Journal of Industrial and Systems Engineering
Vol. 11, special issue on game theory applications, pp. 1-24
Winter (January) 2018

A Game Theoretical Study on Pricing and Production Decisions of Firms

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Abstract

Today, products' lead time adjustment has become one of the main challenges for organizations managers and service companies. Products' lead time has an undeniable impact on firms' inventory control way in order to meet required customer demand. In addition, pricing on manufactured good way, in order to achieve most profit and decrease the side expenditures, is of issues of importance in this context. In this research, market demand, is considered Sensitive to product sales price and amount of inventory-on-hand and we have addressed to analyze the behavior of firms by assuming Being unauthorized of Shortage. It should be noted that in both scenarios, firms' decision making review about appropriate lead time optimal policies, products' price and amount of inventory as the problem decision variables with the aim of maximizing profits and minimizing costs are addressed. In first scenario, only a single firm behavior analysis is considered and in second scenario, two firm's behavior Study when changing in price level, lead time and inventory at exclusive market have been addressed by using Nash equilibrium. In the following, by expressing numerical examples, we have addressed to the problem results analysis and have determined optimal points. Sensitivity analysis clearly Shows Market potential and inventory attraction parameters influence on the decision variables. Eventually, it was found that among two proposed scenarios, single firm's scenario is more profitable than the two firms' scenario. This research can be helpful for planners and industrial managers to optimize existing conditions, achieving maximum profit.

Keywords: Nash game theory, pricing, lead time, inventory controlling.

1-Literature review

In this research, the main and most influential factors in the profitability of the firm, or companies that are in competition with alternative goods production are introduced as the major problem variables and in this section are investigated. These main variables include lead time, pricing and inventory-on-hand. Each of these variables alone could have a significant impact on the profitability of the firm. Evaluation of these variables simultaneous change can have beneficial results in improving performance of firms, individually or competitively, for managers and planning in the industrial sector.

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ISSN: 1735-8272,

Establishing Proportionality among the prices of manufactured goods, products' lead time as well as inventory-on-hand can be a major step in improving performance and increasing firm's profitability.
In this section, we review recent studies in the field of lead time, inventory and pricing in inventory control systems. These studies have been raised in two different categories. First, the research conducted about lead time and its mutual effect on companies' on hand inventory is addressed and then studies conducted about pricing and its mutual effect on inventory is discussed.

1-1- Lead time and inventory-on-hand

One of the very important variables in inventory control systems is lead time. There are numerous studies about the importance of lead time which each has addressed to review its different aspects is focused and has shown its impact on inventory management.

Of these studies, we can refer to Yang et al. (2017). In their research, they adopted a two-level supply chain in which they addressed to investigate the retailer orders optimization and the manufacturer's amount of inventory and compared optimal decisions in centralized and decentralized scenarios and found that in the decentralized chain, at least one member will have excess inventory and customers will benefit from shorter lead time and will outperform the centralized chain. Wang and Disney (2017), in their research addressed to examine strengthening the ordering fluctuations in and inventory in a state-space supply chain and in which they considered lead time as random. They determined exact distribution functions for orders and amount of inventory and examined the conditions for simultaneous inventory reduction and orders variance and found that Simulations of the model with demand real data and lead time will show the model profitability. Heydari and Norouzinasab (2016) applied an incentive policy in a two-echelon supply chain wherein the demand was stochastic and depended on price and lead-time. They analyzed the system decision making with the game theory approach. They demonstrated in numerical examples that proposed method, decreases price and lead-time and also both members' profitability is increased. Roldán et al. (2016) discussed on coordination of the supply chain and mentioned that inventory control and distribution management are affected more than other effective factors. By considering that demand is not deterministic and lead-times are variable, they studied the relationship between available demand information and inventory policies. Xie et al. (2016) combined the inventory-rationing model into markdown instruments by customers segmented with the sensitivity on lead-time. Also, they defined the threshold-type optimal requirement policies and suggested an algorithm to calculate the thresholds. Ju et al. (2015) studied a supply chain model where the retailer orders his required products from two suppliers which one of them is local and reliable and the other is global and under seller. They raised this problem with a dual-sourcing inventory model, with positive lead time and random yield. In addition, by follow the (DOP) policy, they found that the results of their innovative method are close to optimal values. Wu et al. (2015) introduced their model in the form of two modes of one firm and the two firms competing and reviewed lead time and inventory. They showed their results by Nash equilibrium and pointed that growth in lead time increases the Inventory and according to billboard effect, this, will cause more demand. Also they found in the competition mode of two firms, inventory-driven competition increases the lead time.

Amit et al. (2015*)* in their study, considered a model in which demand is also affected by the amount of inventory shown on the shelves in addition to the external uncertainties. In addition, they assumed in their model that demand supply with a large inventory is dominated on demand supply with a less inventory. Finally, they achieved the optimal amount of inventory on the shelves for retailer with numerical calculations. Jian et al. (2015) studied a model in which the classic newsvendor problem by involving the lead time as a control variable is considered. In this model, they considered demand forecasting process and construction cost as basic functions with compressed lead time and finally found that under what circumstances, the trade-off problem can be solved and be profitable for the suggested model. Hammami et al. (2015), in their study, considered a multi-level supply chain with foreign suppliers and multiple production centers and investigated the carbon emissions _with lead time limitation_ And described numerous case studies in the field of greenhouse gases and their emissions in the supply chain. Sarkar et al. (2014) in their model, considering the incentive policies for customers, and policies related to defective goods and goods inspection policy, took order quality and lead time as decision variables. They Assumed lead time as random and optimized the model by minimizing systems' costs. Lin et al. (2014) examined the inventory model Where considered the defective items and lead time. By adopting the distribution-free approach, they could achieve a generalization to the inventory replenishment policies. Song et al. (2013) examined the interaction between the producer and retailer lead time in a minimax model using the Stackelberg game where the demand lead time is assumed free and its mean and variance are known. With a transfer-payment contract and checking the results, they found that this contract divides flexibly system costs between the two parts of the supply chain. El-Wakeel (2012) analyzes a model with the aim of minimizing the annual costs and considered that the lead time follows a uniform distribution and investigated the backorders inventory, when the orders cost unit is a function of the order quantity. Wang and Yan (2009) considered an inventory model in which a supplier gives right to choose the lead time _short or long lead time_ to customers. Orders related to customers who chose a long lead time, can be from later period productions; But Orders related to customers who chose a short lead time, must be supplied from inventory-on-hand. Finally, they showed optimal inventory supply commitment policies. Kaminsky and Kaya (2008), in their study, considered a supply chain network consisting of several producers with central management as external suppliers. They designed a heuristic plan for inventory positioning, orders sequencing and short and reliable lead time and investigated its impact on costs. Wang and Chi (2009), in their paper, addressed in this case which a producer can affect on retailers' pricing decisions and shelf space, using revenue sharing contract. They considered a two-level supply chain with a supplier where demand depends on the retailer's price and shelf space. Finally, they showed by a case study that this contract is properly designed and can increase the profit. Some other related researches are presented by Taleizadeh et al. (2008a,b, 2009, 2010a,b, 2011, 2012, 2013a,b, 2014a,b,c,d).

1-2- Pricing and inventory-on-hand

Other affecting factor is pricing which deciding on its dimensions can have a significant impact on firm's profit. In recent years, several studies have been done in this case. Many of these studies have pointed to the interplay of pricing on inventory control.

For example, Lee and Park (2016), in their research, assumed a supply chain in which two retailers are supplied by one supplier and Retailers compete with each other to supply orders and are allowed to work together by transferring their excess inventory to retailer that it has the inventory shortage. They analyzed their model by the Nash equilibrium and found that even for two quite similar retailers, coordination among transfer price, in rationing game, rather than out of the game, remains in a more limited range of parameter values. Naseri and Hafezalkotob (2016) discussed on transportation problems and considered network flow models to expound the mentioned problem. They emphasized that the product demand is dependent on the price. Thus, they modeled the decision making of pricing and analyzed integration owners with game theory methods. At least they provided the numerical examples to show the results. Alfares and Ghaithan (2016) considered an inventory model which in, the holding cost is dependent on the storage time, demand is dependent on the selling price and the purchase cost is dependent on order size. They created the mathematical model and the optimal solution to solve the problem by considering that demand rate, holding cost and the purchase cost are simultaneously varied. At last the numerical examples showed the influence of various parameters. Noori-daryan and Taleizadeh (2016) by employing the order quantities and selling prices of the manufacturer as the decision variables studied a three-echelon pharmacological supply chain (SC) including multi-distributor of raw materials, a pharmaceutical factory, and multi-drug distributor companies. They considered the Stackelberg game between the members of the chain to maximize the total profit of the supply chain. Lin and Wu (2016), studied price and inventory control in a Taiwan's shrimp supply chain and compares centralized and decentralized supply chain performance and under various scenarios, they addressed to investigate pricing for farmer, wholesaler and market. They concluded that the performance in a centralized supply chain is better. Indefurth and Kiesmüler (2015), in their study considered a periodic review inventory system where the demand is fortuitous and efficiency is random. They analyzed the linear inflationary policies exactly and innovatively by Markov chain and concluded that both methods work properly. Zhang et al. (2015), studies the dynamic pricing strategy and pointed out the inventory replenishment cycle for perishable goods where consumer demand is dependent on the sale price and the number of products available in store. They provided new solutions for administrators, by effective parameters sensitivity analysis. Ahmadi-Javid and Hoseinpour (2015) analyzed and investigated a location-inventory-pricing model in order to design a supply chain distribution network where demand is sensitive to price and inventory capacity. By a numerical example, they found that with the gradual increase possible values for pricing decisions, this model can be a near-optimal solution to solve the inventory-pricing problems, with continued pricing decisions. The approach used in their study could be used for other supply chain planning, with the sensitive-price demand. Chua and Liu (2015), in their study, using a stocking-factor-elasticity approach discussed the pricing of newsvendor problem with multiplied uncertain demand and lost sales. By expressing a numerical example, they found that optimal ordering size is reduced in the presence of uncertain demand for low cost and low price elasticity. Qin et al. (2014), in their study, taking into account the selling price per unit and product quality as effective factors on demand function, they analyzed the pricing and inventory control at the same time on fresh and perishable products and used numerical examples to find the optimal solutions.

