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A New Acceptance Sampling Plan Based on Cumulative Sums of Conforming Run-Lengths

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ABSTRACT

In this article, a novel acceptance-sampling plan is proposed to decide whether to accept or reject a receiving batch of items. In this plan, the items in the receiving batch are inspected until a nonconforming item is found. When the sum of two consecutive values of the number of conforming items between two successive nonconforming items falls underneath of a lower control threshold, the batch is rejected. If this number falls above an upper control threshold, the batch is accepted, and if it falls within the upper and the lower thresholds then the process of inspecting items continues. The aim is to determine proper threshold values and a Markovian approach is used in this regard. The model can be applied in group- acceptance sampling plans, where simultaneous testing is not possible. A numerical example along a comparison study are presented to illustrate the applicability of the proposed methodology and to evaluate its performances in real-world quality control environments.

***Keywords*:** Acceptance Sampling, Quality control, Inspection, Markov process.

1. INTRODUCTION

Scientific sampling plans are the primary tools for quality and performance management in industry today. In an industrial plant, sampling plans are used to decide either to accept or reject a received batch of items. With attribute sampling plans, these accept/reject decisions are based on a count of the number of nonconforming items and in traditional sampling plans, the sample size is assumed constant.

Perhaps the most common acceptance-sampling plan in use today are the single-sample acceptance-sampling plans designed to meet particular points on a Type A or Type B operating characteristic curve (McWilliams et al. 2001). Besides, the idea of using the run-lengths of successive conforming items as an indicator of process performance is used by some authors. Calvin (1983) and Goh (1987) proposed a control chart based on the run-lengths of successive conforming items. Bourke (1991) proposed statistics based on the sums and cumulative sums of such conforming run-lengths

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for the case of 100% inspection. Bourke (2003) proposed a continuous sampling plan for deciding when to switch between the phases of sampling and 100% inspection. His sampling plan was based on the sum of run-lengths of conforming items. Further, Bourke (2002) proposed switching rules based on a cumulative sum of the observed run-lengths of conforming items between successive defective items. Klassen (2001) proposed a credit-based acceptance sampling system. The credit of the producer was defined as the total number of items accepted since last rejection. Niaki and Fallahnezhad (2009) proposed an acceptance-sampling plan based on Bayesian inferences and stochastic dynamic programming. Their objective function was the ratio of the total discounted system cost to the discounted system correct choice probability.

In this paper, a new control policy in an acceptance-sampling plan is introduced, in which Y is defined as the number of conforming items between successive nonconforming items. Then, when the sum of two consecutive values of Y falls below a lower control threshold, the batch is rejected. If this number falls above an upper control threshold, the batch is accepted, and if it falls within the upper and the lower thresholds then the process of inspecting items continues. To the best of authors' knowledge, no attention has been paid so far in the literature to employ the sum of two consecutive values of the number of conforming items to develop a sampling plan for the lot acceptance using the two points on OC curve. The origin of this idea comes from Bourke (2003) to develop a continuous sampling plan for deciding when to switch between the phases of sampling and 100% inspection. While adapting this concept to design an acceptance-sampling plan is the main contribution of this paper, this research has some applications in group-acceptance sampling-plans, when a number of items is considered a group for a truncated life test where they cannot be tested simultaneously in a tester.

The rest of the paper is organized as follows. The required notations are first introduced in Section 1.1. The model development is given in Section 2. A numerical demonstration on the application of the proposed methodology comes next in Section 3. A comparison study comes in Section 4, and finally, we conclude the paper in Section 5.

Notations

The required notations to model the problem follows.

- p : Proportion of nonconforming items in the batch
- δ_1 : Minimum acceptable batch quality-level, i.e. Acceptable Quality Level (AQL)
- δ_2 : Minimum rejectable batch quality-level, i.e. Lot Tolerance Proportion Defective (LTPD)
- ε_1 : Probability of type-I error
- ε_2 : Probability of type-II error
- $\mathbf{P}$: Transition probability matrix

2. MODEL DEVELOPMENT

In an acceptance-sampling plan, assume Y_i is the number of conforming items between the successive $(i-1)^{st}$ and i^{th} nonconforming items. Decision making is based on the value of S_i that is defined as,

$$S_i = Y_i + Y_{i-1} \tag{1}$$

The proposed acceptance sampling policy is defined as follows.

