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Linear Quadratic Differential Game Formulation for Leaderless Formation Control

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Abstract

In this article studies the leaderless formation control problem for a multi-robot system with double integrator dynamics and the interaction dynamics among robots and the formation objective are added together and expressed in terms of individual cost functions. The problem is posed as a linear quadratic differential game. For the non-cooperative mode of play, the open-loop Nash equilibrium solution of the formation control differential game problem is discussed. We show that the existence of a unique Nash equilibrium solution for the formation problem, whose cost functions are Mayer type, depends on the invertibility of a matrix introduced. A triangle formation is tested to justify the models and the results.

Keywords: Linear quadratic formulation, formation control, differential games, non-cooperative players, Nash equilibrium

1- Introduction

A formation is a collective behaviour exhibited by animal groups mostly during foraging and migration. The most common formation is the V (triangle) shape which could be seen among migrating bird flocks. The group flight information reduces the energy required by individual birds (Lissaman and Shollenberger, 1970). Formation control is one of the first problems that arise in multi-robot systems. It is defined as designing control inputs for the robots so that they form and maintain a pre-defined geometric shape. This problem has been studied using different approaches, such as behaviour-based, leader-follower and virtual structure (Balch, and Arkin, 1998), (Das et al., 2002), (Benzerrouk, et al., 2010) and (Xu et al., 2014).

The current literature mostly relies on the leader-follower approach. The leader robot tracks the desired trajectory while the follower robots keep the formation by following the leader with a fixed distance. While the leader-follower approach is a popular design for the formation control, there are limitations. The loss of the leader causes the entire group formation to fail. If the leader only is tracking the desired trajectory without taking the formation error into account, in such situation, the formation can become disjoint and followers can be left behind if they are not able to track the motion of the leader accurately (Ren and Sorensen, 2008).

Moreover, the formation control might also be in the leaderless structure, where all robots have the same role and none of them generates a command target trajectory. In order to maintain a formation, agents in a robot team need to exchange information like relative displacement, velocity etc.

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The formation control problem is more challenging when the robots have different objectives. A robot may choose his objectives based on relative displacement and velocity errors with the immediate neighbours in the graph topology. In other words, each robot could have its own cost function that is only related to neighbours, not the whole team. Thereby, because of the robots' different objectives that are formed individually based on one's local information, analysing the formation control problem falls within the scope of game theory. Game theoretic approaches have been used to solve the formation control problem. In Zhang et al., (2014), formation control problem under directed and time-varying topology is investigated with a class of weakly acyclic game. In Lee et al. (2014), the formation control problem is solved based on the Nash equilibrium strategies incorporating model predictive control (MPC). In Rantzer (2008), the objective of vehicle formation control problem is decomposed into individual objectives for all vehicles using a price in the form of a minmax potential game and the Nash equilibrium is achieved.

In game theory, the differential game is used for the modelling and analysis of conflict in the context of a dynamical system. Each player attempts to control the state of the system to achieve his goal. The interest for using differential game to model and analyse the optimized behaviour in multi-agent systems has been increasing. The use of differential game methodology to analyse multi-agent behaviours, such as power control in cognitive radio in the area of wireless communication (Xu, H. & Zhou, X. (2013)), multi-agent consensus (Semsar-Kazerooni and Khorasani, 2009), swarm foraging (Özgüler and Yıldız, 2014) and pursuit-evasion (Lin et al. 2015) has been reported.

Within differential games, the framework of linear quadratic differential games is used to model problems where the dynamics of the system is described by a set of linear differential equations. In particular, in the area of multi-robot formation control, the differential game has been used in Gu (2008), Lin (2014) and Mylvaganam, and Astolfi (2015). In Gu (2008), the formation control problem is formulated as a linear quadratic tracking problem and the open-loop Nash equilibrium solution is used as a self-enforcing controller for all robots in the formation. In Lin (2014), formation control of a multiple-UAV (unmanned aerial vehicle) system using differential game approach is studied and a distributed Nash strategy design is proposed. Mylvaganam and Astolfi (2015), studied the formation control problem with one leader and N follower fashion taking into account collision avoidance and formulate it in the form of a nonlinear differential game and the approximate solution to the game is given. The linear quadratic differential game in the cooperative and non-cooperative manner has been focused in Engwerda (2005). It is stated that the non-cooperative linear quadratic differential game under the open-loop information structure has a unique Nash equilibrium solution if and only if a set of coupled Riccati differential equations are solvable. Obtaining solutions to the coupled Riccati differential equations associated with linear-quadratic differential games are generally not straight-forward if such solutions exist. The precise integration method is proposed in Peng et al., (2013) to solve the non-cooperative type of these coupled equations.