Chung et al. (2014), considered their model for electronic goods which include multiple period discount. They make decisions for pricing and supply chain inventory in the presence of these discounts and proposed an optimal strategy over the lifetime of the products in the market. Ding et al. (2012), in their study, modeled a Buyback contract, using a pricing and inventory periodic review model and by considering the cost of compensation and setup in a finite planning horizon. Finally, they proposed an optimal policy for decision-maker, in order to maximize total profit. Chao et al. (2012), studied the inventory optimization problem and pricing in a periodic review inventory system where considered cost of setting up and ordering capacity in any given period And showed that the optimal inventory control is characterized by a (s, s', p) policy. pang (2011), in his study, reviewed the optimal pricing policies and inventory control in a periodic-review inventory system with fixed ordering cost and increased demand and considered the possibility of inventory corruptibility over time and also assumed that unmet demand may be saved to some extent. He finally analyzed two sufficient conditions under which the policies are optimal. As seen, none of the articles introduced, studied the effect of three factors including lead time, price and amount of inventory together to maximize firms' profits. We would consider it in this research. In addition, the impact of these three factors on the two firms' activity process as competitive under Nash approach to achieve maximum profit are Among the expected objectives of this research.

This study is an attempt to find a solution to adjust parameters that affect the firm profitability, and thereby offers a new way ahead of managers and planners. In this study, adjusting the products' lead time, determining the optimal price for competing firms' goods and determining appropriate inventory-on-hand, with the aim of maximizing the profitability and minimizing the existing costs. In the next sections, we first propose a definition for the problem and after the introduction of the parameters and decision variables of the problem assumptions, by considering all costs, the problem objective function is modeled. In following, with proving the model objective function concavity, accurate, an adequate and efficient solution method to achieve the optimum values of problem decision variables are provided for each scenario. In Sections 4 and 5, numerical example and sensitivity analysis are expressed respectively, and finally, Section 5 summarizes and concludes the matter.

2-The problem definition

In this research, the modeling of the manufacturing firms profit function has been addressed. In this modeling, we are trying to offer an optimal value for each main problem variables in addition to the maximizing profit, by considering all the factors that directly or indirectly affect the firms' profitability. We developed this problem for two different scenarios. In the first scenario, a manufacturing firm is considered that plans to maximize its average profit by using to determine optimum values for lead time, inventory-on-hand and the price of the product. The mentioned firm, should determine the price and time proportional to its base inventory (I), and to cover the demand without the worry of cost due to late delivery and storage in the warehouse. One of the most effective components in available amount of inventory of manufacturing firms is the lead time. In this research, it is assumed that the shortage is not permitted and profit for firms is always positive and non-zero. In the second model, the previous model has been developed for two competing firms. So that in this model, demand is dependent on the inventory of firms' warehouse and their selling price for a desired good. In this model, firms generate substitute products and this relationship between the price and price elasticity of substitute products prices is obvious in the demand function. To analyze the performance of competing firms in the second scenario, Nash equilibrium have been used. In Nash approach, both firms without the knowledge of the terms of each other, participate in a non-cooperative game and in fact, a management trend between the relationships among the two firms is generated in order to maximize the profits.

The purpose of this paper is to maximize firm profit by using optimal values of decision variables which are selling price, and lead time and inventory.

The decision variables as well as used parameters to model this problem are shown in table (1).

Table 1. Symbols Used in This Issue

Parameters:	
m:	The market potential, $m > 0$;
n :	The elasticity value of demand inventory, $0 < n < -\frac{1}{2}$;
e :	The price elasticity value in demand;
e_i:	The price elasticity value in demand for firm i ;
d :	The demand rate per unit time;
η:	The demand average in lead-time;
θ:	The non-negative coefficient in efficiency $(0,1]$ for assuring from positivity of demand;
v :	The variance of demand in lead-time;
A :	The average backorders;
I :	The total inventory and expected sales value;
c_b:	The back order cost per unit time;
c_h:	The repair and maintenance cost for each product in single time unit;
c_p:	The Production cost for each unit of product in single time unit;
Z:	The Profit function of firm under first scenario;
Z_i :	The Profit function of firm i under the second scenario;
Decision variables:	
S :	The available inventory of firm per unit time;
y :	The sale price for each unit of product in single time unit
t :	Lead time

3-Modeling

In this section of research, profit function modeling method in single firm and two firms' scenarios is addressed. Difference due to sales revenue and costs related to production, backorder cost and holding cost expresses the profit function in each firm. Consequently, the general form of profit function for each problem companies is modeled as follows:

$$\max Z = (y - c_p)d - c_b A - c_h S \tag{1}$$

3-1- Single firm scenario

This section has studied the model of a manufacturing firm alone where the firm is looking to determine appropriate inventory at the lead time (t) with price (y). Firm demand assumes as a function of inventory and products' price.

It is assumed that $0 < n < -\frac{1}{2}$. Note that when $n = -\frac{1}{2}$, shows a mode in which demand is independent of inventory and is only dependent on price. Now if e value equals zero, we can say that the demand is independent of price and inventory which in this case the demand will remain dependent only on market potential parameter. The demand function can be considered as Product of market potential, products' price and inventory-on-hand for firms which, the parameters e and n can be regulator of effects of inventory and price in demand function respectively. The demand function is considered as expression (2). In this expression, the more the n, the inventory affects more on demand.

$$d = mS^{2n+1}y^{-e} \tag{2}$$

In the classic inventory theories, the demand is assumed independent of inventory. The inventory-on-hand of firm (S) is resulted from equation (3) where p shows the random demand with density function $\Omega(p)$ and follows the uniform distribution $U(\eta-\nu, \eta+\nu)$.

$$S = \int_{\eta-\nu}^{I} (I-p)\Omega(p)dp = \frac{1}{2\nu}\left(Ip - \frac{1}{2}p^2\right)\Big|_{p=\eta-\nu}^{p=I} = \frac{1}{4\nu}(I-\eta+\nu)^2 \tag{3}$$

In this model, η shows the demand average in lead time and θ is non-negative factor that can vary in the $0 < \theta \le 1$ interval. This parameter creates this confidence that demand will not be negative. Expressions (4) and (5) represent the mean and variance of demand in lead time respectively:

$$\eta = td = tmS^{2n+1}y^{-e} \tag{4}$$

$$\nu = \theta td = \theta tmS^{2n+1}y^{-e} \tag{5}$$

By substituting equations (4) and (5), amount of inventory and shortage of the firms, are resulted which are shown in the expressions (6) and (7):

$$I = \eta - \nu + 2\sqrt{\nu S} = tmS^{2n+1}y^{-e}(1-\theta) + 2\sqrt{\theta tmS^{2n+2}y^{-e}} \tag{6}$$

$$A = \eta + S - I = tmS^{2n+1}y^{-e} + S - tmS^{2n+1}y^{-e}(1-\theta) - 2\sqrt{\theta tmS^{2n+2}y^{-e}} \tag{7}$$

The profit function of the "single firm" problem which is shown in (8) equation is obtained by putting equations (2) to (7) in equation (1).

$$\begin{aligned}\max Z = {} & mS^{2n+1}y^{1-e} - c_p mS^{2n+1}y^{-e} - c_b tmS^{2n+1}y^{-e} - c_b S + c_b tmS^{2n+1}y^{-g}(1-\theta) + \\ & 2c_b S^{n+1}\sqrt{\theta tmy^{-e}} - c_h S\end{aligned} \tag{8}$$

3-2- Two firms' scenario

In this section, we generalize the previous model for two manufacturing firms. According to the problem assumptions, the two companies under study, substitute products manufacturer. In this case, the customer demand is stated as equation (9).

