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A Multi-Stage Single-Machine Replacement Strategy Using Stochastic Dynamic Programming

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ABSTRACT

In this paper, the single machine replacement problem is being modeled into the frameworks of stochastic dynamic programming and control threshold policy, where some properties of the optimal values of the control thresholds are derived. Using these properties and by minimizing a cost function, the optimal values of two control thresholds for the time between productions of two successive nonconforming products is determined. If this time exceeds the first threshold, the production continues. If it is less than the second one, inspection, repair, or replacement occur. However, if it falls within the control thresholds, then the process of sampling continues. At the end, the application of the proposed methodology is demonstrated using a numerical illustration.

***Keywords*:** Machine Replacement Policy; Control Threshold Policy; Exponential Distribution; Stochastic Dynamic Programming.

1.INTRODUCTION

Optimal management of machine replacement is essential at the national level as well as separate firms, where its importance increase due to major economic trends. In this research, we address a finite-horizon production and repair problem in which both the production and repair actions are included in the model. In other words, the machine produces defective items and the cost of each defective item and repair are known.

In most of production processes, the main reason of not stopping the machine for repair is the time required for repair, during which no production can be done (Singh *et al.* 2004). In this research, it is assumed that there is a single item that is produced on a single machine with a constant rate of producing defective items in each stage. States of the machine may be defined by characteristics of the output, where the time between producing defective items is assumed to follow an exponential distribution with the parameter derived based on the sample of defective items gathered from

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production. The machine state is assumed continuous and its probability density function is determined at the beginning of each stage. Moreover, it is assumed that the rate of producing the defective items increases with a constant factor in each stage. Therefore, the state of the machine at the beginning of the next period depends both on the number of defective items produced in the current stage and the time between productions of successive defective items. The rate of producing the defective items in the next stage is derived by multiplying the rate of producing defective item in the current stage by a constant factor. A state-independent fixed cost is also incurred for each repair and there is a production cost in each stage for each production decision. There is also a fixed cost for the machine at the end of the horizon reflecting the initial cost of the system. The objective is to minimize the expected total cost over the finite horizon.

Grosfeld-Nir (2007) presented a two-state partially observable Markov decision process for machine replacement problem. He proved that “dominance in expectation’’ (the expected profit is larger in a good state than in a bad state) suffices for the optimal policy to be of a control limit (CLT) type: continue if and only if the good state probability exceeds CLT. Iravani and Duenyas (2002) considered integrated decisions of maintenance and production in a production system with a deteriorating machine. They consider infinite horizon and assumed that the production and repair times follow exponential distribution.

Niaki and Fallahnezhad (2007) employed Bayesian inference and stochastic dynamic programming to design a decision-making framework in production environment. Moreover, Fallahnezhad *et al.* (2007) considered an application of Bayesian inference in machine replacement problem. They determined the optimal policy for two series machines by Bayesian inference in the context of the finite mixture model, discussed the analysis of time-to-failure data, and proposed an optimal decision-making procedure for machine replacement strategy. In another research, Fallahnezhad and Niaki (2010) proposed a dynamic programming model of the two-machine replacement problem. Furthermore, Fallahnezhad and Niaki (2011) developed an optimal threshold policy for machine replacement problem based on the number of produced defective items.

In this research, a dynamic programming model is presented for the machine replacement problem. In contrast to previous studies, the time between productions of nonconforming products is selected the decision making variable. A single machine is considered and it is assumed that the time between productions of nonconforming products (t) follows an exponential distribution with the hazard rate λ, i.e. $f(t|\lambda) = \lambda e^{-\lambda t}$. The variable t is the state variable of the dynamic programming model where its probability distribution function is determined by gathering a sample of the time between productions of successive defective items from the machine in each stage. Assuming t_i to denote the time between production of (i-1)th and ith defective item, the probability distribution function of t will be $f(t|\lambda) = \lambda e^{-\lambda t}$ where $\lambda = \frac{1}{\bar{t}}$.