In this paper, we consider the leaderless formation control problem for mobile robots with double integrator dynamics. The interaction dynamics among robots and the formation objective are added together and expressed in terms of individual cost functions. Every individual robot aims to minimize its own cost. Herein, each robot in the formation can be viewed as a player or decision maker of a differential game. Considering the fact that analysing linear-quadratic differential games are generally not straight-forward, we show that the existence of a unique Nash equilibrium solution for the formation problem as a differential game, whose cost functions are Mayer type, depends on the invertibility of a matrix introduced. In the literature, the linear quadratic differential game setting approach has been used in Gu (2008) for the leader-follower formation control problem. The model presented in this paper shows much more straightforwardness in analysing the problem with the Mayer type of cost functions.

The remainder of this paper is organized as follows: Required preliminaries in algebraic graph theory and linear quadratic differential game theory are given in section 2. The formation control problem is modelled as a standard linear quadratic differential game in section 3. The open-loop Nash equilibrium solution is discussed in section 4. The leaderless formation control problem as a Mayer type differential game is discussed in section 5. The simulation results are illustrated in section 6. Conclusion and future work are expressed in section 7.

2- Preliminaries

The statement of formation control problem in this paper was based upon in algebraic graph theory in basic level and the special class of non-cooperative linear quadratic differential game.

2-1- Algebraic graph theory

A directed graph $G = (V, E)$ consists of a set of vertices $V = \{1,2, \dots, m\}$ and a set of edges $E \subseteq \{(i, j): i, j \in V\}$ such as $j \neq i$ and $(i, j) \in E \not\Leftrightarrow (j, i) \in E$ (i.e., the graph has no self-loops and contains only ordered pairs of distinct vertices). This definition can be easily assigned to a networked dynamical system to represent the interconnections between the network's nodes. For the formation control problem the set of vertices V corresponds to the robots in the formation and then the set of edges E represents the interconnections for formation. We make the following assumption about the formation graph $G = (V, E)$.

Assumption 1. The following statements hold for the directed graph G:

a) $G$is time-invariant, i.e., E is a fixed set.

b) each edge $(i, j) \in E$ is assigned a weight $\omega_{ij} > 0$.

c) $G$is connected, i.e., for every pair of vertices $i, j \in V$, from i to j for all $j = 1, \dots, m$, $j \neq i$, there exists a path of (undirected) edges from E. ■

Laplacian graphs are used widely to analyse the behaviour of networked systems. In Dong et al., (2017), the formation problem is solved with a distributed protocol based on the Laplacian matrix. The Laplacian graph L associated with the graph $G = (V, E)$ is a $m \times m$ matrix that is defined as

$$L = DWD^T \tag{1}$$

Where$D$ is the incidence matrix with order $m \times |E|$ and $W = diag(\omega_{ij})$ is a diagonal weight matrix with dimension $|E|$. D's uvth element is 1 if the node u is the head of the edge$v$, -1 if the node u is the tail, and 0, otherwise. The n-dimensional graph Laplacian L can be defined as

$$L = L \otimes I_n \tag{2}$$

Where I_n is the identity matrix of dimension n and $\otimes$ is the Kronecker product that satisfies the following properties:

$$(X \otimes Y)^T = X^T \otimes Y^T,\ (X \otimes Y)(U \otimes V) = (XU) \otimes (YV)$$

For real value matrices X, Y, U and V. Based on these properties, (2) can be rearranged as

$$L = DWD^T \otimes I_n = (D \otimes I_n)(W \otimes I_n)(D \otimes I_n)^T = DWD^T \tag{3}$$

Where $D = D \otimes I_n$ and$W = W \otimes I_n$.

The following lemma expresses the basic properties of the Laplacian graph.

Lemma 1: For the directed graph $G = (V, E)$ with the properties were mentioned in *Assumption 1*, the following statements are true.

a) L is *symmetric* and *positive semi definite*.

b) For a vector $\in R^{nm}$, L holds the property of sum-of-squares:

$$z^T L z = \sum_{(i,j)\in E} \omega_{ij} \|z_j - z_i\|^2 \tag{4}$$

Where $z = [z_1{}^T, \dots, z_m{}^T]^T$, $z_i \in R^n$ for $i = 1, \dots, m$, and $\|.\|$ is the Euclidean norm in R^n.

