efficiency intervals and their variances to rank all DMUs.

Rezaeiani and Foroughi (2018) introduced the concept of the reference frontier share and developed a measure for ranking efficient DMUs. One advantage of this approach to other existing ones is that it can rank the extreme and non-extreme efficient units. Amirkhan et al. (2018) proposed a fuzzy-robust approach to overcome the difficulties and limitations associated with the problems having values for the inputs and outputs of DMUs.

2-1-Common weight approaches in DEA

In classic DEA analysis a set of linear programming should be solved in order to achieve the efficiency scores of all DMUs. More formally *n* LPs should be solved to achieve the efficiency scores of all DMUs in PPS, where n is number of DMUs. In each run, the DMU, which is under assessment, is set free to determine the weights of inputs and outputs criteria in a way that its efficiency score is maximized. This procedure is useful as if a DMU is not efficient with DEA in presence of such freedom, will also not be efficient with any other method. Although a main shortage is occurred. The weight of inputs and outputs are different through runs of DEA. So, this freedom may cause an inefficient DMU is assessed as efficient. Several research works have been dedicated to address this concern. Among them the following researches are interesting to be mentioned here.

Jahanshahloo et al. (2005) proposed a common set of weights (CSW) approach by means of solving only one problem. Jahanshahloo et al. (2005) also presented a method for ranking DMUs. Cook and Zhu (2007) proposed a CSW approach in order to evaluate a set of DMUs which might be under management of certain team. Under such conditions, the multipliers of inputs and outputs were equally set. Cook and Zhu (2007) developed a goal-programming model to calculate common-multiplier set. The important feature of those multiplier set was that it minimized the maximum discrepancy among the within-group scores from their ideal levels. Motivated by the work of Cook and Zhu (2007), Cook et al. (2017) proposed a new DEA-based approach for the benchmarking of DMUs and the setting of targets.

Kao (2010) proposed a CSW approach in order to calculate the Malmquist Productivity Index (MPI). Kao (2010) proposed the approach in order to eliminate the inconsistency caused by using different frontier facets to calculate efficiency in multi-period of planning. Kao (2010) proposed a common-weights DEA model for time-series evaluations to calculate the global MPI so that the productivity changes of all DMUs had a common basis for comparison.

Wang et al. (2011) proposed a new methodology based on regression analysis to seek a common set of weights to fully rank the DMUs. The DEA efficiencies obtained with the most favorable weights to each DMU were treated as the target efficiencies of DMUs and were best fitted with the efficiencies determined by common weights. Two new nonlinear regression models were constructed to optimally estimate the common weights.

Ramón et al. (2012) proposed a DEA approach based on CSW to rank DMUs. The idea of approach proposed by Ramón et al. (2012) was to minimize the deviations of the CSW from the DEA profiles of weights. The proposed approach by Ramón et al. (2012) reduced in particular the differences between the DEA profiles of weights that were chosen, so the CSW method proposed by Ramón et al. (2012) was a representative summary of such DEA weights profiles. Ramezani-Tarkhorani et al., (2014) showed that the criteria used by Liu and Peng (2008) were not theoretically strong enough to discriminate among the common weight approach-efficient DMUs with equal efficiency. Moreover, there was no guarantee that their proposed model can select one optimal solution from the alternative components. The optimal solution was considered to be the only unique optimal solution. The study by Ramezani-Tarkhorani et al., (2014) showed that the proposal by Liu and Peng (2008) was not generally correct. Toloo (2014) proposed a new basic integrated linear programming (LP) model to identify candidate DMUs for being the most efficient unit. Two numerical examples were illustrated to show the variant use of those models in different important cases.

Puro, Yadav and Garg (2017) proposed a new multi-component DEA (MC-DEA) approach using common set of weights methodology based on interval arithmetic and unified production frontier. Using this new methodology, they determined unique weights for measuring interval efficiencies. They considered imprecise data as well as undesirable outputs and shared resources. Gharakhani and Toloie Eshlaghy (2018) proposed a new approach for seeking a common set of weights in dynamic network-DEA models based on the goal programming (GP) technique. The proposed approach makes it possible to monitor dynamic change of the period efficiency.

Jahanshahloo et al. (2018) proposed a new effective method for fixed cost allocation based on the efficiency invariance and common set of weights.

Table 1 presents some selected studies in the literature regarding ranking methods, common weights, and multi-objective DEA models as well as the proposed model of this study.

Table 1. Overview of selected studies on ranking methods in DEA

Author(s)	Year	Main Theme			Number Of Model Run(s)	Discrimination Improvement	Computational Effort	Inconsistent Ranks	Zero Data	Negative Data	Infeasibility Occurrence	Ideal-DMU	Anti-Ideal DMU	Return to Scale	Case Study	Benchmark Instances	Main Contribution
		Ranking Idea	Common Weight	Multi-Objective													
Sexton et al.	1986	☑			Many	☑	High							☑			Cross-Efficiency
Roll et al.	1991		☑		Single	☑	Medium				☑						Common Weight
Ganley and Cubbin	1992		☑		Multiple	☑	Low	☑									Common Weight
Roll and Golany	1993		☑		Single	☑	Low										Common Weight
Andersen and Petersen	1993	☑			Many	☑	High				☑						Modifying PPS
Cook et al.	1993		☑		Single	☑	Low										Common Weight
Tone	2002	☑			Many	☑	High										Slack-Based Measure
Adler et al.	2002	Critical Review of Ranking Methods in DEA-Classification of six ranking methods															
Jahanshahloo et al.	2005		☑		Single	☑	Low										Common Weight
Cook and Zhu	2007		☑	☑	Single	☑	Low					☑					Goal Programming
Amirteimoori	2007	☑			Many	☑	High	☑									efficient & anti-efficient
Liu and Peng	2008		☑		Single	☑	Low							☑		☑	Common Weight
Jahanshahloo et al.	2010		☑		Single	☑	Low	☑				☑		☑			Common Weight
Kao	2010		☑		Multiple	☑	Medium	☑									Multi-period Malmquist
Lee et al.	2011	☑			Many	☑	High				☑			☑			Two-Stage Ranking
Chen and Liang	2011	☑			Single	☑	Medium				☑						Single Model Ranking
Noura et al.	2011	☑			Many	☑	Medium										Effectiveness of DMU
Khalili-Dmaghani et al.	2011	☑			Many	☑	High								☑		Simulation Approach
Lee and Zhu	2011	☑			Many	☑	High							☑			Super Efficiency
Wang et al.	2011	☑	☑		Multiple	☑	Medium					☑					Regression Analysis
Khodabakhshi and Aryavash	2012	☑			Many	☑	High				☑						Min-Max Efficiency
Ramón et al.	2012	☑	☑		Multiple	☑	Medium										Profile of Weights
Foroughi	2013	☑			Many	☑	High							☑			Ranking all RTS
Chen	2013	☑			Many	☑	High										Super efficiency/SBM
Chen et al.	2013	☑			Many	☑	High				☑			☑			directional distance function
Sun et al.	2013		☑		Multiple	☑	Medium	☑				☑	☑				Ideal &Anti-ideal DMU
Hosseinzadeh Lotfi et al.	2013	Critical Review of Ranking Methods in DEA-Classification of ranking methods in seven groups															
Ramezani-Tarkhorani et al.	2014		☑	They showed that the proposal by Liu and Peng (2008) was not generally correct.													
Toloo	2014	☑			Single	☑	Low									☑	Linear Programing
Oukil and Amin	2015	☑			Multiple	☑	High										Hybrid Approach
Khodabakhshi and Aryavash	2015	☑			Multiple	☑	Medium	☑									Optimistic-Pessimistic
Tavassoli et al.	2015	☑			Multiple	☑	High								☑	☑	SBM, SCSC, DEA-DA
Oral et al.	2015	☑			Multiple	☑	High									☑	Cross-Efficiency
Wu et al.	2015	☑			Multiple	☑	High									☑	Pareto Improvement
Proposed Method	-	☑	☑	☑	Single	☑	Low		☑	☑		☑	☑	☑		☑	Combined unique model