- If $S_i \geq U$ then the batch is accepted
- If $S_i \leq L$ the batch is rejected
- If $L < S_i < U$ the process of inspecting the items continues

where U is the upper control and $L \geq 1$ is the lower control threshold.
In stage i of the data gathering process, the index of different states of the Markov model, *j*, is defined as:

- $j = 1$ represents the state where the batch is rejected. In this state, $S_i \leq L$.
- $j = Y_i + 2$ where $Y_i = 0,1,2...,U-1$ represents the state in which the inspection process continues gathering more data. In this state, $L < S_i = Y_i + Y_{i-1} < U$.
- $j = U + 2$ represents the state where the batch is accepted. In this state $S_i \geq U$.

In other word, the acceptance-sampling plan can be expressed by a Markov model, in which the transition probability matrix among the states of the batch can be expressed as

$$p_{jk} = \begin{cases} 1 & j = k = 1 \\ 0 & j = 1, k > 1 \\ P\left(Y_{i+1} \leq L - j + 2\right) & U+2 > j > 1, L \geq j-2, k = 1 \\ 0 & U+2 > j > 1, L < j-2, k = 1 \\ 0 & U+2 > j > 1, U+2 > k > 1, j+k-4 \leq L \\ 0 & U+2 > j > 1, U+2 > k > 1, j+k-4 \geq U \\ P\left(Y_{i+1} = k-2\right) & U+2 > j > 1, U+2 > k > 1, U > j+k-4 > L \\ 1 & j = k = U+2 \\ 0 & j = U+2, k < U+2 \\ P\left(Y_{i+1} \geq U - j + 2\right) & U+2 > j > 1, k = U+2 \end{cases} \tag{2}$$

where, p_{jk} is probability of going from state j to state k in a single step, Y_{i+1} denotes the number of conforming items between the successive nonconforming items, and $P\left(Y_{i+1} = r\right) = \left(1-p\right)^r p$ where p denotes the proportion of nonconforming items in the batch, when the sample size in a sampling without replacement strategy is small compared to the batch size.

The values of p_{jk} are determined based on the relations among the states. For example, when $1 < j < U + 2$, $L \geq j-2$, and $k = 1$, then according to the definition of *j*, it is concluded that $j = Y_i + 2$ and the transition probability of going form state *j* to state *k=1* is equal to the probability of rejecting the batch that is evaluated as follows

$$p_{j1} = P\left(L \geq S_{i+1} = Y_{i+1} + Y_i\right) = P\left(L \geq Y_{i+1} + j - 2\right) = P\left(Y_{i+1} \leq L - j + 2\right) \tag{3}$$

In another case where $1 < j < U + 2$, $1 < k < U + 2,$ and $L < U < j + k - 4$, then $j = Y_i + 2$ and thus

$$\begin{aligned} p_{jk} &= P\left(L < S_{i+1} = Y_{i+1} + Y_i < U, Y_{i+1} = k - 2\right) = P\left(L < j - 2 + Y_{i+1} < U, Y_{i+1} = k - 2\right) = \\ &P\left(L < j - 2 + k - 2 < U, Y_{i+1} = k - 2\right) = P\left(L < j + k - 4 < U, Y_{i+1} = k - 2\right) \end{aligned} \tag{4}$$

In situation in which $1 < j < U + 2$ and $k = U + 2$, we have $j = Y_i + 2$ and hence

$$p_{jU+2} = P\left(S_{i+1} = Y_{i+1} + Y_i \geq U\right) = P\left(Y_{i+1} + j - 2 \geq U\right) = P\left(Y_{i+1} \geq U - j + 2\right) \tag{5}$$

Finally, when $1 < j < U + 2$, $1 < k < U + 2$, *and* $j + k - 4 \geq U$, we have $j = Y_i + 2$ and therefore

$$\begin{aligned} p_{jk} &= P\left(L < S_{i+1} = Y_{i+1} + Y_i < U, Y_{i+1} = k - 2, j + k - 4 \geq U\right) \\ &= P\left(L < j - 2 + Y_{i+1} < U, Y_{i+1} = k - 2, j + k - 4 \geq U\right) \\ &= P\left(L < j + k - 4 < U, j + k - 4 \geq U\right) = 0 \end{aligned} \tag{6}$$

As a result, when $L = 1$ and $U = 3$, for example, the transition probability matrix among the states of the system can be expressed as:

$$\mathbf{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ P\left(Y \leq 1\right) & 0 & 0 & P\left(Y = 2\right) & P\left(Y \geq 3\right) \\ P\left(Y \leq 0\right) & 0 & P\left(Y = 1\right) & 0 & P\left(Y \geq 2\right) \\ 0 & P\left(Y = 0\right) & 0 & 0 & P\left(Y \geq 1\right) \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array} \tag{7}$$

It can be seen the matrix $\mathbf{P}$ is an absorbing Markov chain with states 1 and 5 being absorbing and states 2, 3, 4 being transient.