Proof: These two results are well-known in algebraic graph theory and their proof can be found in Merris, (1994). ■

2-2- Linear quadratic differential game

Consider an m-player game described by the following state equation:

$$\dot{z} = Az + \sum_{i=1}^{m} B_i u_i, \quad z(0) = z_0 \tag{5}$$

where z is the state vector of the game, u_i is a control vector player i can manipulate, z_0 is the initial state of the game, A, B_i $(i = 1, \dots, m)$ are constant matrices of appropriate dimensions and $\dot{z}$ denotes the time derivative of z. The objective for every player is the minimization of his cost by choosing appropriate controls for the underlying dynamical system (5). The cost function player i aims to minimize is:

$$J^i = z(T)^T Q_{iT} z(T) + \int_0^T (z^T Q_i z + \sum_{j=1}^{m} {u_j}^T R_{ij} u_j) dt \tag{6}$$

Where T is the finite time horizon, Q_{iT} and Q_i are symmetric positive semidefinite matrices ($Q_{iT}, Q_i \geq 0$), and R_{ii} are symmetric positive definite ($R_{ii} > 0$). Suppose that the players have noncooperative behavior (i.e., they are assumed not to collaborate in trying to attain their objectives). For the noncooperative linear quadratic finite horizon differential game, the natural solution concept is the Nash equilibrium. Nash equilibrium is a strategy combination of all players with the property that no one can gain lower cost by unilaterally deviating from it. Under the open-loop information structure in the game defined by (5) and (6), the open-loop Nash equilibrium is defined as the set of admissible actions $(u_1^*, \dots, u_m^*)$ if for all admissible $(u_1, \dots, u_m)$ the following inequalities:

$$J^i(u_1^*, \dots, u_{i-1}^*, u_i^*, u_{i+1}^*, \dots, u_m^*) \leq J^i(u_1^*, \dots, u_{i-1}^*, u_i, u_{i+1}^*, \dots, u_m^*)$$

$u_i \in \Gamma_i$ holds for $i = 1, \dots, m$ where Γ_i is the admissible strategy set for player i.

Remark 1: According to Engwerda, (1998), since the stated assumptions in the definition of this game, the cost function J^i $(i = 1, \dots, m)$ is a strictly convex function of u_i for all admissible control functions u_j, $j \neq i$ and for all z_0. This implies that the conditions following from the minimum principle are both necessary and sufficient.

The results of the minimum principle for open-loop Nash equilibrium of the differential game (5) and (6) are given in the following theorem.

Theorem 1. For the linear quadratic differential game (5) and (6), let there exist a solution set $(P_i, i = 1, \dots, m)$ to the coupled Riccati differential equations:

$$\dot{P}_i + P_i A - P_i \sum_{j=1}^{m} S_j P_j + Q_i + A^T P_i = 0, \quad P_i(T) = Q_{iT} \tag{7}$$

Where $S_i = B_i R_{ii}^{-1} B_i^T$. Then, the differential game has a unique open-loop Nash equilibrium given by:

$$u_i^* = -R_{ii}^{-1} B_i^T P_i \Phi(t, 0) z_0 \tag{8}$$

where $\Phi(t, 0)$ satisfies:

$$\dot{\Phi}(t, 0) = (A - \sum_{i=1}^{m} S_i P_i) \Phi(t, 0), \qquad \Phi(0,0) = I. \tag{9}$$

Proof: The proof can be found in Engwerda (2005) for two-player games with adding this note that the proof results can be generalized straightforward to the game in (5) and (6). ■

Theorem 1 provides a set of sufficiency conditions for the existence of open-loop Nash equilibrium solution. However, it can be shown that a Nash equilibrium may exist even if (7) does not admit a solution as indicated in example 6.2 in Engwerda (2005).

3- The formation control problem

Consider a networked system of m mobile robots, each of which is described by a double integrator dynamics as the following:

$$\dot{q}_i = v_i, \ \dot{v}_i = u_i \tag{10}$$

Where $q_i, v_i, u_i \in R^n$ $(i = 1, \dots, m)$ are n-dimensional coordinates (e.g., $n = 2, 3$), velocity and control vectors, respectively. In this model control u_i is considered input for robot i, and hence the formation control problem is defined as designing control inputs for the robots so that they form and maintain a pre-defined geometric shape. The control objective for leaderless formation control can be expressed as to design u_i such that

$$\|q_i - q_j - d_{ij}\| \to 0 \tag{11}$$

As $t \to T$ and $(i, j) \in E$ where $d_{ij} \in R^n$ is the desired distance vector between i and j. To control the group of robots to keep the formation cohesively, the secondary objective is:

$$\|\dot{q}_i - \dot{q}_j\| \to 0. \tag{12}$$

The formation and velocity error vector can be considered in form of:

$$\|q_i - q_j - d_{ij}\| + \|\dot{q}_i - \dot{q}_j\| \tag{13}$$

and then, the whole group formation and velocity error can be expressed in quadratic form as:

$$\sum_{(i,j)\in E} \omega_{ij}(\|q_i - q_j - d_{ij}\|^2 + \|\dot{q}_i - \dot{q}_j\|^2). \tag{14}$$