According to the past literature, including ranking methods in DEA, common weight approaches in DEA, and multi-objective approaches in DEA, there are several shortcomings and pitfalls in the existing methods. These issues are summarized as follows:

(1) In order to improve the discrimination power of DEA, usually procedures including several models should be solved. Moreover, high computational efforts are usually required.

(2) Ranking DMUs based on targets formed based on observed DMUs can make suitable drivers for all DMUs to progress. Ideal DMU can be assumed as best target formed based on observed DMUs while anti-ideal DMU can be assumed as worst target formed based on observed DMUs. There is no unique model in literature to rank DMUs based on their situation in comparison with both ideal and anti-ideal DMUs in the PPS. So, distance of a given DMU cannot be calculated simultaneously from ideal DMU and anti-ideal DMU.

(3) The classic two-phase approaches usually distinct the efficient and inefficient DMUs in the first phase, and then in the second phase the efficient DMUs are ranked using an extra procedure in order to recognize strong efficient and weak-efficient DMUs.

(4) Some of the existing super-efficiency approaches in the literature may have infeasibility issues. For instance, the most common ranking procedure by Andersen and Petersen (1993) may cause infeasibility issues while ranking DMUs. So, most of them required more customizations and extensions in order to resolve the infeasibility issues.

(5) The ranking methods are so sensitive about return to scale conditions. Under such situations most of them may report infeasibility under variables return to scale (VRS) conditions.

(6) The existing ranking method cannot handle DMUs with negative of zero data, while this may occur in real cases.

In this paper, based on the method proposed by Sun et al. (2013), we are going to propose an approach in order to resolve the above mentioned issues. The main contributions of the proposed method in this study in comparison with the existing methods in the literature are summarized as follows:

(1) Enhancing the discrimination power of DEA models through a single model.

(2) Determining a common set of weights for all inputs and outputs.

(3) Proposing a unique model in order to fully rank all DMUs in PPS with low number of DMUs.

(4) Considering both ideal and anti-ideal DMUs as best and worst targets in a unique Model.

(5) Illustrating the performance of proposed approach in comparison with existing approach using several numerical examples adopted from literature.

(6) Extending the proposed approach in order to handle zero and negative data.

(7) Ranking any number of DMUs in a single run of the model.

(8) No sensitivity about return to scale assumptions.

(9) No chance for generating infeasible solutions.

3- The proposed Approach

In order to make a better understanding of the basis of the model of this study, the models by Sun et al., (2013) is revisited here briefly.

3-1- Ranking models proposed by Sun et al. (2013)

The models proposed by Sun et al., (2013) rank DMUs through two approaches. The first approach is based on comparing the DMUs by an ideal decision making unit (IDMU) and the second approach is based on comparing the DMUs by an anti-ideal decision making unit (ADMU).

3-1-1- Ranking by Ideal Decision Making Unit (IDMU)

Suppose n DMUs consume m different inputs to produce s different outputs which can be shown by x_{ij} and y_{rj}, $r=1,2, \dots, s$; $j=1,2, \dots, n$; $i=1,2, \dots, m$, respectively. Similar to traditional DEA models, it is assumed that all data are positive. The data are arranged in matrices such that the input matrix has m rows and n columns, and the output matrix has s rows and n columns. The smallest values in each row of input matrix are considered as the inputs of the IDMU. In the same way, the largest values in each row of output matrix are considered as the outputs of the IDMU. The IDMU can be formed as (1)-(3).

$$\text{IDMU} = (xmin_i, ymax_r) \tag{1}$$

$$xmin_i = \min \{ x_{ij} \mid j = 1,\ldots,n \} , (i = 1,\ldots,m) \tag{2}$$

$$ymax_r = \max \{ y_{rj} \mid j = 1,\ldots,n \} , (r = 1,\ldots,s) \tag{3}$$

The IDMU is assumed as reference set for other DMUs. The objective function determines the minimum distance between each DMU and IDMU. In the constraints, the efficiency of IDMU is considered to be equal to one. At the same time, each DMU seeks to maximize its efficiency score (Sun et al., 2013)

3-1-2- Ranking by anti-ideal decision making unit (ADMU)

The second approach is based on comparing the DMUs by an Anti-Ideal Decision Making Unit (ADMU). Again, the input matrices have *m* rows and *n* columns and the output matrices have *s* rows and *n* columns. The largest values in each row of input matrices are considered as the inputs of the ADMU. The smallest values in each row of output matrices are considered as the outputs of ADMU. The ADMU can be formed as (4)-(6).

$$ADMU = (xmax_i, ymin_r) \tag{4}$$

$$xmax_i = \max \{ x_{ij} \mid j = 1,\ldots,n \} , (i = 1,\ldots,m) \tag{5}$$

$$ymin_r = \min \{ y_{rj} \mid j = 1,\ldots,n \} , (r = 1,\ldots,s) \tag{6}$$

The ADMU is assumed as reference set for other DMUs. The objective function determines the minimum distance between each DMU and ADMU. In the constraints, the efficiency of ADMU is considered to be equal to one. At the same time, each DMU seeks to maximize its efficiency (Sun et al., 2013).

3-2- Proposed approach

Suppose *n* DMUs, each one consumes *m* different inputs to produce *s* different outputs which can be shown by x_{ij} and y_{rj}, $r=1,2, \ldots, s$; $j=1,2, \ldots, n$; $i=1,2, \ldots, m$, respectively. Combining the IDMU and ADMU approaches considering a common weight procedure, a new integrated model is developed to fully rank DMUs. The proposed comprehensive model is developed as (7)-(16).

$$Min\ Z = \sum_{j=1}^{n} \delta_j + \sum_{j=1}^{n} \sigma_j \tag{7}$$

S.t.