Analyzing the above absorbing Markov chain requires to rearrange the single-step transition probability matrix in the following form:

$$P = \begin{bmatrix} \mathbf{A} & \mathbf{O} \\ \mathbf{R} & \mathbf{Q} \end{bmatrix} \tag{8}$$

where $\mathbf{A}$ is the identity matrix representing the probability of staying in a state defined as

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{9}$$

$\mathbf{O}$ is the probability matrix of escaping an absorbing state (always zero) that is defined as

$$\mathrm{O} = \begin{array}{c} \\ 1 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \tag{10}$$

Q is a square matrix containing the transition probabilities of going from a non-absorbing state to another non-absorbing state defined as

$$\mathrm{Q} = \begin{array}{c|ccc} & 2 & 3 & 4 \\ \hline 2 & 0 & 0 & P(Y=2) \\ 3 & 0 & P(Y=1) & 0 \\ 4 & P(Y=0) & 0 & 0 \end{array} \tag{11}$$

and **R** is the matrix containing all probabilities of going from a non-absorbing state to an absorbing state (i.e., accepted or rejected batch) that is defined as follows

$$\mathbf{P} = \begin{array}{c|cc} & 1 & 5 \\ \hline 2 & P(Y \le 1) & P(Y \ge 3) \\ 3 & P(Y \le 0) & P(Y \ge 2) \\ 4 & 0 & P(Y \ge 1) \end{array} \tag{12}$$

Rearranging the **P** matrix in the latter form yields the following:

$$\mathbf{P} = \begin{array}{c|ccccc} & 1 & 5 & 2 & 3 & 4 \\ \hline 1 & 1 & 0 & 0 & 0 & 0 \\ 5 & 0 & 1 & 0 & 0 & 0 \\ 2 & P(Y \le 1) & P(Y \ge 3) & 0 & 0 & P(Y=2) \\ 3 & P(Y \le 0) & P(Y \ge 2) & 0 & P(Y=1) & 0 \\ 4 & 0 & P(Y \ge 1) & P(Y=0) & 0 & 0 \end{array} \tag{13}$$

Bowling et al. (2004) proposed an absorbing Markov chain model for determining the optimal process means. According to their method, the fundamental matrix **M** containing the expected number of transitions from a non-absorbing state to another non-absorbing state before absorption occurs can be obtained using the following equation

$$\mathbf{M} = (\mathbf{I} - \mathbf{Q})^{-1} \tag{14}$$

For the above numerical example, i.e., when $L=1$ and $U=3$, the fundamental matrix $\mathbf{M}$ can be obtained as:

$$\mathbf{M} = (\mathbf{I} - \mathbf{Q})^{-1} = \left(\begin{array}{c|ccc} & 2 & 3 & 4 \\ \hline 2 & 1 & 0 & -P(Y=2) \\ 3 & 0 & 1-P(Y=1) & 0 \\ 4 & -P(Y=0) & 0 & 1 \end{array}\right)^{-1} \tag{15}$$

where **I** is the identity matrix.

Matrix $\mathbf{F}$, the absorption probability matrix containing the long run probabilities of the transition from a non-absorbing state to an absorbing state, can be obtained as follows (Bowling et al. 2004):

$$\mathbf{F} = \mathbf{M} \times \mathbf{R} \tag{16}$$

Again when $L = 1$ and $U = 3$, the elements of $\mathbf{F}$ $\left(f_{jk} \quad ; \quad j = 2,3,4 \quad ; \quad k = 1,5\right)$ represent the probabilities of the batch being accepted and rejected, respectively, given that the initial state is $j = 2,3,4$. In this case, the probability of accepting the batch is obtained as:

$$\begin{aligned} &\text{Probability of accepting the batch=} \\ &\sum_{j=2}^{\infty} \mathrm{P}\left(\text{Accepting the batch} \middle| \text{the initial state is } j\right) \times \mathrm{P}\left(\text{the initial state is } j\right) \\ &= \sum_{j=2}^{4} f_{j5} P\left(Y = j-2\right) + P\left(Y \geq 3\right) \end{aligned} \tag{17}$$