Remark 2: It can be verified easily from Lemma 1(b) that L holds the following property:

$$\sum_{(i,j)\epsilon E} \omega_{ij}(z_j - z_i) = z^T DW.$$

Using the results in Lemma 1 and Remark 2, the formation and velocity error (14) can be transformed into a matrix representation as the following:

$$\sum_{(i,j)\epsilon E} \omega_{ij}(\|q_i - q_j - d_{ij}\|^2 + \|\dot{q}_i - \dot{q}_j\|^2)$$

$$= \sum_{(i,j)\epsilon E} \omega_{ij}(\|q_i - q_j\|^2 - 2(q_i - q_j)^T d_{ij} + \|d_{ij}\|^2 + \|\dot{q}_i - \dot{q}_j\|^2)$$

$$= q^T DWD^T q - 2q^T DWd + d^T Wd + v^T DWD^T v$$

$$= \begin{bmatrix} q \\ 1 \\ v \end{bmatrix}^T \begin{bmatrix} \mathcal{L} & -\mathcal{DW}d & 0 \\ -(\mathcal{DW}d)^T & d^T\mathcal{W}d & 0 \\ 0 & 0 & \mathcal{L} \end{bmatrix} \begin{bmatrix} q \\ 1 \\ v \end{bmatrix}$$

Where $q = [q_1{}^T, \dots, q_m{}^T]^T \in R^{nm}$ and $v = [\dot{q}_1{}^T, \dots, \dot{q}_m{}^T]^T \in R^{nm}$. This expression can be shown in compact form:

$$\begin{bmatrix} q \\ 1 \\ v \end{bmatrix}^T \begin{bmatrix} \mathcal{L} & -\mathcal{DW}d & 0 \\ -(\mathcal{DW}d)^T & d^T\mathcal{W}d & 0 \\ 0 & 0 & \mathcal{L} \end{bmatrix} \begin{bmatrix} q \\ 1 \\ v \end{bmatrix} = z^T Q z \tag{15}$$

where $z = [q^T, 1, v^T]^T \in R^{2nm+1}$, $d = col[d_{ij}] \in R^{nm}$ is the whole formation desired distance vector and $Q = \begin{bmatrix} \mathcal{L} & -\mathcal{D}\mathcal{W}d & 0 \\ -(\mathcal{D}\mathcal{W}d)^T & d^T\mathcal{W}d & 0 \\ 0 & 0 & \mathcal{L} \end{bmatrix}$. *Formation matrix* Q has a property which is given below.

Remark 3. Since $z(t)^T Q z(t) \geq 0$, formation matrix Q is *positive semi definite* ($Q \geq 0$).

On the other hand, let z and $u = [u_1{}^T, \ldots, u_m{}^T]^T \in R^{2nm}$ be the state and control vectors, respectively. Differentiating the state vector z with respect to t we have:

$\dot{z} = [\dot{q}^T, 0, \dot{v}^T]^T = [v^T, 0, u^T]^T$

and expressing $\dot{z}$ in terms of state-space representation yields:

$$\dot{z} = Az + \sum_{i=1}^{m} B_i u_i \tag{16}$$

Where $A \begin{bmatrix} 0 & I_{nm} \\ 0 & 0 \end{bmatrix} \in R^{2nm+1}$, $B_i = [0, b_i{}^T]^T \in R^{(2nm+1)\times n}$ and $b_i = [0 \ldots I_n \ldots 0]^T \in R^{nm\times n}$. Linear differential state equation (16) is the dynamics of the group of the mobile robots.

In the formation graph G each vertex (robot) $i \in V$ can choose its own weight matrix W_i regarding with the immediate neighbours in the graph topology. Suppose that in a formation graph with 4 nodes and $|E| = 3$ the set of neighbors of vertex 2 are 1 and 3. Then, W_2 will be selected as

$$W_2 = \begin{bmatrix} \omega_{21} & 0 & 0 \\ 0 & \omega_{23} & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the entries of W_i (i.e., ω_{ij} for all j such that $(i,j)\epsilon E$) can be selected to reflect a motion situation, for instance, individual robots can choose these entries based on the fuel level in their tank. When individual robots have chosen their weight matrix based on a variety of factors, individual robot i will have its own L_i and Q_i matrices.