$$\sum_{i=1}^{m} \omega_i x_{ij} - \delta_j = \sum_{r=1}^{s} \mu_r y_{rj} \ , \ j = 1,\ldots,n \tag{8}$$

$$\sum_{i=1}^{m} \omega_i xmin_i = 1 \tag{9}$$

$$\sum_{i=1}^{m} \mu_r ymax_r = 1 \tag{10}$$

$$\sum_{i=1}^{m} \varphi_i x_{ij} + \sigma_j = \sum_{r=1}^{s} \gamma_r y_{rj} \ , \ j = 1,\ldots,n \tag{11}$$

$$\sum_{i=1}^{m} \varphi_i xmax_i = 1 \tag{12}$$

$$\sum_{i=1}^{m}\gamma_r ymin_r = 1 \tag{13}$$

$$\omega_i \ ,\varphi_i \geq \varepsilon, \quad \forall i \tag{14}$$

$$\mu_r \ ,\gamma_r \geq \varepsilon, \quad \forall r \tag{15}$$

$$\delta_j ,\sigma_j \quad free\ in\ sign, \quad \forall j \tag{16}$$

The objective function (7) minimizes the summation of distances between DMUs and IDMU and the summation of distances between DMUs and ADMU, concurrently. The set of constraints (8), which is written for all DMUs, guaranty the relative efficiency scores of DMUs be less than or equal to 1 considering the IDMU. Constraints (9)-(10) express that the efficiency score of IDMU is equal to 1. The set of constraints (11), which is written for all DMUs, guaranty the relative efficiency scores of DMUs be less than or equal to 1 considering ADMU. Constraints (12)-(13) express that the efficiency of ADMU should be equal to 1. Set of constraints (14) express that all input variables are non-negative. Set of constraints (15) express that all output variables are non-negative. Set of constraints (16) defines the free in sing variables. It is notable that ε is a non-Archimedean small positive value which prevents the multiplier to be zero. The proposed model (7)-(16) is linear programming and its global optimum solution can be achieved using existing optimization software packages.

The main outputs of model (7)-(16) include two different sets of common weights (i.e., $\omega_i \ ,\mu_r$, and $\varphi_i \ ,\gamma_r$), the sum of distances between DMUs and IDMU (i.e., $\sum_{j=1}^{n}\delta_j$) , and the sum of distances between DMUs and ADMU (i.e., $\sum_{j=1}^{n}\sigma_j$). Having these data would make it possible to evaluate the relative efficiency of each DMU. It is worth mentioning that all these data can be achieved in a single run of the model while in the model proposed by Sun et al., (2013) these results should be achieved through running two sets of models.

3-3- Theoretical properties of proposed model

Theorem 1. The constraints of the proposed model (7)-(16) constitute a non-empty convex set.

Proof. It is clear that the constraints of model (7)-(16) constitute a non-empty set which we call it Ω. If (ω,μ) and (ω',μ') belong to Ω, then for every $\alpha \in \ 0,1 \ : (\alpha\omega + (1-\alpha)\omega', \alpha\mu + (1-\alpha)\mu') \in \Omega$. In the same way, if (φ,γ) and (φ',γ') belong to Ω, for every $\beta \in (0,1): (\beta\varphi + (1-\beta)\varphi', \beta\gamma + (1-\beta)\gamma') \in \Omega$. Hence, Ω is a convex set. This completes the proof of the theorem 1.

Theorem 2. The objective function (7) of the model (7)-(16) is a convex set in the defined range.

Proof. The Hessian matrix of the objective function (1) is a zero matrix. Since the Hessian matrix of the objective function (1) is positive definite, the objective function (1) is a strictly convex function. This completes the proof of the Theorem 2.

Theorem 3. The model (7)-(16) is always feasible independent of values of inputs and outputs.

Proof. Suppose an arbitrary solution of the model (7)-(16) as follows:

$$\omega_i = \frac{1}{\mathrm{m}\times\mathrm{xmin}_\mathrm{i}} , \quad \forall i \tag{17}$$

$$\varphi_i = \frac{1}{\mathrm{m}\times\mathrm{xmax}_\mathrm{i}} , \quad \forall i \tag{18}$$

$$\mu_r = \frac{1}{\mathrm{r}\times\mathrm{ymax}_r}, \quad \forall r \tag{19}$$

$$\gamma_r = \frac{1}{\mathrm{r}\times\mathrm{ymin}_r}, \quad \forall r \tag{20}$$

$$\delta_j=\left(\frac{1}{m}\times\sum_{i=1}^{m}\frac{x_{ij}}{\mathrm{xmin_i}}\right)-\left(\frac{1}{r}\times\sum_{r=1}^{s}\frac{y_{rj}}{\mathrm{ymax}_r}\right),\quad \forall j \tag{21}$$

$$\sigma_j=\left(\frac{1}{r}\times\sum_{r=1}^{s}\frac{y_{rj}}{\mathrm{ymin}_r}\right)-\left(\frac{1}{m}\times\sum_{i=1}^{m}\frac{x_{ij}}{\mathrm{xmax_i}}\right),\quad \forall j \tag{22}$$

By substituting the equations (17), (19), and (21) into the set of constraints (8), we have:

$$\sum_{i=1}^{m}\frac{1}{\mathrm{m\times xmin_i}}x_{ij}-\left(\left(\frac{1}{m}\times\sum_{i=1}^{m}\frac{x_{ij}}{\mathrm{xmin_i}}\right)-\left(\frac{1}{r}\times\sum_{r=1}^{s}\frac{y_{rj}}{\mathrm{ymax}_r}\right)\right)=\sum_{r=1}^{s}\frac{1}{\mathrm{m\times xmin_i}}y_{rj}\ ,\ j=1,\dots,n$$

Inserting the equation (17) into the constraints (9), yields:

$$\frac{1}{m}\times\sum_{i=1}^{m}\frac{1}{\mathrm{xmin_i}}xmin_i=1$$

Using the equation (19) into the constraints (10), we get:

$$\frac{1}{r}\times\sum_{i=1}^{m}\frac{1}{\mathrm{ymax}_r}ymax_r=1$$

Substituting the equations (18), (20), and (22) into the set of constraints (11), we obtain:

$$\sum_{i=1}^{m}\frac{1}{\mathrm{m\times xmax_i}}x_{ij}+\left(\frac{1}{r}\times\sum_{r=1}^{s}\frac{y_{rj}}{\mathrm{ymin}_r}\right)-\left(\frac{1}{m}\times\sum_{i=1}^{m}\frac{x_{ij}}{\mathrm{xmax_i}}\right)=\sum_{r=1}^{s}\frac{1}{\mathrm{r\times ymin}_r}y_{rj}\ ,\ j=1,\dots,n$$

Inserting the equation (18) into the constraints (12), yields:

$$\frac{1}{m}\times\sum_{i=1}^{m}\frac{1}{\mathrm{xmax_i}}x\max_i=1$$

Using the equation (20) into the constraints (13), we get:

$$\frac{1}{r}\times\sum_{i=1}^{m}\frac{1}{\mathrm{ymin}_r}y\min_r=1$$

It is clear that all constraints (8)-(13) are satisfied through proposed arbitrary solution. It is notable that δ_j,σ_j are free in sign variables. Thus, the solution is in the feasible region of the constraints of the model. Hence, independent of the inputs and outputs of the DMUs, there always exists one feasible solution for model (7)-(16). This completes the proof of the theorem 3.

Theorem 4. The model (7)-(16) has an optimum solution.

Proof. The proof of the theorem is based on the theorems (1) to (3). The Ω is a non-empty set based on theorem (1) and the objective function is strictly convex based on theorem (2). Hence, the model (7)-(16) is a convex linear programming. Furthermore, according to theorem (3) the model (7)-(16) always has at least one feasible solution. So, the solution obtained by solving the model (7)-(16) is optimum.

Theorem 5. The term $\delta_j+\sigma_j$ for IDMU is equal to zero at optimality of model (7)-(16).