The decision-making process is based on the control limit policy, where the concept of control threshold policy is employed. It means to perform inspection, repair, or replace if and only if the random variable t (the time between productions of nonconforming products) is less than a threshold. If and only if t exceeds another threshold, the production continues. Otherwise, taking more samples is the result of falling t within the control thresholds. In the model development, a cost function is minimized to determine the optimal policy. Since t is a random variable, to select the optimal decision in each stage, the value of $E(t)$ are compared with determined optimal control thresholds and the appropriate decision will be chosen.

The rest of the paper is organized as follows. The notations are given in Section 2. Mathematical formulation and solution comes in Section 3. Section 4 contains a numerical demonstration of the application of the proposed methodology. We conclude the paper in Section 5.

2. NOTATION

The following notations are used throughout this paper,

- n An index used for a decision-making stage, $n = 0, 1, 2, \ldots$, where $n = 0$ indicates the initial stage
- t_i The time between productions of (i-1)th and ith defective item in each stage
- t State variable of Dynamic programming model. Variable t denotes the time between productions of defective items by the machine that its probability distribution function follows an exponential distribution with rate λ
- λ The rate of producing defective item that its value is equal to $\frac{1}{\bar{t}}$ where $\bar{t}$ denotes the average of t_i values in each stage
- d_n Lower threshold for time between productions of nonconforming items in stage n
- $d_n^{'}$ Upper threshold for time between productions of nonconforming items in stage n
- A Coefficient of cost of inspection, repair, or replacement in each stage
- B Coefficient of operation cost of continuing production in each stage
- C Cost of gathering a sample of defective items in each stage
- $V(n)$ Total cost of the system where n decision making stages is remained
- B Degradation coefficient of the rate of producing defective items in each stage.

3. THE MODEL

As mentioned earlier, the proposed decision-making process is based on the time between productions of nonconforming products. In stage n of the decision-making process, if the time between production of nonconforming items (t) is less than d_n, the machine should be repaired or replaced. In this case, the cost of repairing or replacing is assumed a function of the expected value of the random variable t. If t is more than $d_n^{'}$, then the machine is assumed to be in acceptable condition. In this case, its operation cost is assumed a function of the rate of producing nonconforming items that is proportional to the inverse of the expected time between productions of nonconforming items. In other words, the conditional expected value of the cost when t is less than or equal to d_n is

$$\text{Cost of repairing/replacing the machine} = A\,E\left(t \mid t \le d_n\right) = A\left(\frac{\int_0^{d_n} t\,f(t)\,dt}{F(d_n)}\right) \tag{1}$$

where $F(d_n) = 1 - e^{-\lambda d_n}$ is the cumulative probability distribution function of $f(d_n)$.

Similarly, the conditional expected value of the cost when t is greater than or equal to $d_n^{'}$ is obtained as

$$\text{Operation cost of the machine} = \frac{B}{E\left(t \left| t \geq d_n^{'}\right.\right)} = B\left(\frac{\left(1-F\left(d_n^{'}\right)\right)}{\int_{d_n^{'}}^{\infty} t\, f\,(t)\,dt}\right) \tag{2}$$

where $F\left(d_n^{'}\right) = 1 - e^{-\lambda d_n^{'}}$ is the cumulative probability distribution function of $f\,\left(d_n^{'}\right)$.