Using the finite horizon cost function concept in the optimal control theory we can define the formation and velocity error expression $z^T Q z$ in form of a finite horizon quadratic cost function for robot i to minimize as the following:

$$J^i = z(T)^T Q_{iT} z(T) + \int_0^T (z^T Q_i z + \sum_{j=1}^{m} u_j{}^T R_{ij} u_j) dt \tag{17}$$

Where

$$Q_{iT} = \begin{bmatrix} \mathcal{L}_{iT} & -\mathcal{D}\mathcal{W}_{iT}d & 0 \\ -(\mathcal{D}\mathcal{W}_{iT}d)^T & d^T\mathcal{W}_{iT}d & 0 \\ 0 & 0 & \mathcal{L}_{iT} \end{bmatrix}, Q_i = \begin{bmatrix} \mathcal{L}_i & -\mathcal{D}\mathcal{W}_i d & 0 \\ -(\mathcal{D}\mathcal{W}_i d)^T & d^T\mathcal{W}_i d & 0 \\ 0 & 0 & \mathcal{L}_i \end{bmatrix}, L_{iT} = DW_{iT}D^T,$$

$L_i = DW_iD^T$, $\mathcal{W}_{iT} = W_{iT} \otimes I_n$, $W_{iT} = diag[\omega_{ij}]$, $\mathcal{W}_i = W_i \otimes I_n$, $W_i = diag[\mu_{ij}]$ and $\mu_{ij} \geq 0$, $R_{ij} > 0$ are the weight parameters.

Remark 4: From (17) and (14), we can see that the term for terminal formation and velocity error is

$$z(T)^T Q_{iT} z(T) = \sum_{(i,j)\epsilon E} \omega_{ij} (\|q_i(T) - q_j(T) - d_{ij}\|^2 + \|\dot{q}_i(T) - \dot{q}_j(T)\|^2)$$

and the term for formation and velocity error for the entire process is

$z^T Q_i z = \sum_{(i,j)\epsilon E} \mu_{ij} (\|q_i - q_j - d_{ij}\|^2 + \|\dot{q}_i - \dot{q}_j\|^2)$.

Remark 5: Notice from *Remark 3* that the matrices Q_{iT} and Q_i are positive semidefinite. With this in mind and considering that all matrices are real value and symmetric as well as R_{ii} is positive definite (by definition), $J^i(u)$ is then a strictly convex function of u_i for all admissible control functions u_j, $j \neq i$ and for all z_0 (from (16)).

By state equation (16) and cost functions (17), it can be seen that the leaderless formation control problem is transformed into a standard linear quadratic differential game problem given in (5) and (6). For this formation game, considering players of the game in a non-cooperative manner as well as that, all players make their decision based on initial state $z(0) = z_0$, (i.e., the open-loop information structure is available in the game), the open-loop Nash equilibrium solution in Theorem 1 can be used as the formation control strategy for all the robots. In the next section, open-loop Nash equilibrium for the formation control is described in details.

4- Open-loop Nash formation control

In the past section, the leaderless formation control problem was considered as a non-cooperative linear quadratic differential game. This game under the open-loop information structure admits an open-loop Nash equilibrium solution given in the theorem 1. Moreover, the equilibrium associated state trajectory is given by:

$$z^* = [I0 \dots 0\ ]e^{M(T-t)}\begin{bmatrix} I \\ Q_{1T} \\ \vdots \\ Q_{mT} \end{bmatrix} H(T)^{-1} z_0 \qquad (18)$$

where

$$H(T) = [I0 \dots 0\ ]e^{MT}\begin{bmatrix} I \\ Q_{1T} \\ \vdots \\ Q_{mT} \end{bmatrix} \text{and } M \begin{bmatrix} -A & S_1 & \dots & S_m \\ Q_1 & A^T & 0 & 0 \\ \vdots & 0 & \ddots & 0 \\ Q_m & 0 & 0 & A^T \end{bmatrix}.$$

As it seen the equilibrium associated state trajectory exists if $H(T)$ is invertible. Additionally, it can be shown that the invertibility of $H(T)$ is equivalent to the existence of a unique open-loop Nash equilibrium (Engwerda, (2005)). We have the following result for the formation control problem based on the minimum principle conditions.

Theorem 2. Suppose that the formation control problem as a differential game in (16) and (17) satisfies the existence condition of unique open-loop Nash equilibrium as stated in Theorem 1. Then, $H(T)$ is invertible for all $T > 0$.