According to theorem (4), we know that the model (7)-(16) is a convex linear programming, so the optimum solution of the model is occurred on bounds of active constraints. In a bound solution of linear programming the value of slacks and surplus variables of active constraints are equal to zero. The sets of constraints (8) and (11) can be re-written as follows:

$$\sum_{i=1}^{m}\omega_i x_{ij}\ge\sum_{r=1}^{s}\mu_r y_{rj}\ ,\ j=1,\dots,n$$

$$\sum_{i=1}^{m}\varphi_i x_{ij}\le\sum_{r=1}^{s}\gamma_r y_{rj}\ ,\ j=1,\dots,n$$

So, δ_j and σ_j can be assumed as surplus and slack variables in sets of constraints (8) and (11), respectively. We know that for an efficient DMU, the weighted sum of inputs should be equal to weighted sum of outputs. Hence, $\sum_{i=1}^{m}\omega_i x_{ij} = \sum_{r=1}^{s}\mu_r y_{rj}$, $j = 1,\ldots,n$ and $\sum_{i=1}^{m}\varphi_i x_{ij} = \sum_{r=1}^{s}\gamma_r y_{rj}$, $j = 1,\ldots,n$, this means that δ_j and σ_j are equal to zero for such DMUs. Consequently $\delta_j + \sigma_j$ is equal to zero. This completes the proof of the theorem 4.

4- Numerical examples

Four numerical examples have been taken from the literature in order to illustrate the applicability and efficacy of model (7)-(16). All the numerical examples are coded in LINGO software and executed on a Pentium 4, 2.67 GHz computer with RAM 4.0 GB using MS-Windows 7.0. The developed software codes for proposed approach of this study and the method proposed by Sun et al. (2013) are presented in Appendix A and Appendix B, respectively.

4-1- First example: The Asian companies producing lead bars

The data set of this example has been taken from Chang and Chen (2008), which was also used by Sun et al. (2013).

Table 2. Data for first example adopted from Chang and Chen (2008) and Sun et al. (2013)

DMU	Input		Output	
	X1	X2	Y1	Y2
1	43.08	17.446	19.385	84
2	9.85	19.941	22.849	88
3	7.92	48.461	52.693	82.4
4	75.15	58.269	70.537	96
5	56.92	62.148	70.769	91.92
6	137.38	31.835	44.517	97.23
7	61.54	49.231	76.308	90
8	29.54	76.8	65.969	97
9	289.23	138.462	153.846	98
10	19.69	54.154	64	92

According to table 2, this example includes two inputs and two outputs. The inputs include the value of equipment in terms of one hundred thousand dollars (X_1) and the value of sold goods in terms of million dollars (X_2). The outputs include the outcome obtained by the selling in terms of million dollars (Y_1) and the mean of output (Y_2). The results of implementation of classic CCR model, the models proposed by Sun et al. (2013), and the proposed model (7)-(16) are presented in table 3.

Table 3. The results of first example adopted from Chang and Chen (2008) and Sun et al. (2013)

DMU	CCR model		Sun et al., (2013)				proposed model of this study			
	Score	Rank	Model(5)	Rank	Model(8)	Rank	W-MU	Rank	Fi-Gama	Rank
1	1	1	0.857	1	8.091	1	0.857	1	8.090	1
2	1	1	0.785	2	7.416	2	0.786	2	7.417	2
3	1	1	0.303	5	2.859	5	0.303	5	2.858	5
4	0.799	8	0.293	7	2.771	7	0.293	7	2.769	7
5	0.808	7	0.263	8	2.487	8	0.263	8	2.486	8
6	1	1	0.544	3	5.134	3	0.544	3	5.131	3
7	1	1	0.325	4	3.075	4	0.325	4	3.072	4
8	0.722	9	0.225	9	2.124	9	0.225	9	2.123	9
9	0.717	10	0.126	10	1.192	10	0.126	10	1.189	10
10	1	1	0.302	6	2.857	6	0.302	6	2.855	6

In the second and third columns of table 3, the values of efficiency and ranking obtained from traditional CCR model are presented. Based on this model, The DMUs 4, 5, 8, and 9 are inefficient and other DMUs are efficient. As it is clear the discrimination power of classic CCR model is weak in this case.

In the fourth to seventh columns of table 3, the values of efficiency scores and the rankings obtained from the model proposed by Sun et al. (2013) are presented. The ranking is done based on model (5) and model (8) in Sun et al. (2013). The proposed procedure by Sun et al. (2013) presents a full ranking for DMUs in this example. The discrimination power of the classic CCR model is improved. The obtained results from the solution of both models (5) and (8) in Sun et al. (2013) is similar. Although Sun et al. (2013) have noted that these results are not necessarily similar for all cases. It is notable that two sets of models should be solved in proposed approach by Sun et al. (2013) and the results of both models are not merely similar.

In eighth to eleventh columns of table 3, the values of efficiency and the rankings obtained from the model (7)-(16) in this study are presented. It can be concluded from contents of Table 3 that the rankings obtained from the model (7)-(16) are consistent with the results of proposed models by Sun et al. (2013) while a single model is run. Therefore, the proposed model (7)-(16) in this study has the same characteristics of proposed models by Sun et al. (2013) and enhance the computational efforts.

4-2- Second example: The flexible production systems

The data set of the second example is from Shang and Sueyoshi (1995) which was also used by Sun et al., (2013).

Table 4. The data of second example: flexible production system (adopted from Shang and Sueyoshi (1995); Sun et al., (2013))

DMU	Input			Output		
	X_1	X_2	Y_1	Y_2	Y_3	Y_4
1	17.02	5	42	45.3	14.2	30.1
2	16.46	4.5	39	40.1	13	29.8
3	11.76	6	26	39.6	13.8	24.5
4	10.52	4	22	36	11.3	25
5	9.5	3.8	21	34.2	12	20.4
6	4.79	5.4	10	20.1	5	16.5
7	6.21	6.2	14	26.5	7	19.7
8	11.12	6	25	35.9	9	24.7
9	3.67	8	4	17.4	0.1	18.1
10	8.93	7	16	34.3	6.5	20.6
11	17.74	7.1	43	45.6	14	31.1
12	14.85	6.2	27	38.7	13.8	25.4

As can be seen in table 4, the data set includes two inputs and four outputs. The inputs are operational expenses and annual amortization (X_1) in terms of one hundred thousand dollars, and the required space for workshop in each system (X_2), in terms of square feet. Outputs include the improvements in qualitative interests (Y_1), goods which are being produced (Y_2), the means of delayed works (Y_3), and the mean of outputs (Y_4).

The second example has been solved using models proposed by Sun et al., (2013) and the proposed model (7)-(16) and the results are summarized in table 5.