In case when t falls in between d_n and $d_n^{'}$, more samples are taken and the decision-making continues to the next stage, where the cost becomes $P\left(d_n \leq t \leq d_n^{'}\right)\alpha\left(V\,(n-1)+C\right)$, in which α is the discount factor to consider the cost associated with the next stage (n-1) in the current stage (n). Thus, the total system cost at stage n of the decision-making process becomes

$$\begin{aligned} V\,(n) = {} & A\,E\left(t \left| t \leq d_n\right.\right) P\left(t \leq d_n\right) + \frac{B}{E\left(t \left| t_n \geq d_n^{'}\right.\right)} P\left(t \geq d_n^{'}\right) + \\ & P\left(d_n^{'} \geq t \geq d_n\right)\alpha\left(V\,(n-1)+C\right) \end{aligned} \tag{3}$$

Eq. (3) can be rewritten as

$$V\,(n) = A\int_0^{d_n} tf\,(t)\,dt + \frac{B\left(1-F\left(d_n^{'}\right)\right)^2}{\int_{d_n^{'}}^{\infty} tf\,(t)\,dt} + P\left(d_n \leq t \leq d_n^{'}\right)\alpha\left(V\,(n-1)+C\right) \tag{4}$$

Since the objective is to determine the thresholds so that the cost is minimized, by differentiating Eq. (4) with respect to d_n and $d_n^{'}$ and setting both zero, we get

$$\frac{\partial V\,(n)}{\partial d_n} = 0 \;\Rightarrow\; A\,d_n f\,\left(d_n\right) - \alpha\left(V\,(n-1)+C\right) f\,\left(d_n\right) = 0$$

$$\Rightarrow d_n = \frac{\alpha\left(V\,(n-1)+C\right)}{A} \tag{5}$$

and

$$\frac{\partial V\,(n)}{\partial d_n^{'}} = 0 \;\Rightarrow$$

$$\frac{-2f\left(d_n^{'}\right)\left(\int_{d_n^{'}}^{\infty} t\,f\,(t)\,dt\right)B\left(1-F\left(d_n^{'}\right)\right)+B\,d_n^{'}f\left(d_n^{'}\right)\left(1-F\left(d_n^{'}\right)\right)^2}{\left(\int_{d_n^{'}}^{\infty} t\,f\,(t)\,dt\right)^2}+f\left(d_n^{'}\right)\alpha\left(V\left(n-1\right)+C\right)=0$$

$$\Rightarrow f\left(d_n^{'}\right)\left(\frac{-B\left(2\left(\frac{\int_{d_n^{'}}^{\infty} t\,f\,(t)\,dt}{\left(1-F\left(d_n^{'}\right)\right)}\right)-d_n^{'}\right)}{\left(\frac{\int_{d_n^{'}}^{\infty} t\,f\,(t)\,dt}{\left(1-F\left(d_n^{'}\right)\right)}\right)^2}+\alpha\left(V\left(n-1\right)+C\right)\right)=0 \tag{6}$$

Since $E\left(t\,\middle|\,t\geq d_n^{'}\right)=\frac{\int_{d_n^{'}}^{\infty} t\,f\,(t)\,dt}{\left(1-F\left(d_n^{'}\right)\right)}$, Eq. (6) results in

$$\frac{B\left(2E\left(t\,\middle|\,t\geq d_n^{'}\right)-d_n^{'}\right)}{\left(E\left(t\,\middle|\,t\geq d_n^{'}\right)\right)^2}=\alpha\left(V\left(n-1\right)+C\right) \tag{7}$$

Hence

$$\left(E\left(t\,\middle|\,t\geq d_n^{'}\right)\right)^2\alpha\left(V\left(n-1\right)+C\right)-2B\,E\left(t\,\middle|\,t\geq d_n^{'}\right)+B\,d_n^{'}=0 \tag{8}$$

Eq. (8) provides the optimal value of $d_n^{'}$.