Proof: Engwerda and Weeren (1995) shows that if a set of coupled Riccati differential equations (7) has a solution on $[0, T]$, this solution is:

$$P_i = G_i(T-t)H(T-t)^{-1}$$

where

$$G_i(T-t) = [00 \dots I \dots 0\ ]e^{M(T-t)}\begin{bmatrix} I \\ Q_{1T} \\ \vdots \\ Q_{mT} \end{bmatrix}.$$

It can be seen that the invertibility of $H(.)$ is vital to obtain P_i. This proves the theorem.■

Remark 6: In example 6.2 in Basar and Olsder (1999) for the two-player linear quadratic game, it has been shown that for horizon length $T = 0.1$ matrix $H(T)$ is not invertible, whereas the calculations showed that for $T = 0.11$ matrix $H(T)$ is invertible. Then, invertibility of $H(T)$ obviously depends on chosen horizon length T and matrix M.

In fact, there is no general set of conditions (on the parameters of the linear quadratic Nash game) that would guarantee either the existence of a solution to (7) or the invertibility of $H(T)$. One special case is the games with weakly coupled players. For a sufficiently small $\varepsilon > 0$, let $W_{iT} = \varepsilon(diag[\omega_{ij}])$ and $W_i = \varepsilon(diag[\mu_{ij}])$ which implies that for $\varepsilon = 0$ the game decomposes into m completely decoupled optimal control problems, one for each player.

In the next section, we discuss the problem in which the weight matrices Q_i are not included in the cost functions, so-called Mayer type cost functions.

5- Invertibility of the matrix $H(T)$

To simplify the problem, the weight matrices $W_i = diag[\mu_{ij}]$ could be set zero. Then, the quadratic cost functions in (17) will be simplified as

$$J^i = z(T)^T Q_{iT} z(T) + \int_0^T \sum_{j=1}^m u_j{}^T R_{ij} u_j dt \tag{19}$$

Where it means that robot i will try to minimize a weighted sum of the terminal formation errors and velocity errors while minimizing its control effort made during the entire formation control process. A similar quadratic performance index for the formation control has been used in Lee (2014). Before we present the next theorem for the formation game problem in (16) and (19), the following definitions are introduced first.
Define

$S = [S_1 \dots S_m\ ], V_i = [0 \dots I \dots 0\ ],$

$Q_T = \begin{bmatrix} Q_{1T} \\ \vdots \\ Q_{mT} \end{bmatrix},$

$D = \begin{bmatrix} A^T & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & A^T \end{bmatrix}$

and $W = I - A(T-t) + [S(T-t) + (SD - AS)\frac{(T-t)^2}{2} - ASD\frac{(T-t)^3}{6}]Q_T$.

The matrix M for the differential game problem in (16) and (19) is

$M = \begin{bmatrix} -A & S \\ 0 & D \end{bmatrix}.$

Theorem 3. The formation control problem defined as a differential game by (16) and (19) admits a unique open-loop Nash equilibrium for every initial state if and only if the matrix W is invertible.
Proof: We observe that

$$A^2 = D^2 = M^4 = 0.$$

The power series for the matrix exponential $e^{M(T-t)}$ results

$$e^{M(T-t)} = \begin{bmatrix} I - A(T-t) & S(T-t) + (SD - AS)\frac{(T-t)^2}{2} - ASD\frac{(T-t)^3}{6} \\ 0 & I + D(T-t) \end{bmatrix}.$$

Then it is verified easily

$H(T-t) = I - A(T-t) + [S(T-t) + (SD - AS)\frac{(T-t)^2}{2} - ASD\frac{(T-t)^3}{6}]Q_T,$
$H(T) = W$

and

$$G_i(T-t) = V_i(I + D(T-t))Q_T.$$

This result proves the theorem. ■

6- Simulation results

We test a triangle formation shape with a team of four robots in 2-dimensional coordinates (e.g., $n = 2, m = 4$) (see Figure 1). In the formation graph the set of edges E and the incidence matrix D of the triangle shape are:

$$E = \{e_{12}, e_{14}, e_{23}\},\ D = [-1100-10010-110\ \ ].$$

The weight matrices in the cost functions are formed based on the neighbourhood relation, such that for this triangle formation shape it is seen from Figure1, the neighbours of robot 1 are robot 2 and 4, the neighbours of robot 2 are robot 1 and 3, the neighbour of robot 3 is robot 2 and the neighbour of robot 4 is robot 1. We select the following formation cost weight matrices:

$$W_{1T} = [500050000\ ],\ W_{2T} = [500000005\ ], W_{3T} = [000000005\ ], W_{4T} = [000050000\ ].$$

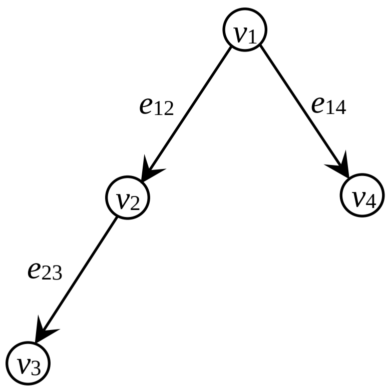

Fig 1. Triangle shape and formation graph.