Table 5. The ranking results of second example: flexible production system (adopted from Shang and Sueyoshi (1995); Sun et al., (2013))

DMU	Sun et al., (2013) model				proposed model of this study			
	Model(5)	Rank	Model(8)	Rank	W-MU score	Rank	Fi-Gama score	Rank
1	0.7356	3	2.6029	3	0.7356	3	2.9189	3
2	0.8091	1	2.8622	1	0.8091	1	3.2108	1
3	0.4989	8	1.7672	8	0.4989	8	1.9799	8
4	0.7637	2	2.7018	2	0.7636	2	3.0304	2
5	0.6559	4	2.3239	4	0.6559	4	2.6030	4
6	0.3733	10	1.3188	10	0.3733	10	1.4815	10
7	0.3882	9	1.3722	9	0.3882	9	1.5406	9
8	0.5030	6	1.7780	6	0.5030	6	1.9960	6
9	0.2764	12	0.9736	12	0.2764	12	1.0970	12
10	0.3596	11	1.2704	11	0.3596	11	1.4269	11
11	0.5352	5	1.8935	5	0.5352	5	2.1238	5
12	0.5006	7	1.7726	7	0.5006	7	1.9864	7

In the second to fifth columns of table 5, the value of efficiency scores and the rankings obtained from the models proposed by Sun et al. (2013) are presented. It is clear that the full ranking is achieved and the discrimination power of the model is very high as a distinctive rank is assigned to each DMU. The ranking of models proposed by Sun et al. (2013) are similar. In sixth to ninth columns of table 5, the values of efficiency and the rankings obtained from the proposed model (7)-(16) are presented. The rankings obtained from the proposed model (7)-(16) are consistent with the results of models proposed by Sun et al. (2013). These results achieved through a single run which decreases the computational efforts of proposed model (7)-(16) in comparison with models proposed by Sun et al. (2013).

4-3- Third example: The efficiency of 15 baseball players

This example is adopted from Washio and Yamada (2013). According to Table 6, the data set of this example includes one input and three outputs for each baseball player. The input (X_1) is the number of presence. The outputs are all bases (Y_1), run bats (Y_2), and possessed bases (Y_3).

Table 6. The data for third example: baseball players' example (adopted from Washio and Yamada (2013))

DMU	Input		Output	
	X1	Y1	Y2	Y3
1	133	52	15	1
2	118	34	20	5
3	147	52	12	8
4	158	55	19	3
5	139	48	7	2
6	168	65	17	14
7	98	33	12	1
8	119	43	12	5
9	146	47	17	2
10	114	42	14	5
11	151	38	4	10
12	144	44	17	3
13	138	28	12	19
14	120	38	6	5
15	141	45	14	2

This example has also been solved using classic CCR model, models proposed by Washio and Yamada (2013), and the proposed model (7)-(16) in this study. The results are summarized in table 7.

Table 7. The ranking results of baseball players' example (adopted from Washio and Yamada (2013))

DMU	CCR model		Washio and Yamada (2013)		proposed model of this study			
	Score	Ran	Aggressive	Rank based	W-MU	Rank	Fi-Gama	Rank
1	1.000	1	3	5	0.589	1	2.346	1
2	1.000	1	5	3	0.434	13	1.729	13
3	0.911	9	7	7	0.533	5	2.122	5
4	0.951	6	6	6	0.525	6	2.089	6
5	0.884	11	12	14	0.521	7	2.072	7
6	1.000	1	1	1	0.583	2	2.321	2
7	0.938	7	8	9	0.508	8	2.020	8
8	0.932	8	4	4	0.545	4	2.168	4
9	0.895	10	9	10	0.485	9	1.932	9
10	1.000	1	2	2	0.555	3	2.211	3
11	0.711	15	15	15	0.379	14	1.510	14
12	0.873	12	10	11	0.461	12	1.833	12
13	1.000	1	14	8	0.306	15	1.217	15
14	0.814	14	13	13	0.477	11	1.900	11
15	0.839	13	11	12	0.481	10	1.915	10

In the second and third columns of table 7, the efficiency scores and rankings obtained from CCR model are presented, respectively. Based on CCR model, DMUs 1, 2, 6, 10, and 13 are evaluated efficient and other DMUs are evaluated inefficient. CCR model cannot distinguish the efficient DMUs.

In the fourth and fifth columns of table 7, the rankings obtained from cross efficiency models proposed by Washio and Yamada (2013) are presented. Unfortunately, the ranking results of proposed models by Washio and Yamada (2013) are not similar. In the sixth to ninth columns of table 7, the value of efficiency scores and the ranking obtained from the proposed model (7)-(16) of the current research are presented. It is clear that the rankings achieved through both ideal and anti-ideal approaches are similar. But, the rankings achieved by proposed model (7)-(16) are dissimilar to the rankings of the model proposed by Washio and Yamada (2013), because these two models are inherently different.

4-4- Fourth example: The efficiency of 12 decision-making units

This example is adopted from a study conducted by Khodabakhshi and Aryavash (2012). According to Table 8, the data of this example include three inputs and two outputs for DMUs.

Table 8. The data for fourth example: adopted from Khodabakhshi and Aryavash (2012)

DMU	Input			Output	
	X1	X2	X3	Y1	Y2
1	350	39	9	67	751
2	298	26	8	73	611
3	422	31	7	75	584
4	281	16	9	70	665
5	301	16	6	75	445
6	360	29	17	83	1070
7	540	18	10	72	457
8	276	33	5	74	590
9	323	25	5	75	1074
10	444	64	6	74	1072
11	323	25	5	25	350
12	444	64	6	104	1199

This example has been solved using the proposed model by Sun et al., (2013) and the proposed model (7)-(16). The results are presented in table 9.

Table 9. The ranking results of fourth example: adopted from Khodabakhshi and Aryavash (2012)

DMU	proposed model of this study				Sun et al., (2013)			
	W-MU	Rank	Fi-Gama	Rank	Model 5	Rank	Model 8	Rank
1	0.508	8	3.311	6	0.508	8	3.311	6
2	0.650	4	3.163	8	0.650	4	3.163	8
3	0.471	9	2.135	10	0.471	9	2.135	10
4	0.661	2	3.651	5	0.661	2	3.651	5
5	0.661	3	2.281	9	0.661	3	2.281	9
6	0.612	7	4.586	2	0.612	7	4.586	2
7	0.354	11	1.306	12	0.354	11	1.306	12
8	0.711	1	3.298	7	0.711	1	3.298	7
9	0.616	6	5.130	1	0.616	6	5.130	1
10	0.442	10	3.725	4	0.442	10	3.725	4
11	0.205	12	1.672	11	0.205	12	1.672	11
12	0.622	5	4.166	3	0.622	5	4.166	3

In the second to fifth columns of table 9, the results of the proposed model (7)-(16) (7-16) are presented. In the sixth to ninth columns of Table 9, the results of the models suggested by Sun et al. (2013) are shown. There are two important points about these results. First, the two models have produced similar results. Second, a comparison between the data mentioned in the third and fifth columns (also seventh and ninth columns) of Table 9 shows that the results obtained from the two approaches are not similar. This example shows that the similarity of the rankings does not necessarily occur in all examples. This case was not addressed by Sun et al., (2013). Sun et al., (2013) argue that the results of their model are often similar. They cannot give an exception in this area. So we can conclude that the proposed approach by Sun et al., (2013) may also present conflictive ranks. The fourth numerical example provides clear evidence for this claim.

In this Section, four benchmark instances were solved using different procedures, including the proposed model (7)-(16). The results showed that the proposed model (7)-(16) is comparable with existing methods in the literature while the computations burden has substantially been decreased. On the other hand the most similar model to ours is the model proposed by Sun et al. (2013) which required to solve two models one for ideal and the other for anti-ideal cases in order to summarize the result. The proposed model of this study is a unique model incorporating both ideal and anti-ideal DMUs which yields to low computational and setting efforts. Moreover no conflictive ranks are reported by the proposed model of this study. In the next Section, the proposed model (7)-(16) is extended for non-negative and zero inputs and outputs and an algorithmic procedure is developed in order to solve a DEA problem using proposed approach.