To ensure optimal values of d_n and $d_n^{'}$ minimize the cost function, it is necessary to prove that the second derivative is positive in d_n and $d_n^{'}$. Differentiating twice Eq. (4) with respect to d_n and $d_n^{'}$ results in the following equation

$$\frac{\partial^2 V\left(n\right)}{\partial^2 d_n}=A\,f\left(d_n\right)+A\,d_n f\,'\left(d_n\right)-\alpha\left(V\left(n-1\right)+C\right)f\,'\left(d_n\right) \tag{9}$$

Since $d_n=\frac{\alpha\left(V\left(n-1\right)+C\right)}{A}$, then $A\,d_n f\,'\left(d_n\right)-\alpha\left(V\left(n-1\right)+C\right)f\,'\left(d_n\right)=0$. Therefore,

$\frac{\partial^2 V(n)}{\partial^2 d_n} = A f(d_n) > 0$. Hence, it is concluded that the optimal value of d_n minimizes the objective function.

Similarly, assuming $h\left(d_n^{'}\right) = E\left(t \middle| t \geq d_n^{'}\right)$, the first derivative of the cost function with respect to $d_n^{'}$ can be rewritten as

$$\frac{\partial V(n)}{\partial d_n^{'}} = f\left(d_n^{'}\right)\left(\frac{-B\left(2h\left(d_n^{'}\right) - d_n^{'}\right)}{h\left(d_n^{'}\right)^2} + \alpha\left(V(n-1) + C\right)\right) \tag{10}$$

Defining $r\left(d_n^{'}\right)$ as

$$r\left(d_n^{'}\right) = \frac{-B\left(2h\left(d_n^{'}\right) - d_n^{'}\right)}{h\left(d_n^{'}\right)^2} + \alpha\left(V(n-1) + C\right) \tag{11}$$

The second derivative of the cost function with respect to $d_n^{'}$ becomes

$$\frac{\partial^2 V(n)}{\partial^2 d_n^{'}} = f\left(d_n^{'}\right) r'\left(d_n^{'}\right) + f'\left(d_n^{'}\right) r\left(d_n^{'}\right) \tag{12}$$

Since $d_n^{'}$ is the optimal solution therefore $r\left(d_n^{'}\right) = 0$, it is concluded that

$$\frac{\partial^2 V(n)}{\partial^2 d_n^{'}} = f\left(d_n^{'}\right) r'\left(d_n^{'}\right) \tag{13}$$

Moreover, since the function $f\left(d_n^{'}\right)$ is positive, it is only required to show $r'\left(d_n^{'}\right) > 0$. By differentiating the function $r\left(d_n^{'}\right)$ with respect to $d_n^{'}$, the following equation is resulted

$$\begin{aligned} r'\left(d_n^{'}\right) &= B\,\frac{-\left(2h'\left(d_n^{'}\right) - 1\right) h\left(d_n^{'}\right) + 2h'\left(d_n^{'}\right)\left(2h\left(d_n^{'}\right) - d_n^{'}\right)}{h^3\left(d_n^{'}\right)} \\ &= \frac{h\left(d_n^{'}\right) + 2h'\left(d_n^{'}\right)\left(h\left(d_n^{'}\right) - d_n^{'}\right)}{h^3\left(d_n^{'}\right)} \end{aligned} \tag{14}$$

Also by differentiating the function $h\left(d_n^{'}\right)$ with respect to $d_n^{'}$, we have

$$h'\left(d'_n\right)=\frac{\left(1-F\left(d'_n\right)\right)\left(-d'_n\left(f\left(d'_n\right)\right)\right)+f\left(d'_n\right)\int_{d'_n}^{\infty}t\,f\,(t)\,dt}{\left(1-F\left(d'_n\right)\right)^2}$$

$$=\frac{-d'_n f\left(d'_n\right)+f\left(d'_n\right)\dfrac{\int_{d'_n}^{\infty}t\,f\,(t)\,dt}{\left(1-F\left(d'_n\right)\right)}}{\left(1-F\left(d'_n\right)\right)}=f\left(d'_n\right)\frac{-d'_n+h\left(d'_n\right)}{\left(1-F\left(d'_n\right)\right)} \tag{15}$$

Since $d'_n < h\left(d'_n\right)=E\left(t\,\middle|\,t\geq d'_n\right)$, it is concluded that $h'\left(d'_n\right)>0$. Therefore, $h'\left(d'_n\right)\left(h\left(d'_n\right)-d'_n\right)>0$. Since $h\left(d'_n\right)$ is always positive , $r'\left(d'_n\right)>0$. Hence $\frac{\partial^2 V\,(n)}{\partial^2 d'_n}>0$ and it is proved that the optimal value of d'_n minimizes the cost function.