The desired offset vectors of the formation shape among the robots are:

$$d_{12} = [-2-4\ ], d_{14} = [2-4\ ], d_{23} = [-2-4\ ].$$

The initial positions of the robots are:

$$q_1(0) = [-10\ ], q_2(0) = [10\ ], q_3(0) = [20\ ], q_4(0) = [-30\ ].$$

The initial velocities of the robots are:

$$\dot{q}_1(0) = [02\ ], \dot{q}_2(0) = [03\ ], \dot{q}_3(0) = [01.5\ ], \dot{q}_4(0) = [01\ ].$$

The weight matrices for the control effort in the cost functions are selected to be the identity matrix of dimension 2, e.g., $R_{11} = R_{22} = R_{33} = R_{44} = [1001\ ]$.
The finite horizon time is selected to be $T = 7$s and the sample time for simulation is considered to be 0.1s. With the given parameters for the triangle formation, we observe that $H(7)$ is invertible. Therefore, the game has a unique open-loop Nash equilibrium that can be calculated through (8) and then, the state trajectory for the robots are given by (18).

The robots' trajectories in the $x - y$ plane are shown in figure 2. The results show that the desired triangle shape formation among the four robots is achieved at the specified finite horizon time. The formation graph in Figure 1 and the desired offset vectors of the formation shape among the robots show that to form the desired shape: robot 1 and 2 have to achieve to relative distance 2 and 4 in the x and y position, respectively, robot 1 and 4 have to achieve to relative distance -2 and 4 in the x and y

position, respectively, and robot 2 and 3 have to achieve to relative distance 2 and 4 in the x and y position, respectively. Moreover, all robots in the order given above have to achieve to zero relative velocities on both x and y-axis. The position and velocity error between the robots' trajectories and their desired configuration in the formation are illustrated in figure 3 and figure 4, respectively. It can be seen that all the mentioned relative distances and also zero relative velocities are realized.

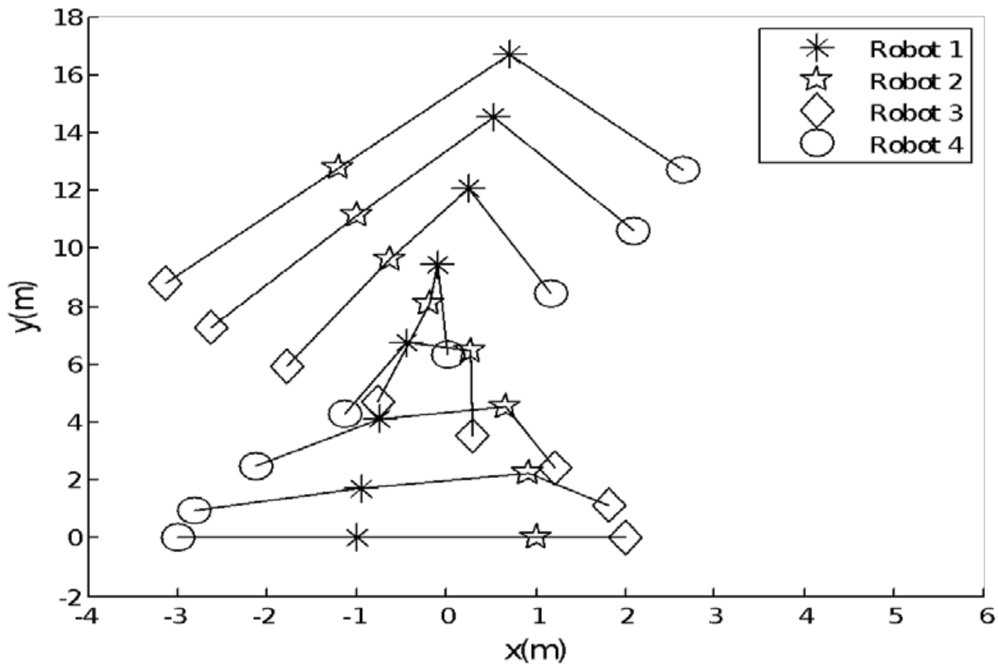

Fig 2. Progression of four robots in the triangle formation in the $x - y$ plane.