$$d=\sum_{i=1}^{2} d_i \tag{9}$$

The level of customers' demand depends on the amount of inventory and price of each firm. Each firm meets a part of the market demand according to the inventory on its hand. So we have for $i=1,2$:

$$d_i = mS_i(S_i+S_{3-i})^{2n} y_i^{-e_i} y_{3-i}^{e_{3-i}} \tag{10}$$

Now we can define demand mean values and demand variance per unit of time, in expressions (11) and (12).

$$\eta_i = t_i d_i = t_i m S_i (S_i+S_{3-i})^{2n} y_i^{-e_i} y_{3-i}^{e_{3-i}} \tag{11}$$

$$v_i = \theta t_i d_i = \theta t_i m S_i (S_i+S_{3-i})^{2n} y_i^{-e_i} y_{3-i}^{e_{3-i}} \tag{12}$$

The amount of inventory and shortage of firm (i), are resulted by placing two expressions (11) and (12), as (13) and (14) expressions.

$$I_i=\eta_i - v_i + 2\sqrt{v_i S_i} = (1-\theta)t_i m S_i (S_i+S_{3-i})^{2n} y_i^{-e_i} y_{3-i}^{e_{3-i}} + 2\sqrt{\theta t_i m}\, y_i^{\frac{-e_i}{2}} y_{3-i}^{\frac{e_{3-i}}{2}} S_i (S_i+S_{3-i})^n \tag{13}$$

$$A_i=\eta_i + S_i - I_i = S_i + \theta t_i m S_i (S_i+S_{3-i})^{2n} y_i^{-e_i} y_{3-i}^{e_{3-i}} - 2\sqrt{\theta t_i m}\, y_i^{\frac{-e_i}{2}} y_{3-i}^{\frac{e_{3-i}}{2}} S_i (S_i+S_{3-i})^n \tag{14}$$

The problem profit function for firm (i), is resulted by substituting equations (9) to (14) in equation (1) as follows, if $i=1,2$:

$$\begin{aligned}\max Z_i &= (y_i - c_{pi} - \theta c_b t_i) m S_i (S_i+S_{3-i})^{2n} y_i^{-e_i} y_{3-i}^{e_{3-i}} \\ &\quad + 2c_b S_i (S_i+S_{3-i})^n \sqrt{\theta m t_i}\, y_i^{\frac{-e_i}{2}} y_{3-i}^{\frac{e_{3-i}}{2}} - (c_h + c_b) S_i\end{aligned} \tag{15}$$

4- Solution method

In this section, we will examine the problem separately in two scenarios. In the "single firm scenario", maximum point of profit function is determined against optimal values of decision variables; while in the "two firms Scenario", with the aim of achieving maximum profit, the behavior of two companies is analyzed by using Nash equilibrium fully.

4-1- Single firm scenario

In the model presented in this scenario, the problem objective function depends on products' price, inventory and products' lead time variables. To achieve the optimal value for each of them, the profit function shown in equation (7) should be derived with respect to y, S and t. In this way, in each derivation, impact of variable which is derived with respect to it on the profit function, can be shown.

$$\frac{\partial Z}{\partial t} = -c_b m S^{2n+1} y^{-e} + c_b m S^{2n+1} y^{-g}(1-\theta) + c_b S^{n+1} t^{-\frac{1}{2}} \sqrt{\theta m y^{-e}} \tag{16}$$

$$\frac{\partial Z}{\partial S} = (2n+1)mS^{2n}y^{1-e} - (2n+1)c_p mS^{2n}y^{-e} - (2n+1)c_b tmS^{2n}y^{-e} - c_b$$
$$+ (2n+1)c_b tmS^{2n}y^{-e}(1-\theta) + 2(n+1)c_b S^n\sqrt{\theta tmy^{-e}} - c_h \tag{17}$$

$$\frac{\partial Z}{\partial y} = (1-e)mS^{2n+1}y^{-e} + ec_p mS^{2n+1}y^{-e-1} + ec_b tmS^{2n+1}y^{-e-1}$$
$$- ec_b tmS^{2n+1}y^{-e-1}(1-\theta) - ec_b S^{n+1}\sqrt{\theta mt}\,y^{-\frac{e}{2}-1} \tag{18}$$

Then for finding the optimal value of each product variables, including price, lead time and inventory-on-hand, equations (16) to (18) which are derivatives of this objective function with respect to each of these three variables, are equal to zero. Finally, the optimal values of decision variables are obtained by finding the roots of the above equations. By putting the equation (16) equal to zero and simplifying it, the expression (19) will be obtained. Expression (19) shows the impact of price and inventory on the lead time.

$$t^* = \frac{y^{*e}}{\theta m S^{*2n}} \tag{19}$$

To find an expression related to the optimal products' price value and inventory independently from each other, we can substitute the (19) equation in (17) and (18) expressions. (20) and (21) expressions, show optimal price and inventory for manufacturing firm respectively.

$$y^* = \frac{ec_p}{e-1} \tag{20}$$

$$S^* = \left[\frac{c_h}{(2n+1)m\left[\left(\frac{ec_p}{e-1}\right)^{1-e} - c_p\left(\frac{ec_p}{e-1}\right)^{-e}\right]}\right]^{\frac{1}{2n}} \tag{21}$$

The optimal value of products' lead time for the firm, can be obtained independent of other variables, by substituting equations (21) and (20) in equation (19). Optimal expressions obtained for each decision variables indicate that only the parameters related to Production cost for each unit of product in single time and price elasticity are effective on the optimal value of products' price variable. In addition, we find that the firm's inventory has been correlated inversely with market potential parameter and is related directly with production cost. Thus, the increase in market potential, increases demand and consequently reduces the firms' inventory.

4-2- Two firms' scenario

In this section, for analysis of firms' behavior, approaches of game theory have been used. In fact, the models proposed in this scenario are a generalization of the previous scenario model. In this scenario, two substitute products manufacturers will compete to meet the customers' demands. Here, customer demand depends on the amount of inventory-on-hand and the selling price of products for each firm too, and each firm provides in turn a part of the market demand. The Game theory has been used in order to analyze mathematical models posed in competitive conditions and the decision makers logical, to realize the main objective of the problem which is to maximize corporate profits. Games Theory, with the aim of achieving an optimal solution for business examines the firms' behavior in a game in which the decisions of a firm about setting parameters and setting variables are dependent on the choice of other companies. In this episode, the Nash approach is used for modeling the firms' performance.

4-2-1- Nash approach

In this section, Nash equilibrium is used to investigate two manufacturing companies in order to achieve best output of the game and to determine most optimal conditions for each manufacturing firm.

In Nash approach, each firm is trying to maximize its profit function. In this approach, the two companies are not aware of each other and with no knowledge of each other, participate in a non-cooperative game; therefore, according to equation (15), for each of these firms, there is a separate profit function. Z_i is the profit function of ith firm, if $i=1,2$. For investigation of variables conditions in each firm, the profit function of ith firm is derived separately with respect to y_i, t_i and S_i variables. Thus, to find each decision variables optimally, other variables are assumed constant and interested variable conditions are examined. For firm i, if $i=1,2$ we have:

$$\frac{\partial Z_i}{\partial t_i}=-c_b\theta m S_i(S_i+S_{3-i})^{2n}y_i^{-e_i}y_{3-i}^{g_{3-i}}+c_b t_i^{-\frac{1}{2}}\sqrt{\theta m}\,y_i^{-\frac{e_i}{2}}y_{3-i}^{\frac{e_{3-i}}{2}}S_i(S_i+S_{3-i})^n \tag{22}$$

$$\begin{aligned}\frac{\partial Z_i}{\partial y_i}=\;&mS_i(S_i+S_{3-i})^{2n}y_{3-i}^{e_{3-i}}(1-e_i)y_i^{-e_i}+\left(-c_{pi}-c_b\theta t_i\right)mS_i(S_i+S_{3-i})^{2n}\left(-e_i\right)y_{3-i}^{e_{3-i}}y_i^{-e_i-1}\\&-e_i c_b\sqrt{t_i\theta m}S_i(S_i+S_{3-i})^n y_{3-i}^{\frac{e_{3-i}}{2}}y_i^{-\frac{e_i}{2}-1}\end{aligned} \tag{23}$$

$$\begin{aligned}\frac{\partial Z_i}{\partial S_i}=\;&(y_i-c_{pi}-c_b\theta t_i)my_i^{-e_i}y_{3-i}{}^{e_{3-i}}\left[\left(S_i+S_{3-i}\right)^{2n}+2n\left(S_i+S_{3-i}\right)^{2n-1}S_i\right]\\&+2c_b\sqrt{t_i\theta m y_i^{-e_i}y_{3-i}{}^{e_{3-i}}}\left[\left(S_i+S_{3-i}\right)^n+n\left(S_i+S_{3-i}\right)^{n-1}S_i\right]-(c_b+c_h)\end{aligned} \tag{24}$$

As mentioned in the previous sections, each of these firms are substitute products producers and seek to supply their inventory to meet the customers demand. Hence, we can consider the common goal of both companies to respond to required market demand; it means that If we consider the first and second companies inventory as S_1 and S_2, Inventory required to meet the total market demand is shown with S and is equal to the sum of the firms' inventory ($S=S_1+S_2$). To clarify the problem, the share of each of the firms in supplying this inventory is defined λ_1 and λ_2 coefficients and we know that $\lambda_1+\lambda_2=1$. Now we can rewrite the problem profit function - that is, equation (15) - by introducing $S\lambda_1$ and $S\lambda_2$ instead of S_1 and S_2 as equation (25).

$$\begin{aligned}\max Z_i=\;&(y_i-c_{pi}-\theta c_b t_i)m\lambda_i S^{2n+1}y_i^{-e_i}y_{3-i}^{e_{3-i}}\\&+2c_b\lambda_i S^{n+1}\sqrt{\theta m t_i}\,y_i^{-\frac{e_i}{2}}y_{3-i}^{\frac{e_{3-i}}{2}}-(c_h+c_b)S\lambda_i\end{aligned} \tag{25}$$

Finally, by taking the derivative of the profit function shown in the expression (25) and incorporating λ_i In expressions (22) to (24), optimal values for both firms, can be achieved.