5-The algorithmic approach for solving a DEA problem with Zero and Negative Data

The following algorithmic procedure is proposed to solve a DEA problem using proposed approach.

Step 1. Pre-screening of Data. If the input and output data are all positive, go to step 3. If there exists at least one negative or equal to zero value go to step 2.

Step 2. Dealing with negative or zero data. If there exists a zero value go to step 2.1. If there exist negative data go to step 2.2.

Step 2.1. Zero data. In such cases, the zero data are replaced by a very small non-Archimedes value called ε.

Step 2.2. Negative Data. Transform negative inputs and outputs using following relations (Ali and Seiford, 1992).

$$\tilde{X}_{ij} = \left| Min\, X_{ij} \right| + X_{ij} + c, \quad \forall i, j \tag{23}$$

$$\tilde{Y}_{rj} = \left| Min Y_{ij} \right| + Y_{ij} + c, \quad \forall r, j \tag{24}$$

Where, $\tilde{X}_{ij}$ and $\tilde{Y}_{rj}$ are mapping of inputs and outputs, and c is a constant value.

Step 3. Solve the problem. Solve the problem by proposed model (7)-(16).

Step 4. Check rankings. If both rankings are similar, then stop and report the solutions; otherwise, Spearman rank test should be run on different ranking series in order to check whether there is meaningful statistical correlation between these two ranking series. If there is significant statistical difference between the two ranking series, stop and report the solutions; otherwise, go to step 5.

Step 5. Ask decision-makers' preferences. Accomplish the Hurwicz evaluation criterion on the basis of certain level of optimism (Wang and Yang, 2007).

Figure 1 illustrates the schematic algorithmic view of proposed approach.

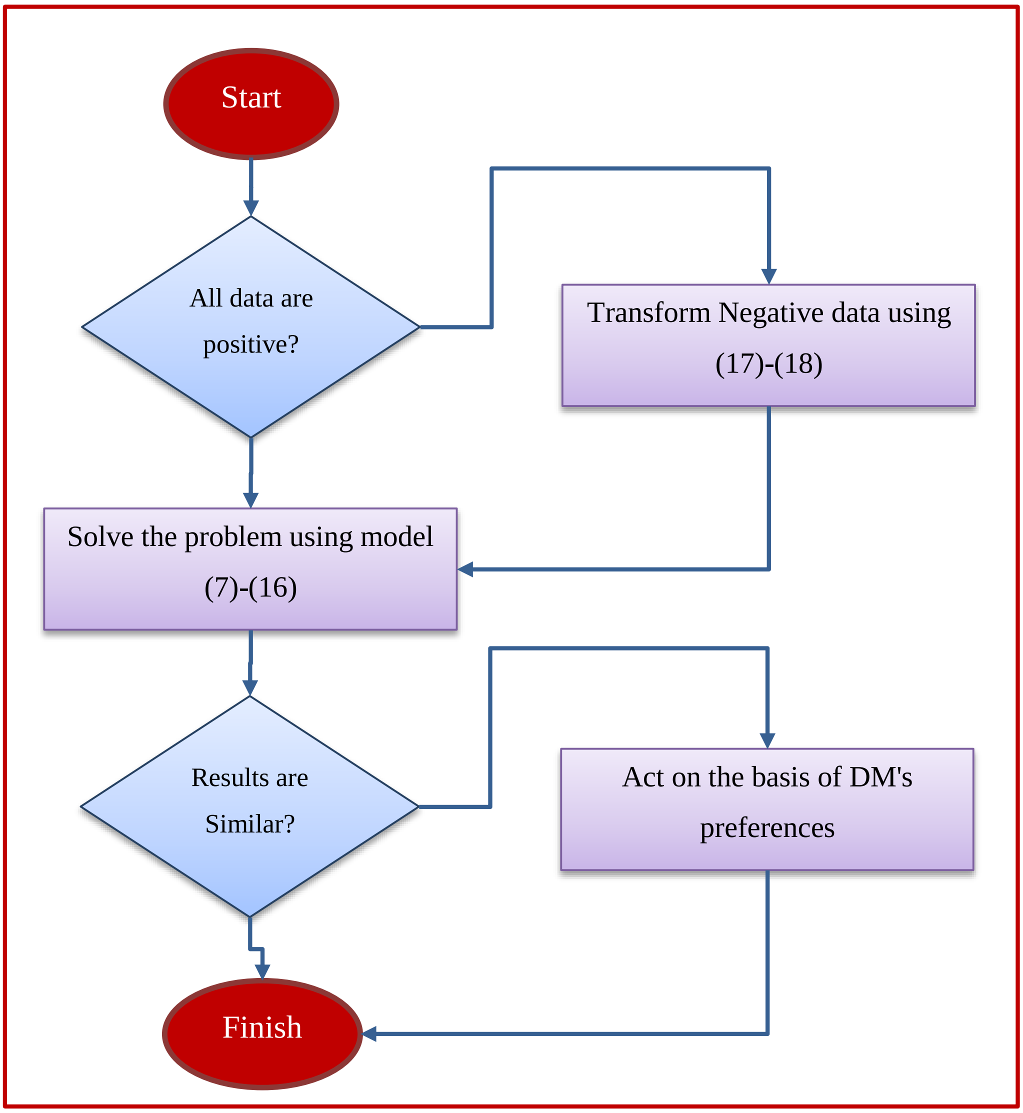

Fig 1. The algorithmic approach for the proposed model

To illustrate the application and order of the proposed algorithmic approach, an example with positive, negative, and zero data is presented.

5-1- A numerical example with positive, negative, and zero data

This example is adopted from a study conducted by Sharp et al., (2007). As can be seen in table 10, the data of this example include two inputs and three outputs.

Table 10. The example with positive, negative and zero data adopted from Sharp et al., (2007)

DMU	Input		Output		
	X1	X2	Y1	Y2	Y3
1	1.03	-0.05	0.56	-0.09	-0.44
2	1.75	-0.17	0.74	-0.24	-0.31
3	1.44	-0.56	1.37	-0.35	-0.21
4	10.80	-0.22	5.61	-0.98	-3.79
5	1.30	-0.07	0.49	-1.08	-0.34
6	1.98	-0.10	1.61	-0.44	-0.34
7	0.97	-0.17	0.82	-0.08	-0.43
8	9.82	-2.32	5.61	-1.42	-1.94
9	1.59	0.00	0.52	0.00	-0.37
10	5.96	-0.15	2.14	-0.52	-0.18
11	1.29	-0.11	0.57	0.00	-0.24
12	2.38	-0.25	0.57	-0.67	-0.43
13	10.30	-0.16	9.56	-0.58	0.00

The data of the first input (X_1) are all positive. In the second input (X_2), the data are negative or zero. The data of the first output (Y_1) are all positive. The data of the second (Y_2) and third output (Y_3) are negative or zero.

The data of the first input and first output are all positive and do not need to be transformed. In order to transform the data of the second input, second output, and third output, the equations (17)-(18) are used. The c value is set to 0.01. Table 11 presents the transformed data.