3.1 Method of determining d'_n

To evaluate $E\left(t\,\middle|\,t\geq d'_n\right)$, the following integral needs to be calculated

$$\int_{d'_n}^{\infty}tf\,(t)\,dt=\int_{d'_n}^{\infty}t\,\lambda e^{-\lambda t}\,dt=-t\,e^{-\lambda t}\Big|_{d'_n}^{\infty}+\int_{d'_n}^{\infty}e^{-\lambda t}\,dt$$

$$=d'_n e^{-\lambda d'_n}+\left(-\frac{1}{\lambda}e^{-\lambda t}\Big|_{d'_n}^{\infty}\right)=d'_n e^{-\lambda d'_n}+\frac{1}{\lambda}e^{-\lambda d'_n}=\left(d'_n+\frac{1}{\lambda}\right)e^{-\lambda d'_n} \tag{16}$$

Since $1-F\left(d'_n\right)=e^{-\lambda d'_n}$, we have

$$E\left(t\,\middle|\,t\geq d'_n\right)=\frac{\int_{d'_n}^{\infty}t\,f\,(t)\,dt}{\left(1-F\left(d'_n\right)\right)}=d'_n+\frac{1}{\lambda} \tag{17}$$

Replacing Eq. (17) in Eq. (7) results in

$$\frac{B\left(2E\left(t\,\middle|\,t\geq d'_n\right)-d'_n\right)}{\left(E\left(t\,\middle|\,t\geq d'_n\right)\right)^2}=\alpha\left(V\,(n-1)+C\right)$$

$$\Rightarrow\left(d'_n+\frac{1}{\lambda}\right)^2\alpha\left(V\,(n-1)+C\right)-2B\left(d'_n+\frac{1}{\lambda}\right)+Bd'_n=0 \tag{18}$$

Eq. (18) can be rewritten as

$$d_n^{'2}\alpha\left(V\left(n-1\right)+C\right)+\left(\frac{2}{\lambda}\alpha\left(V\left(n-1\right)+C\right)-B\right)d_n^{'}+\frac{\alpha\left(V\left(n-1\right)+C\right)-2B\lambda}{\lambda^2}=0 \quad (19)$$

That can be solved to obtain the optimal value of $d_n^{'}$.

Moreover, in order to evaluate the optimal cost of the system, since $\int_0^{d_n} t f(t)dt=\frac{1}{\lambda}-\left(d_n+\frac{1}{\lambda}\right)e^{-\lambda d_n}$ and $\frac{B\left(1-F\left(d_n^{'}\right)\right)^2}{\int_{d_n^{'}}^{\infty} t f(t)dt}=\frac{Be^{-\lambda d_n^{'}}}{\left(d_n^{'}+\frac{1}{\lambda}\right)}$, the function $V\left(n\right)$ can be rewritten as

$$V\left(n\right)=A\left(\frac{1}{\lambda}-\left(d_n+\frac{1}{\lambda}\right)e^{-\lambda d_n}\right)+\frac{Be^{-\lambda d_n^{'}}}{\left(d_n^{'}+\frac{1}{\lambda}\right)}+\left(e^{-\lambda d_n}-e^{-\lambda d_n^{'}}\right)\alpha\left(V\left(n-1\right)+C\right) \quad (20)$$

Inserting the optimal values of d_n and $d_n^{'}$ in Eq. (20), results in the optimal total system cost.

In the next section, a numerical illustration is given to demonstrate the application of the proposed methodology.