By putting the derivative of the problem profit function with respect to the lead time variable equal to zero, according to equation (22) and simplifying it, Expression (26) is obtained which shows effect of products' price and inventory-on-hand for two firms, on the lead time variable.

$$t_i^{N*} = \frac{(y_i^{N*})^{e_i} (y_{3-i}^{N*})^{-e_{3-i}}}{\theta m \left(S^{N*}\right)^{2n}} \quad , \quad i = 1,2 \tag{26}$$

The optimum values for products' price and inventory, independently of each other by putting the equation (26) in equations (23) and (24) and simplifying these terms will be equal to:

$$y_i^{N*} = \frac{c_{pi} e_i}{e_i - 1} \quad , \qquad i = 1,2 \tag{27}$$

$$\mathrm{S}^{N*} = \left[\frac{c_h}{\left(\frac{c_{pi} e_i}{e_i - 1} - c_{pi} \right) m \left(\frac{c_{pi} e_i}{e_i - 1} \right)^{-e_i} \left(\frac{c_{p3-i} e_{3-i}}{e_{3-i} - 1} \right)^{e_{3-i}} (1 + 2n)} \right]^{\frac{1}{2n}} \tag{28}$$

The optimal value of lead time variable of i th firm is obtained by putting equations (27) and (28) in equation (26).

5-Computational and practical results

In this section of research, using numerical examples, we address to analyze the proposed models. Table (2) contains examples for a single firm model. Tables (3) and (4) show examples of Nash approach.

5-1- Single firm scenario

In this section, by expressing numerical examples for single firm model, the optimal values of decision variables are shown.

Example 1: Values of parameters applied in this example and results are shown in table (2).

Table 2. Parameters and values y^*, S^* and t^* obtained from them for model of an individual firm

m	n	c_p	c_b	c_h	e	θ	y^*	S^*	t^*	Z
20000	-0.25	0.8	0	0.5	8	0.1	0.9143	21915000	1.1429	10957000
20000	-0.25	0.8	0	1	8	0.1	0.9143	5478600	0.5714	5478600
20000	**-0.25**	**0.6**	**0**	**0.7**	**11**	**1**	**0.6600**	**6857100000**	**0.0429**	**4800000000**
20000	-0.3	0.7	0	0.6	8.5	1	0.7933	3825300	0.0622	3442800
20000	-0.3	0.6	0	0.7	8	1	0.6857	14785000	0.0490	15525000
20000	-0.3	0.6	0	0.7	8.5	0.9	0.6800	20320000	0.0508	21336000
20000	-0.35	0.5	0	0.4	9	0.1	0.5625	28733000	0.4688	26818000

5-2- Two firms' scenario

In expressed example of this section, the optimal values for inventory, products' price and products' lead time variables and amount of profit for each firm as well as total profit of two firms are shown in Nash approach. In the numerical examples of this research, $\lambda_1 = 0.4$ is considered.

Example 2: If parameters values are considered as $m = 20000$, $n = -0.3$, $c_{p1} = 0.6$, $c_{p2} = 0.5$, $c_b = 0$, $c_h = 0.7$, $e_1 = 12$, $e_2 = 9$, $\theta = 1$, the decision variables values of two firms are expressed by following Nash approach as table (3).

Table 3. Values t_i^{N*}, S_i^{N*} and y_i^{N*}" under Nash approach for $i = 1,2$ ($\lambda_2 = 1 - \lambda_1$)

λ_1	t_1^{N*}	t_2^{N*}	y_1^{N*}	y_2^{N*}	S_1^{N*}	S_2^{N*}	Z_1	Z_2	Z_1+Z_2
0.4	0.0312	0.0259	0.6545	0.5625	15596	23394	16376	40068	56444

Example 3: If parameters values are considered as $m = 20000$, $n = -0.25$, $c_{p1} = 0.6$, $c_{p2} = 0.5$, $c_b = 0$, $c_h = 0.7$, $e_1 = 11$, $e_2 = 8$, $\theta = 1$, the decision variables values of two firms are expressed by following Nash approach as table (4).

Table 4. Values t_i^{N*}, S_i^{N*} and y_i^{N*} under Nash approach for $i = 1,2$ ($\lambda_2 = 1 - \lambda_1$)

λ_1	t_1^{N*}	t_2^{N*}	y_1^{N*}	y_2^{N*}	S_1^{N*}	S_2^{N*}	Z_1	Z_2	Z_1+Z_2
0.4	0.0429	0.0517	0.66	0.5714	354480	531720	**248136**	**362490**	**610626**

In Nash approach numerical example, from two raised examples, we considered example which led to more profit as the optimal decision which is shown bold in Table (4) and in the sensitivity analysis section, we will address to its analysis more.

6- Sensitivity analysis

In this section, we address to analyze parameter affecting on model and effect of each parameter in the optimal values of decision variables and the problem profit value are shown. The values which were set as the optimal values of decision variables in the numerical example and 0.25, 0.5 and 0.75, more and less than these values are selected for the sensitivity analysis.

The result of this study is shown in table (5). Also figures related to effect of each parameter influencing on the model, for single firm scenario and two firms' scenario are depicted in figure (1) to figure (6) charts. Finally, figure (7) and figure (8), show total profit changes of two-firms against m and n parameters in Nash approach and the comparison between mentioned scenarios respectively. According to the results of the sensitivity analysis shown in the table (5), we found that as is obvious in the column related to two firms total profit $(\sum Z_i^N)$, the resulted profit of the Nash approach is less than the total profit of single firm in column (Z) and it shows that when one firm makes decisions based on its optimal values about its main variables, total profit is more than when two firms are active to supply the market demand without information from each other (figure 8). For a better analysis of optimal values of decision variables, parameters values e_1, e_2 that are price elasticity are considered as fixed. As shown in the expressions (27), the basic price variable in this model is only dependent on the parameters e_i, c_{pi} , so in the cited examples, its remains unchanged. Furthermore, by examining table (5) more, we can find that in “two firms’ scenario” (by considering $\lambda_1=0.4$), the amount of inventory-on-hand for the first firm is always less than the second one. This leads to reduce costs related to warehousing and the firms incur fewer costs for warehousing. Also, according to mentioned Table, the products’ lead time in all raised modes for firm1 is less.

Table7. Changes on parameters *m* and *n,* and their impacts on decision variables in both scenarios

%parameters changes		**First scenario: Single firm**				**Second scenario: two firm**						
		Single firm				**Nash Approach) $\lambda_1 = 0.4$ (**						
		y^*	S^*	t^*	Z	y_1^{N*}	y_2^{N*}	S_1^{N*}	S_2^{N*}	t_1^{N*}	t_2^{N*}	$\sum Z_i^N$
m	+75	0.66	2.1×10^{10}	0.0429	1.47×10^{10}	0.66	0.5714	1085600	1628400	0.0429	0.0517	1870000
	+50	0.66	1.54×10^{10}	0.0429	1.08×10^{10}	0.66	0.5714	797580	1196400	0.0429	0.0517	1373900
	+25	0.66	1.07×10^{10}	0.0429	7.5×10^{9}	0.66	0.5714	553870	830810	0.0429	0.0517	954100
	-25	0.66	3.86×10^{9}	0.0429	2.7×10^{9}	0.66	0.5714	199390	299090	0.0429	0.0517	343477
	-50	0.66	1.71×10^{9}	0.0429	1.2×10^{9}	0.66	0.5714	88620	132930	0.0429	0.0517	152656
	-75	0.66	4.29×10^{8}	0.0429	3×10^{8}	0.66	0.5714	22155	33232	0.0429	0.0517	38164
n	+75	0.66	1.94×10^{41}	0.0750	1.94×10^{40}	0.66	0.5714	2.17×10^{25}	3.25×10^{25}	0.0750	0.0905	5.08×10^{24}
	+50	0.66	2.38×10^{20}	0.0643	5.55×10^{19}	0.66	0.5714	1.59×10^{12}	2.39×10^{12}	0.0643	0.0775	8.98×10^{11}
	+25	0.66	2.36×10^{13}	0.0536	9.92×10^{12}	0.66	0.5714	61735000	92602000	0.0536	0.0646	63468000
	-25	0.66	4.67×10^{7}	0.0321	5.44×10^{7}	0.66	0.5714	14460	21691	0.0321	0.0388	41648
	-50	0.66	1.43×10^{6}	0.0214	3×10^{6}	0.66	0.5714	1464.6	2196.8	0.0214	0.0258	7608.6
	-75	0.66	8.56×10^{4}	0.0107	4.2×10^{5}	0.66	0.5714	205.3906	308.0859	0.0107	0.0129	2493.5

In this section, figures related to the sensitivity analysis are depicted which help to clear the issue. Figure (1) to figure (3) show variables changes of “single firm” scenario. These charts depict effect of the market potential parameters and demand inventory elasticity in the main problem variables. As obvious in the charts, right plot of the mentioned figures show the effect of n parameter in decision variables which by increasing the inventory elasticity amount, in being fixed of other parameters, the lead time is increasing with fixed slope, the price remains fixed and the inventory changes is so that at first, it increases with very gentle slope and from $n = -0.12$, this increase continues with more slope. Left plot of mentioned figures show the effect of m parameter in any variables of model decision. By increasing the market potential, the firm inventory increases with incremental slope and this parameter does not have any effect on price and lead time variables.