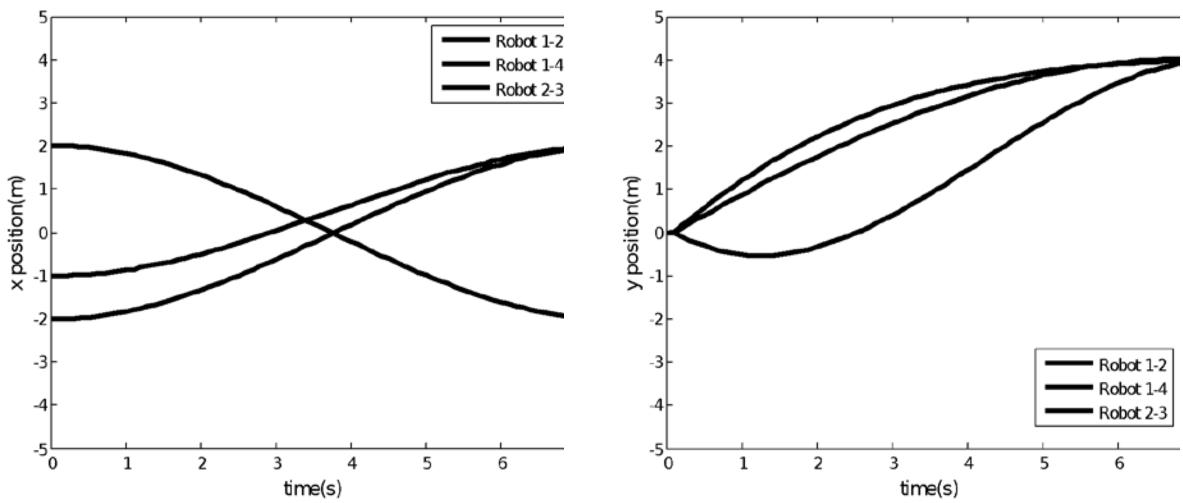

Fig 3. Formation errors on the x-axis and on the y-axis.

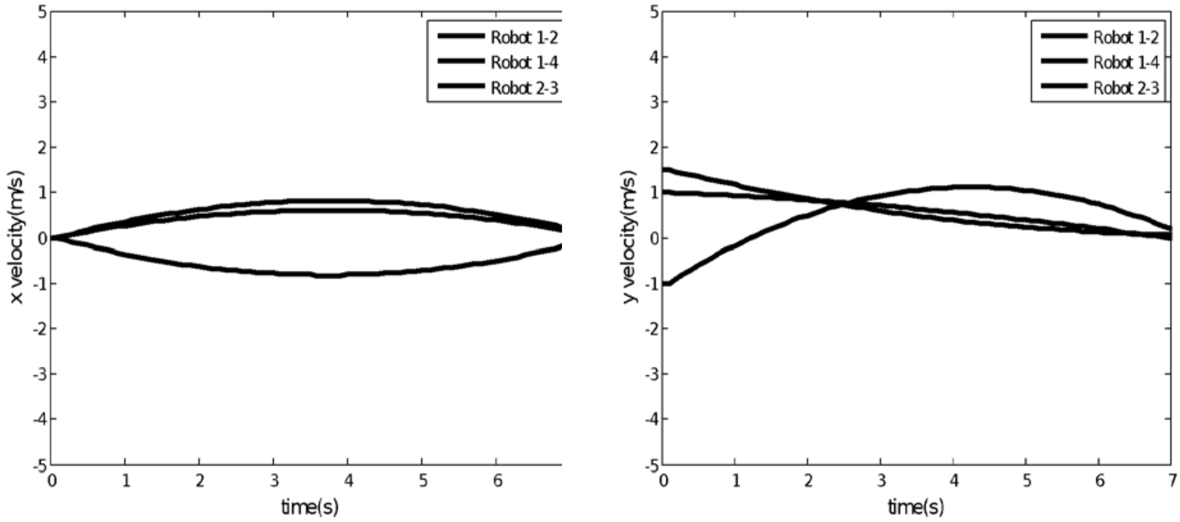

Fig 4. Relative velocities on the x-axis and on the y-axis.

7- Conclusion and future works

In this paper, the leaderless formation control problem for a multi-robot system with double integrator dynamics is modelled as a non cooperative linear quadratic differential game. Through the use of algebraic graph theory, we formed a formation matrix and used it totransform the cost functions into a standard quadratic form. The system dynamics is stated as a linear state equation. The formation problem became a linear quadratic differential game. Assuming that the robots play a Nash strategy for the formation game, the open-loop Nash equilibrium solution is utilized as the formation control strategy. We illustrated the results through testing a triangle formation shape by a group of four robots. Taking the collision avoidance constraint in the formulation of formation game into account is our next purpose.

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