Table 11. The transformed data for example with positive, negative, and zero adopted from Sharp et al., (2007)

DMU	Input		Output		
	X1	X2	Y1	Y2	Y3
1	1.03	2.28	0.56	1.34	3.36
2	1.75	2.16	0.74	1.19	3.49
3	1.44	1.77	1.37	1.08	3.59
4	10.80	2.11	5.61	0.45	0.01
5	1.30	2.26	0.49	0.35	3.46
6	1.98	2.23	1.61	0.99	3.46
7	0.97	2.16	0.82	1.35	3.37
8	9.82	0.01	5.61	0.01	1.86
9	1.59	2.33	0.52	1.43	3.43
10	5.96	2.18	2.14	0.91	3.62
11	1.29	2.22	0.57	1.43	3.56
12	2.38	2.08	0.57	0.76	3.37
13	10.30	2.17	9.56	0.85	3.80

The proposed model (7)-(16) is applied and the results are presented in table 12.

Table 12. The ranking results for example with positive, negative, and zero adopted from Sharp et al., (2007)

DMU	The proposed model			
	W-MU	Rank	Fi-Gama	Rank
1	0.833	2	1.168	11
2	0.509	7	1.629	8
3	0.636	5	3.681	5
4	0.000	13	12.643	3
5	0.679	4	1.031	13
6	0.446	8	3.433	6
7	0.887	1	1.805	7
8	0.048	12	2661.552	1
9	0.551	6	1.061	12
10	0.155	10	4.668	4
11	0.704	3	1.221	10
12	0.361	9	1.303	9
13	0.094	11	20.949	2

It can be concluded form table 12 that the results of rankings based on comparison with ADMU and IDMU are different. Spearman rank test was conducted to check if there is meaningful significant correlation between these two rankings. The results of Spearman rank test are presented in table 13.

Table 13. The results of Spearman correlation test related to two rankings

			W-MU score	Fi-Gamma score
Spearman's rho	**WMUscore**	Correlation Coefficient	1.000	-.720**
		Sig. (2-tailed)	.	.006
		N	13	13
	FiGammascore	Correlation Coefficient	-.720**	1.000
		Sig. (2-tailed)	.006	.
		N	13	13
**. Correlation is significant at the 0.01 level (2-tailed).				

The results, as shown in table 13, indicate that there is no statistically significant correlation between these two rankings at a confidence level of 0.99. Hence, the algorithm will go on with DM's preferences. The results of Hurwicz evaluation criterion (Wang and Yang, 2007) are presented in table 14.

Table 14. The ranking results of decision making units based on various levels of optimism

DMU	α=1		α=0.75		α=0.5		α=0.25		α=0	
	Hurwic	**Rank**	**Hurwic**	**Rank**	**Hurwic**	**Rank**	**Hurwic**	**Rank**	**Hurwic**	**Rank**
1	0.833	2	0.917	8	1.000	9	1.084	10	1.168	11
2	0.509	7	0.789	10	1.069	8	1.349	8	1.629	8
3	0.636	5	1.397	4	2.158	5	2.920	5	3.681	5
4	0.000	13	3.161	3	6.321	3	9.482	3	12.643	3
5	0.679	4	0.767	11	0.855	11	0.943	12	1.031	13
6	0.446	8	1.193	6	1.940	6	2.686	6	3.433	6
7	0.887	1	1.116	7	1.346	7	1.576	7	1.805	7
8	0.048	12	665.424	1	1330.80	1	1996.17	1	2661.55	1
9	0.551	6	0.678	12	0.806	13	0.934	13	1.061	12
10	0.155	10	1.283	5	2.411	4	3.540	4	4.668	4
11	0.704	3	0.834	9	0.963	10	1.092	9	1.221	10
12	0.361	9	0.597	13	0.832	12	1.068	11	1.303	9

Table 14 implies that the result of Hurwicz evaluation is equal to ranking of IDMU when the optimism level is set equal to 1. The result of Hurwicz evaluation is equal to ranking of ADMU when the optimism level is set equal to 0. Other rankings are also achieved based on optimism level 0.75, 0.5, and 0.25.

6- Conclusion

In classic DEA models, a full analysis of efficiency scores requires solving a linear programming for each DMU. One of the problems of classic DEA models is that each DMU can select its desired weights in order to maximize its relative efficiency score. So, the weights of inputs and outputs may vary during different runs. Therefore, the efficiency of DMUs are measured by different weights, which is assumed as one of the shortages of the classic DEA models. In this way, there exists the probability of evaluating several inefficient DMUs as efficient. Another problem is that the different efficient DMUs cannot be distinguished; consequently a full ranking cannot be achieved and discrimination power of classic models reduces. To overcome these problems, the researchers have proposed a lot of models which rank DMUs by using different approaches including the set of common weights. The current research was an attempt to develop a more efficient comprehensive approach to rank DMUs by comparing them with dummy Ideal DMU and virtual Anti-Ideal DMU, concurrently. The main contributions of the this research in comparison with the existing approaches such as Sun et al., (2013) were the concurrent usage of: 1) the common weight method in order to reduce the chance of inefficient DMUs to be evaluated as efficient; 2) virtual ideal and anti-ideal DMUs concurrently to improve the discrimination power of DEA models; 3) a full ranking method to rank the production possibility sets (PPS) with low number of DMUs; 4) an algorithmic approach to handle DMUs with non-positive and zero inputs/outputs; 5) Ranking all DMUs in a single run which reduced the computational efforts effectively. Moreover, the proposed approach had no sensitivity about return to scale assumptions. It ranked any number of DMUs in a single run and this caused low computational efforts. It is shown that the proposed method generates at least one feasible solution independent of the value of inputs and outputs. Finally, its performance was tested using several numerical examples adopted from literature.

The properties of the proposed approach were discussed through several theorems. The applicability and efficacy of proposed approach was tested by four benchmark numerical examples adopted form the relevant literature. A numerical example with negative and zero data was also solved using proposed approach. The results of proposed approach and those of existing methods were compared. The results show that the proposed approach was reliable, promising, and competitive among the existing methods while the computational burden of the proposed approach was less than existing methods.

In general the number of linear programming (LP) required to be solved for a full ranking analysis in DEA is assumed as the main computational difficulty. In comparison with classic ranking method in DEA, the number of required LPs to be solved is very limited in the proposed method of this study. Moreover, in comparison with the recent approaches such as Sun et al. (2013), the proposed method of this study has some benefits. The model proposed by Sun et al. (2013) is required to be distinctively solved for both ideal and anti-ideal cases in order to summarize the result. We have also shown that the ranks achieved by Sun et al. (2013) may be conflictive. While the proposed model of this study uses a unique model incorporating both ideal and anti-ideal DMUs which yields to low computational and setting efforts. Moreover, as in a single run all ranks are achieved, so no conflictive ranks are reported by the proposed model of this study.

The proposed model of this study does not consider the slack variables into account. The slack variables can give more accurate information on ranks of DMUs whenever similar ranks are seen. So the same rank was not reported for none of the numerical examples. The proposed model of this study was developed considering no-orientation toward inputs or outputs, while in real cases ranking may be done considering a certain orientation in presence of some constraints or preferences. The distances of given DMU form both ideal and anti-deal DMUs is the basis of ranking score of the DMU. There is no parameter in the model to tune the sensitivity of a unit of distance from ideal and anti-ideal DMUs and a unit of both distances has the same value. The proposed model of this study can handle crisp data, while in real problems uncertainty in form of fuzzy and random data is a main challenge.