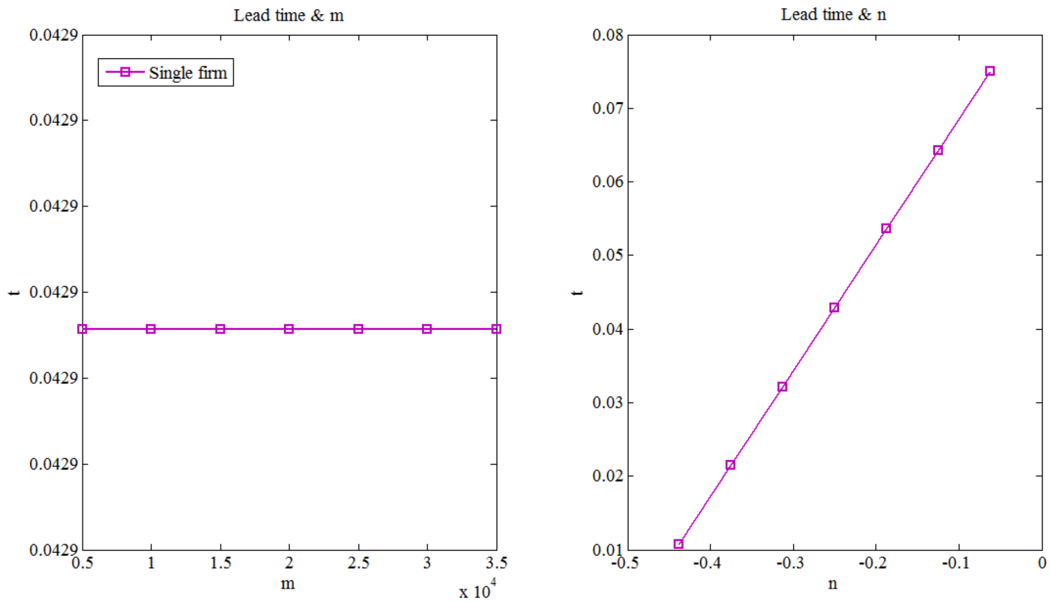


Fig. 1. Effects of m and n changes on the lead-time in single firm's scenario

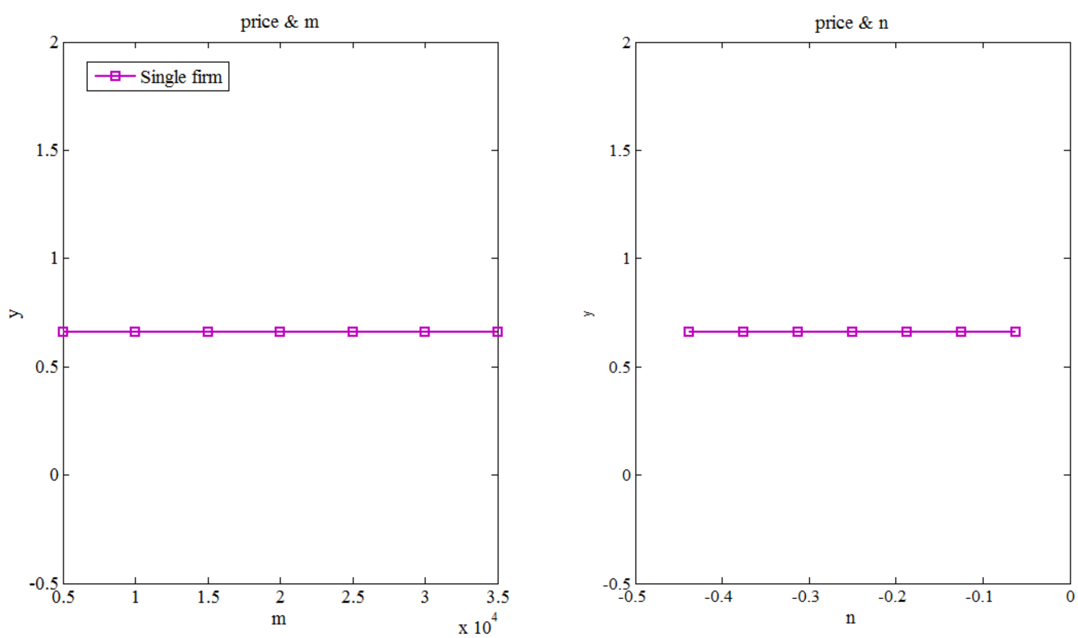


Fig. 2. Effects of m and n changes on the product's price in single firm's scenario

In figure (4) to figure (6), m and n parameters changes on the model decision variables in two-firms' scenario are shown. These charts illustrate the behavior of two firms in Nash approach. Changes related to any firms are shown with different colors. Figure (7) shows total profit in "two-firm scenario" related to Nash approach. In this figures, it is obvious that total profit of two firms' increases by increasing m parameter value with incremental slope and increasing in n variable value is increased with gentle slope at first and then is done by steep slope. Figure (8) shows the comparison between the profits of "single firm's scenario" and "two firms' scenario" in the presence of market potential changes.