4. NUMERICAL ILLUSTRATION

Consider a production system with a nonconforming item production rate of $\lambda=0.1$. It means that a sample of the time between productions of successive defective items, t_i, is gathered where $\bar{t}=10$. Assume t_i denotes the time between production of (i-1)$_{th}$ and ith defective item then the probability distribution function of t will be $f\left(t\,\middle|\,\lambda\right)=\lambda e^{-\lambda t}$ where $\lambda=\frac{1}{\bar{t}}$.

The sampling cost of one observation from the time of producing nonconforming items is $C=1$, the total cost of the system at the start of the process is $V\left(0\right)=60$, the coefficient of cost of machine inspection, repair, or replacement is $A=10$ in each stage, the coefficient of operation cost of continuing production is $B=500$ in each stage, and the discount factor is $\alpha=0.9$ and degradation coefficient is $\beta=1.2$.

Using equations (5) and (19) for the problem at hand results in

$$d_1=\frac{\alpha\left(V\left(0\right)+C\right)}{A}=5.49$$

$$54.9d_1^{'2} + 598d_1^{'} - 4510 = 0 \;\rightarrow\; d_1^{'} = 5.12 \tag{21}$$

Thus, it is concluded that when the expected time between production of nonconforming items in the first decision making stage is less than $d_1 = 5.49$, then the machine should be inspected or possibly repaired or replaced. If the expected time between production of nonconforming products is more than $d_1^{'} = 5.12$, it is concluded that machine is in good condition and no inspection action is required. Since $d_1^{'} = 5.12 \le \bar{t} = 10$, it is concluded that machine is in good condition and no inspection action is required. Moreover, by inserting the optimal values of d_1 and $d_1^{'}$ into Eq. (20) results in $V(1) = 29.16$.

Now assume the production rate of producing nonconforming item is $\lambda = 0.1$ in stage two. It means that a sample of the time between productions of successive defective items, t_i, is gathered where $\bar{t} = 10$. In this case first we need to determine the value of $V(1)$ for evaluating the value of $V(2)$. Since we know that the rate of producing the defective items increases by a constant factor $\beta = 1.2$ therefore in the first stage of decision making we have $\lambda = 0.1 \times 1.2 = 0.12$. Now we can evaluate $V(1)$ using the proposed method that will be equal to 30.46.

Now we can determine the thresholds in the second stage of decision making. In the case we have $\lambda = 0.1$ and by continuing the above process, the following results are obtained

$$d_2 = \frac{\alpha\left(V(1) + C\right)}{A} = 2.83$$

$$28.3d_2^{'2} - 28.1d_2^{'} - 6367 = 0 \;\rightarrow d_2^{'} = 15.5 \rightarrow V(2) = 22.86 \tag{22}$$

Therefore, when the expected time between production of nonconforming items in the second decision-making stage is less than $d_2 = 2.83$, the machine should be repaired or possibly replaced. If it is more than $d_2^{'} = 22.86$, the machine is in a good condition and no repair or replacement action is required. Note that as the number of stages increases, the difference between the thresholds increases resulting in a lower total system cost. Since $d_2^{'} = 22.86 \ge \bar{t} = 10 \ge d_2 = 2.83$, the decision of sampling more defective items is optimal and we continue to the third stage. This process can be carried on to determine the optimal values of the thresholds in all decision-making stages.

5. CONCLUSION

In this article, a stochastic dynamic programming model was utilized to design a control threshold policy for the machine replacement problem. In order to determine the optimal policy, the inspection cost in combination with the cost of continuing the production process was minimized. The important result of this method is that as the number of inspection stages increases, the difference between the control thresholds increases and the total system cost becomes less. In other words, the cost that arises from initial large values associated with not performing any action decreases in the subsequent stages. For further research, different cost functions can be assumed. Moreover, the optimal replacement problem when there are multiple machines may be considered.

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