As mentioned before, ranking methods in DEA usually suffer from heavy computational efforts such as high number of LPs needed to be solved in order to make the analysis complete. This situation is not interesting based on managerial perspective. The proposed model of this study can rank any number of DMUs in a single run. Even the PPSs with low number of DMUs can be ranked by the proposed method of this study. This property make the usage of the proposed approach more general and it fits with a large number of real life and managerial applications. High sensitivity and infeasibility are other issues may occur in real life applications of ranking procedures. The proposed method of this study assures feasible solutions independent of the values of inputs and outputs. Conflictive ranking is another issue which reveals in real life application. No conflictive rank is generated by proposed method of this study, so it can be easily used in any managerial application such as project selection, portfolio selection, human resource performance assessment, energy planning, healthcare evaluation, and economy and banking performance measurement in which ranking is important.

Ranking DMUs considering a virtual target formed based on observed DMUs provides more accurate benchmarking pattern while the best observed DMU in the PPS should also improve itself. This means that we are not focusing to rectify all DMUs with the best observed DMU as was done in classic ranking method but we are pushing all DMUs, including the best observed one toward a virtual target formed based on observed DMUs. This property of proposed approach provides a sense of improvement and dynamics for all DMUs in managerial systems. Moreover, in some managerial cases we may interest to rank the DMUs not only based in their distance from ideal DMU but the distance from anti-ideal (the worst case) is important. The risk seeker mangers would like to rank DMUs based on the best observed DMU while the risk averse managers would like to rank DMUs based on the worst observed DMUs. The proposed approach of this study include both insights in order to rank DMUs.

In future investigations it might be possible to consider uncertainty in data through intervals, fuzzy, or probabilistic data. Incorporating the slack variables will yield some benefits in favor of calculating the efficiency scores as well. Extending the proposed model of this study for specific orientation toward inputs or outputs will yield to more specialized applications. Incorporating a parameter in order to distinct the importance of distances from ideal and anti-ideal DMUs is another interesting research scheme.

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Appendix A: Lingo codes for proposed approach in this study

```
MODEL:
SETS:
DMU/1..10/:SIGMA,DELTA;
INPUT/1..2/:W,Xmin,Xmax,Fi;
OUTPUT/1..2/:MU,Ymax,Ymin,Gamma;
LINK1(INPUT, DMU):X;
LINK2(OUTPUT, DMU):Y;
ENDSETS
DATA:
X=@OLE('D:\DEA-MODEL-EX1.xlsx','X');
Y=@OLE('D:\DEA-MODEL-EX1.xlsx','Y' );
EPSILON=0.000001;
ENDDATA
! The objective function of EXTENDED model;
MIN=@SUM(DMU(J): SIGMA(J))+@SUM(DMU(J): DELTA(J));
! The constraints of EXTENDED model;
@FOR(INPUT(I):
Xmin(I)=@MIN(DMU(J): X(I,J));
Xmax(I)=@MAX(DMU(J): X(I,J))
);

@FOR(OUTPUT(R):
Ymax(R)=@MAX(DMU(J): Y(R,J));
Ymin(R)=@MIN(DMU(J): Y(R,J))
);

@FOR(DMU(J):
@SUM(INPUT(I):W(I)*X(I,J))-SIGMA(J)=@SUM(OUTPUT(R):MU(R)*Y(R,J));
@SUM(INPUT(I):Fi(I)*X(I,J))+DELTA(J)=@SUM(OUTPUT(R):Gamma(R)*Y(R,J))
);

@SUM(INPUT(I):W(I)*Xmin(I))=1;
@SUM(INPUT(I):Fi(I)*Xmax(I))=1;

@SUM(OUTPUT(R):MU(R)*Ymax(R))=1;
@SUM(OUTPUT(R):Gamma(R)*Ymin(R))=1;

@FOR(INPUT(I):
        W(I)>= EPSILON;
        Fi(I)>= EPSILON
);

@FOR(OUTPUT(R):
        MU(R)>= EPSILON;
        Gamma(R)>= EPSILON
);

DATA:

@OLE('D:\DEA-MODEL-EX1.xlsx','B8:E8')=@WRITEFOR(INPUT(U): W(U),MU(U));

@OLE('D:\DEA-MODEL-EX1.xlsx','B10:E10')=@WRITEFOR(OUTPUT(k): Fi(k),Gamma(k));

@OLE('D:\DEA-MODEL-EX1.xlsx','B13:K13')=@WRITEFOR(DMU(J): @Sum(OUTPUT(R):
MU(R)*Y(R,J))/(@SUM(INPUT(I):W(I)*X(I,J))));

@OLE('D:\DEA-MODEL-EX1.xlsx','B15:K15')=@WRITEFOR(DMU(J): @Sum(OUTPUT(R):
Gamma(R)*Y(R,J))/(@SUM(INPUT(I):Fi(I)*X(I,J))));
```

ENDDATA
END

Appendix B: Lingo Codes for Method proposed by Sun et al., 2013

```
MODEL:
SETS:
DMU/1..12/: SIGMA,finsol;
INPUT/1..3/:W,Xmin;
OUTPUT/1..2/:MU,Ymax;
LINK1(INPUT, DMU):X;
LINK2(OUTPUT, DMU):Y;
ENDSETS
DATA:
X=@OLE('D:\DEA-MODEL-EX4-1.xlsx','X');
Y=@OLE('D:\DEA-MODEL-EX4-1.xlsx','Y' );
EPSILON=0.000001;
ENDDATA

SUBMODEL OBJmodel3: ! the objective function of model (3) ;
MIN=IDEALD;
IDEALD=@SUM(DMU(J): SIGMA(J));
ENDSUBMODEL

SUBMODEL CONSTmodel3: ! the constraints of model (3) ;
@for(INPUT(I):
Xmin(I)=@min(DMU(J): X(I,J))
);
@for(OUTPUT(R):
Ymax(R)=@max(DMU(J): Y(R,J))
);
@FOR(DMU(J):
@SUM(INPUT(I):W(I)*X(I,J))-SIGMA(J)=@SUM(OUTPUT(R):MU(R)*Y(R,J))
);
@SUM(INPUT(I):W(I)*Xmin(I))=1;
@SUM(OUTPUT(R):MU(R)*Ymax(R))=1;
@FOR(INPUT(I):
        W(I)>= EPSILON
);

@FOR(OUTPUT(R):
        MU(R)>= EPSILON
);

@FOR(DMU(J):
        SIGMA(J)>= 0
);
ENDSUBMODEL


CALC:
@SOLVE(OBJmodel3,CONSTmodel3);
ENDCALC

DATA:
@OLE('D:\DEA-MODEL-EX4-1.xlsx','B9')=@WRITEFOR(INPUT(t): W(t)); ! only for Model3;
@OLE('D:\DEA-MODEL-EX4-1.xlsx','B12')=@WRITEFOR(OUTPUT(p): MU(p)); ! only for Model3;
@OLE('D:\DEA-MODEL-EX4-1.xlsx','B14')=@WRITE(IDEALD); ! only for Model3;
ENDDATA
END
```