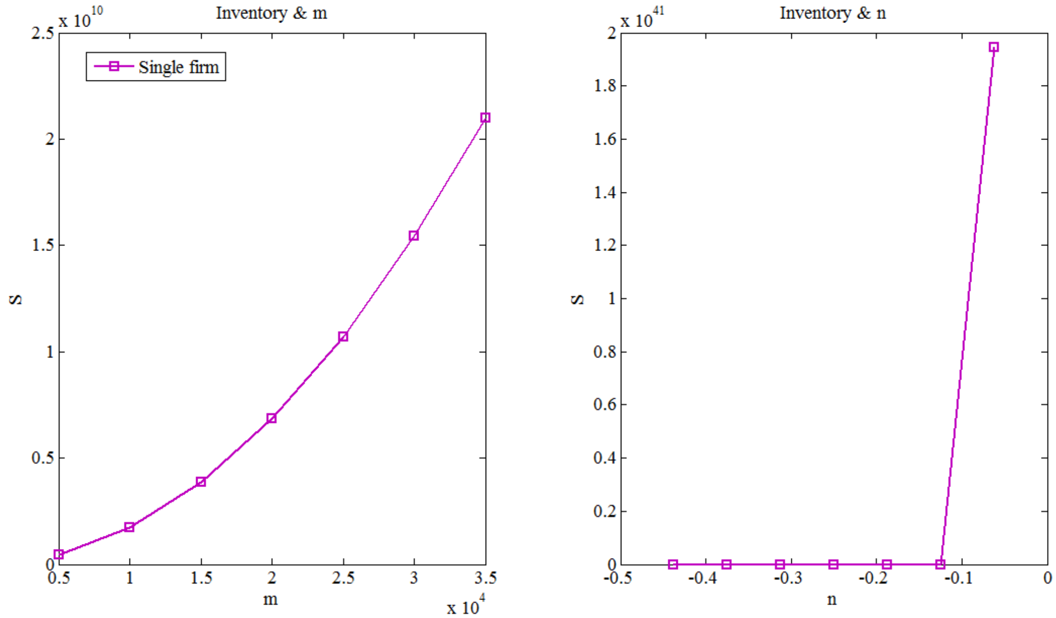


Fig. 3. Effects of m and n changes on the inventory in single firm's scenario

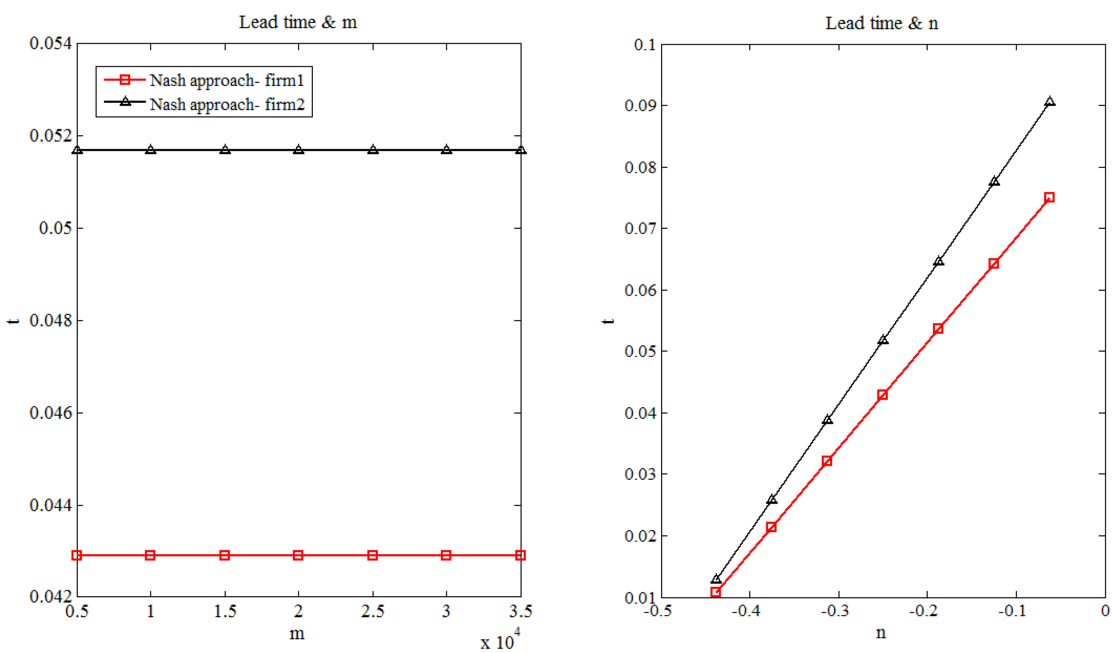


Fig. 4. Effects of m and n changes on lead-time under Nash approach

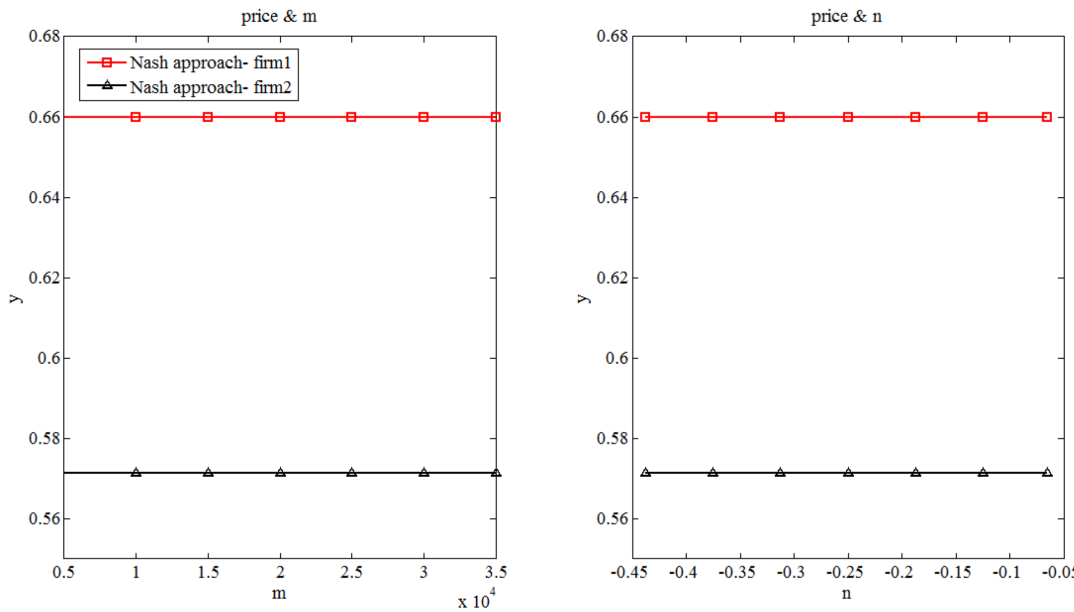


Fig. 5. Effects of m and n changes on product's price under Nash approach

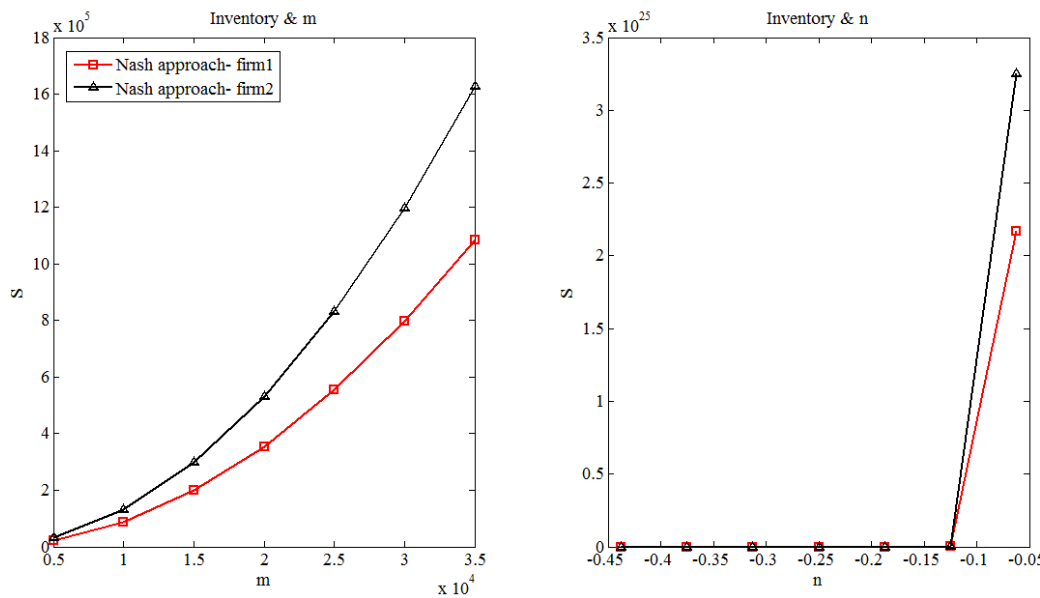
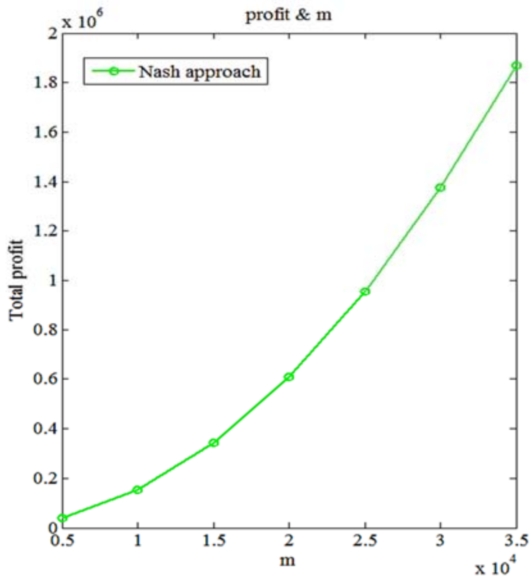


Fig. 6. Effects of m and n changes on inventory under Nash approach

In this chart, it is obvious that by increasing the market potential, the total profit of both scenarios increases with the unstable slopes (e.g. When the market potential is 2×10^4, the profit of "single firm's scenario" and "two firms' scenario" are 4.8×10^9 and 6.106×10^5 respectively. By increasing the market potential to 3×10^4, it can be seen that the profits increases to 1.08×10^{10} and 1.374×10^6 respectively.). According to the results, the increasing profit is greater in "single firm's scenario". As mentioned in previous sections, firms generate substitute products in "two firms' scenario". This matter, affected on demand function through price elasticity parameters and causes the less profit for this scenario compared with the "single firm's scenario".



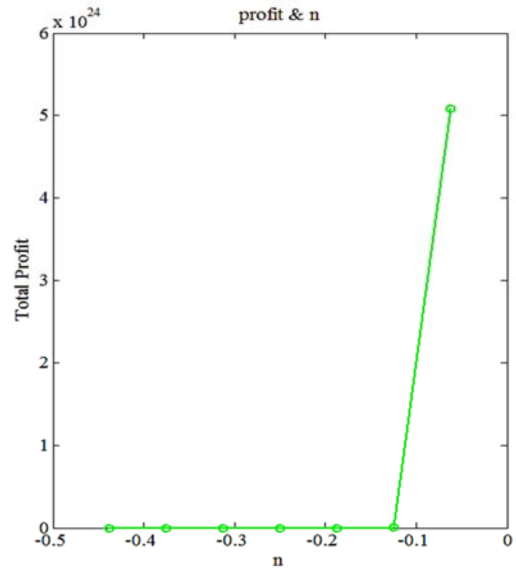


Fig. 7. Effects of m and n changes on profit under Nash approach for two firms' scenario

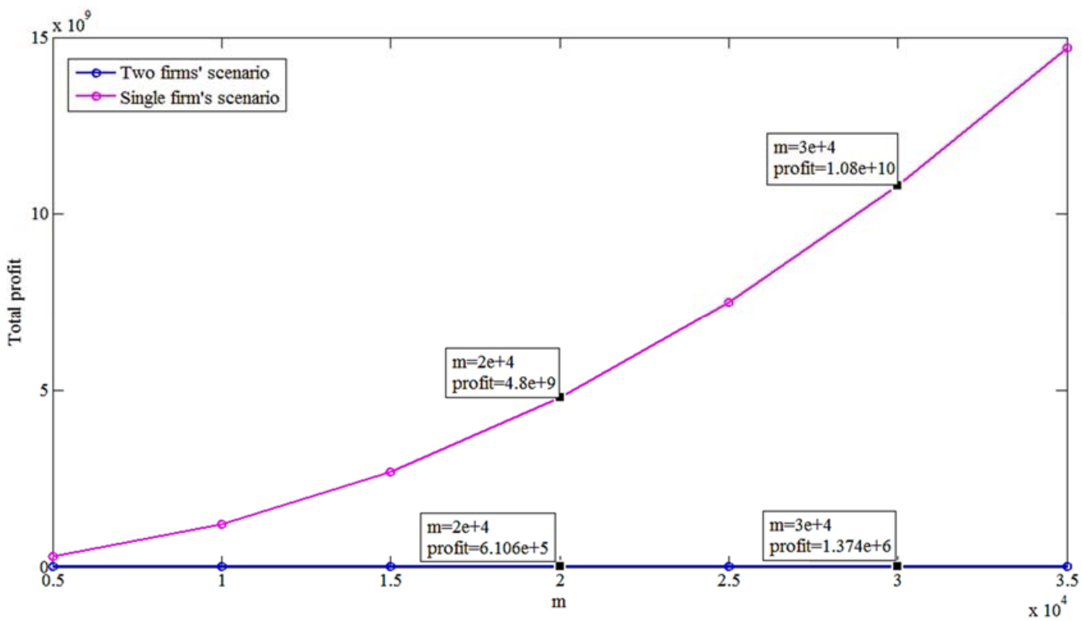


Fig. 8. Comparison between the profits of "single firm's scenario" and "two firms' scenario" in the presence of market potential changes

7- Conclusion

In this study, the firms' decisions about inventory amount and production were investigated. The market demand is assumed sensitive to Price and inventory on hand investigated. This study contains the pricing, inventory and delivery decisions for manufacturing firms. In two different scenarios, the analysis of firms' behavior is investigated. In "single firm scenario", we investigated the impacts of main problem variables, i.e. price, lead time and inventory, on the behavior of a firm," two firms Scenario" analyzes the behavior of manufacturing firms in the non-inclusive market. It should be noted that firms considered in the "two firms Scenario", are substitute products producers and by using Nash equilibrium, we reviewed their performance. In this paper, we found that to obtain most profit, adjusting lead time, inventory-on-hand and products' price is necessary. Among influencing parameters, we can refer to the market potential parameters, inventory elasticity, price elasticity and parameters related to production cost and maintenance costs which in numerical example and sensitivity analysis section, their effect on the firms' profitability level are addressed. Finally, it was found that by adjusting the market potential and demand inventory elasticity parameters that are more effective compared to other parameters, we can achieve maximum profit and lead time management. In addition, in the numerical examples also between two mentioned scenarios discussed in this paper, the "single firm scenario" leads to greater profitability. This research can be generalized in the various aspects. In this model, we can consider the demand function for the manufacturing firms as probable. In addition, in the model discussed in this paper, in addition to the price and inventory variables, demand will also be affected by other factors such as quality. We can also extend this study for supply chains, with several suppliers, under more scenarios.

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Appendices

Appendix A: Proving the concavity of the firm profit function under "single firm" scenario

According to the model presented in above study, it is necessary to prove that the firm's profit function is concave. For this purpose, it's sufficient to form its Hessian matrix and its minors to be addressed. If the function is concave, the first minor is negative and other minors are positive and negative as decussate. To show this, at first the Hessian matrix for this function is formed for (A-1).

$$H=\begin{bmatrix} \frac{\partial^2 Z}{\partial y^2} & \frac{\partial^2 Z}{\partial y \partial t} & \frac{\partial^2 Z}{\partial y \partial S} \\ \frac{\partial^2 Z}{\partial t \partial y} & \frac{\partial^2 Z}{\partial t^2} & \frac{\partial^2 Z}{\partial t \partial S} \\ \frac{\partial^2 Z}{\partial S \partial y} & \frac{\partial^2 Z}{\partial S \partial t} & \frac{\partial^2 Z}{\partial S^2} \end{bmatrix} < 0 \tag{A-1}$$

Now we can obtain above Matrix elements as below by the deriving of the interested function:

$$\frac{\partial^2 Z}{\partial y \partial t}=\frac{\partial^2 Z}{\partial t \partial y}=\theta e c_b m S^{2n+1} y^{-e-1}-\frac{1}{2} t^{\frac{-1}{2}} e c_b S^{n+1} \sqrt{\theta m}\, y^{-\frac{e}{2}-1} \tag{A-2}$$

$$\frac{\partial^2 Z}{\partial y^2}=-e(1-e) m S^{2n+1} y^{-e-1}+e(-e-1) c_p m S^{2n+1} y^{-e-2}+$$
$$\theta e(-e-1) c_b t m S^{2n+1} y^{-e-2}-e(-\frac{e}{2}-1) c_b S^{n+1} \sqrt{\theta t m}\, y^{(-\frac{e}{2}-2)} \tag{A-3}$$

$$\frac{\partial^2 Z}{\partial y \partial S}=\frac{\partial^2 Z}{\partial S \partial y}$$
$$=(2n+1) m S^{2n}\left[(1-e) y^{-e}+e c_p y^{-e-1}+e c_b t y^{-e-1} \theta\right]-e(n+1) c_b S^n \sqrt{\theta t m}\, y^{-\frac{e}{2}-1} \tag{A-4}$$

$$\frac{\partial^2 Z}{\partial t^2}=-\frac{1}{2} c_b S^{n+1} t^{-\frac{3}{2}} \sqrt{\theta m y^{-e}} \tag{A-5}$$

$$\frac{\partial^2 Z}{\partial t \partial S}=\frac{\partial^2 Z}{\partial S \partial t}$$
$$=-\theta c_b m(2n+1) S^{2n} y^{-e}+c_b(n+1) S^n t^{-\frac{1}{2}} \sqrt{\theta m y^{-e}} \tag{A-6}$$

$$\frac{\partial^2 Z}{\partial S^2}=2n(2n+1) m S^{2n-1} y^{1-e}-2n(2n+1) c_p m S^{2n-1} y^{-e}-2n(2n+1) \theta c_b t m S^{2n-1} y^{-e}+$$
$$2n(n+1) c_b S^{n-1} \sqrt{\theta t m y^{-e}} \tag{A-7}$$

Finally, the minors signs has been addressed, we consider The first minor that has been shown with $|H|_1$, as the term (A-8) and with negativity condition of this expression, we discuss the next function minors. Then with positivity condition of second minor which is illustrated by (A-9) and by assuming negativity of third minor which is shown in the (A-10) and the determinant of Hessian matrix (A-1), the desired profit function strictly is considered concave.

$$\begin{aligned}
\left|H\right|_1 &= \frac{\partial^2 Z}{\partial y^2} < 0 \\
&= -e(1-e)mS^{2n+1}y^{-e-1} - e(e+1)c_p mS^{2n+1}y^{-e-2} - \theta e(e+1)c_b tmS^{2n+1}y^{-e-2} \\
&\quad + e(\frac{e}{2}+1)c_b S^{n+1}\sqrt{\theta tm}\, y^{(-\frac{e}{2}-2)} < 0
\end{aligned} \tag{A-8}$$

$$\begin{aligned}
\left|H\right|_2 &= \frac{\partial^2 Z}{\partial y^2}\times\frac{\partial^2 Z}{\partial t^2} - \frac{\partial^2 Z}{\partial y\partial t}\times\frac{\partial^2 Z}{\partial t\partial y} \geq 0 \\
&= -\begin{bmatrix} e(1-e)mS^{2n+1}y^{-e-1} + e(e+1)c_p mS^{2n+1}y^{-e-2} \\ +\theta e(e+1)c_b tmS^{2n+1}y^{-e-2} - e(\frac{e}{2}+1)c_b S^{n+1}\sqrt{\theta tm}\, y^{(-\frac{e}{2}-2)} \end{bmatrix}\times \\
&\quad \left[-\frac{1}{2}c_b S^{n+1}t^{-\frac{3}{2}}\sqrt{\theta m y^{-e}}\right] - \left[\theta e c_b mS^{2n+1}y^{-e-1} - \frac{1}{2}t^{\frac{-1}{2}} e c_b S^{n+1}\sqrt{\theta m}\, y^{-\frac{e}{2}-1}\right]^2 > 0
\end{aligned} \tag{A-9}$$

$$\begin{aligned}
\left|H\right|_3 &= \frac{\partial^2 Z}{\partial y^2}\frac{\partial^2 Z}{\partial t^2}\frac{\partial^2 Z}{\partial S^2} + 2\frac{\partial^2 Z}{\partial t\partial S}\frac{\partial^2 Z}{\partial y\partial S}\frac{\partial^2 Z}{\partial y\partial t} - \\
&\quad \left[\frac{\partial^2 Z}{\partial y^2}\left(\frac{\partial^2 Z}{\partial t\partial S}\right)^2 + \left(\frac{\partial^2 Z}{\partial y\partial t}\right)^2\frac{\partial^2 Z}{\partial S^2} + \left(\frac{\partial^2 Z}{\partial y\partial S}\right)^2\frac{\partial^2 Z}{\partial t^2}\right] < 0
\end{aligned} \tag{A-10}$$

Appendix B: Proving the concavity of the i firm profit function in Nash Approach under "two firms" scenario

According to the solutions presented in above study, it is necessary to prove that the firm's profit function is concave under the Nash Approach. For this purpose, its Hessian matrix is formed according to (B-1) Matrix and its minors to be addressed. As defined before, $S_1 = S\lambda_1$ and $S_2 = S\lambda_2$. By considering S_1 and S_2 in the functions, the profit function of each firm is concave when first minor is negative and other minors are positive and negative as decussate. So, for the firm i ($i = 1,2$), we have:

$$H = \begin{bmatrix} \frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i^2} & \frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i\partial t_i} & \frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i\partial S_i} \\ \frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i\partial y_i} & \frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i^2} & \frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i\partial S_i} \\ \frac{\partial^2 Z_{\mathrm{i}}}{\partial S_i\partial y_i} & \frac{\partial^2 Z_{\mathrm{i}}}{\partial S_i\partial t_i} & \frac{\partial^2 Z_{\mathrm{i}}}{\partial S_i^2} \end{bmatrix} < 0 \tag{B-1}$$

Now we calculate Hessian matrix elements. For this purpose, it is sufficient to derive the profit function of each firm with respect to decision variables. For firm $i = 1,2$, we do the following:

$$\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i \partial t_i} = \frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i \partial y_i} = e_i y_i^{-e_i-1} y_{3-i}^{e_{3-i}} c_b \theta m S_i (S_i + S_{3-i})^{2n} + c_b t_i^{-\frac{1}{2}} \sqrt{\theta m} (-\frac{e_i}{2}) y_i^{-\frac{e_i}{2}-1} y_{3-i}^{\frac{e_{3-i}}{2}} S_i (S_i + S_{3-i})^n \tag{B-2}$$

$$\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i^2} = m S_i (S_i + S_{3-i})^{2n} \begin{bmatrix} -e_i y_i^{-e_i-1} y_{3-i}{}^{e_{3-i}} (1-e_i) \\ +(-c_{pi} - c_b \theta t_i)(-e_i) y_{3-i}^{e_{3-i}} (-1-e_i) y_i^{-e_i-2} \end{bmatrix} + e_i c_b \sqrt{t_i \theta m} S_i (S_i + S_{3-i})^n (\frac{e_i}{2}+1) y_i^{-\frac{e_i}{2}-2} y_{3-i}^{\frac{e_{3-i}}{2}} \tag{B-3}$$

$$\begin{aligned} \frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i \partial S_i} &= \frac{\partial^2 Z_{\mathrm{i}}}{\partial S_i \partial y_i} \\ &= \left[y_{3-i}^{e_{3-i}} (1-e_i) y_i^{-e_i} + (-c_{pi} - c_b \theta t_i)(-e_i) y_i^{-e_i-1} y_{3-i}{}^{e_{3-i}} \right] m \\ &\quad \times \left[(S_i + S_{3-i})^{2n} + 2n (S_i + S_{3-i})^{2n-1} S_i \right] \\ &\quad - e_i c_b \sqrt{t_i \theta m} y_i^{-\frac{e_i}{2}-1} y_{3-i}{}^{\frac{e_{3-i}}{2}} \left[(S_i + S_{3-i})^n + n (S_i + S_{3-i})^{n-1} S_i \right] \end{aligned} \tag{B-4}$$

$$\frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i^2} = -\frac{1}{2} c_b t_i^{-\frac{3}{2}} \sqrt{\theta m y_i^{-e_i} y_{3-i}{}^{e_{3-i}}} S_i (S_i + S_{3-i})^n \tag{B-5}$$

$$\begin{aligned} \frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i \partial S_i} &= \frac{\partial^2 Z_{\mathrm{i}}}{\partial S_i \partial t_i} \\ &= -c_b \theta m y_i^{-e_i} y_{3-i}^{e_{3-i}} \left[(S_i + S_{3-i})^{2n} + 2n (S_i + S_{3-i})^{2n-1} S_i \right] \\ &\quad + c_b t_i^{-\frac{1}{2}} \sqrt{\theta m} y_i^{-\frac{e_i}{2}} y_{3-i}^{\frac{e_{3-i}}{2}} \left[(S_i + S_{3-i})^n + n (S_i + S_{3-i})^{n-1} S_i \right] \end{aligned} \tag{B-6}$$

$$\begin{aligned} \frac{\partial^2 Z_{\mathrm{i}}}{\partial S_i{}^2} &= (y_i - c_{pi} - c_b \theta t_i) m y_i^{-e_i} y_{3-i}{}^{e_{3-i}} \left[4n (S_i + S_{3-i})^{2n-1} + 2n(2n-1)(S_i + S_{3-i})^{2n-2} S_i \right] + \\ &\quad 2 c_b \sqrt{t_i \theta m y_i^{-e_i} y_{3-i}{}^{e_{3-i}}} \left[2n (S_i + S_{3-i})^{n-1} + n(n-1)(S_i + S_{3-i})^{n-2} S_i \right] \end{aligned} \tag{B-7}$$

Finally, in accordance with the functions concavity conditions, we analyze the minors sign. As shown in equation (B-8), we apply the less or equal to zero condition on the first matrix minor. For the second minor which (B-9) indicates that, greater than zero condition is considered and the determinant of Hessian matrix of above function _which is third minor of function_ is calculated according to the expression (B-10) negativity condition is applied on it. With these conditions, the mentioned function will be strictly concave.

$$\left|H\right|_1=\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i^{\ 2}}<0$$

$$=mS_i(S_i+S_{3-i})^{2n}\left[-e_i y_i^{-e_i-1}y_{3-i}^{\ \ e_{3-i}}(1-e_i)+(-c_{pi}-c_b\theta t_i)(-e_i)y_{3-i}^{e_{3-i}}(-1-e_i)y_i^{-e_i-2}\right] \tag{B-8}$$

$$+e_ic_b\sqrt{t_i\theta m}S_i(S_i+S_{3-i})^n(\frac{e_i}{2}+1)y_i^{-\frac{e_i}{2}-2}y_{3-i}^{\frac{e_{3-i}}{2}}<0$$

$$\left|H\right|_2=\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i^{\ 2}}\times\frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i^{\ 2}}-\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i\partial t_i}\times\frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i\partial y_i}>0$$

$$=\begin{bmatrix} mS_i(S_i+S_{3-i})^{2n}\left[-e_i y_i^{-e_i-1}y_{3-i}^{\ \ e_{3-i}}(1-e_i)+(-c_{pi}-c_b\theta t_i)(-e_i)y_{3-i}^{e_{3-i}}(-1-e_i)y_i^{-e_i-2}\right] \\ +e_ic_b\sqrt{t_i\theta m}S_i(S_i+S_{3-i})^n(\frac{e_i}{2}+1)y_i^{-\frac{e_i}{2}-2}y_{3-i}^{\frac{e_{3-i}}{2}} \end{bmatrix}$$

$$\times\left[-\frac{1}{2}c_bt_i^{-\frac{3}{2}}\sqrt{\theta m y_i^{-e_i}y_{3-i}^{\ \ -e_{3-i}}}S_i(S_i+S_{3-i})^n\right] \tag{B-9}$$

$$-\begin{bmatrix} e_i y_i^{-e_i-1}y_{3-i}^{e_{3-i}}c_b\theta mS_i(S_i+S_{3-i})^{2n} \\ +c_bt_i^{-\frac{1}{2}}\sqrt{\theta m}(-\frac{e_i}{2})y_i^{-\frac{e_i}{2}-1}y_{3-i}^{\frac{e_{3-i}}{2}}S_i(S_i+S_{3-i})^n \end{bmatrix}^2>0$$

$$\left|H\right|_3=\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i^2}\frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i^2}\frac{\partial^2 Z_{\mathrm{i}}}{\partial S_i^2}+2\frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i\partial S_i}\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i\partial S_i}\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i\partial t_i}-$$

$$\left[\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i^{\ 2}}\left(\frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i\partial I_i}\right)^2+\left(\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i\partial t_{\ i}}\right)^2\frac{\partial^2 Z_{\mathrm{i}}}{\partial S_i^2}+\left(\frac{\partial^2 Z_{\mathrm{i}}}{\partial y_i\partial S_i}\right)^2\frac{\partial^2 Z_{\mathrm{i}}}{\partial t_i^2}\right]<0 \tag{B-10}$$