Also the expected number of inspected items will be determined as follows,

$$\begin{aligned} &\text{Expected number of inspected items} = \sum_{j=2}^{U+1} \begin{pmatrix} \left(\text{the number of inspected items in state j}\right) \\ \left(\text{the number of visits to state j}\right) \end{pmatrix} \\ &= \sum_{j=2}^{U+1} \left(j-2\right) m_{jj} \end{aligned} \tag{18}$$

where m_{jj} represents the expected number of times in the long-run the transient state j is occupied before absorption occurs (i.e., before accepting or rejecting the batch).

This new acceptance-sampling plan should fulfill two constraints of the first and the second types of errors. The probability of Type-I error shows the probability of rejecting the batch when the nonconforming proportion of the batch is acceptable. The probability of Type-II error is the probability of accepting the batch when the nonconforming proportion of the batch is not acceptable. Then, on the one hand if $p = \delta_1 = AQL$, the probability of rejecting the batch will be less than ε_1 and on the other hand, in case where $p = \delta_2 = LTPD$, the probability of accepting the batch will be less than ε_2. Hence,

$$\begin{aligned} &p = \delta_1 \rightarrow \text{Probability of accepting the batch} \geq 1 - \varepsilon_1 \\ &p = \delta_2 \rightarrow \text{Probability of accepting the batch} \leq \varepsilon_2 \end{aligned} \tag{19}$$

From the inequalities in (19), the proper values of the thresholds L and U are determined.

In the next section, a numerical example is given to demonstrate the application of the proposed acceptance-sampling plan.

3. A NUMERICAL EXAMPLE

Consider an acceptance sampling problem with the parameters $L=1$, $U=3$, $\delta_1 = AQL = 0.2$, $\delta_2 = LTPD = 0.5$, $\varepsilon_1 = 0.2$, and $\varepsilon_2 = 0.4$.

The appropriateness of the thresholds L and U depends on the existing proportion of the batch nonconformities. To show this, in an in-control situation assume $p = \delta_1 = 0.2$. Then, we have

$$\mathbf{M} = (\mathbf{I}-\mathbf{Q})^{-1} = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\overset{\begin{matrix} 2 & \quad\quad 3 & \quad\quad 4 \end{matrix}}{\begin{bmatrix} 1 & 0 & -P(Y=2) \\ 0 & 1-P(Y=1) & 0 \\ -P(Y=0) & 0 & 1 \end{bmatrix}^{-1}} = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\overset{\begin{matrix} 2 & \quad 3 & \quad 4 \end{matrix}}{\begin{bmatrix} 1.03 & 0 & 0.13 \\ 0 & 1.19 & 0 \\ 0.21 & 0 & 1.03 \end{bmatrix}}$$

The long-run absorption probability matrix, $\mathbf{F}$, is obtained as:

$$\mathbf{F} = \mathbf{M}\times\mathbf{R} = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\overset{\begin{matrix} 1 & \quad 5 \end{matrix}}{\begin{bmatrix} 0.37 & 0.63 \\ 0.24 & 0.76 \\ 0.07 & 0.93 \end{bmatrix}}$$

In this case, the probability of accepting the batch and expected number of inspected items is calculated as:

$$\text{Probability of accepting the batch} = \sum_{j=2}^{4} f_{j5} P(Y = j-2) + P(Y \geq 3) = 0.88$$

$$\text{Expected number of inspected items} = \sum_{j=2}^{4} (j-2)\, m_{jj} = 1.19 + 2(1.03) = 3.25$$

Now since $0.88 \geq 1-\varepsilon_1 = 0.8$, the proposed acceptance sampling plan with the thresholds $L=1$ and $U=3$ satisfies the constraint on the Type-I error.

In an out-of-control condition when $p = \delta_2 = 0.5$, then

$$\mathbf{M} = (\mathbf{I}-\mathbf{Q})^{-1} = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\overset{\begin{matrix} 2 & \quad\quad 3 & \quad\quad 4 \end{matrix}}{\begin{bmatrix} 1 & 0 & -P(Y=2) \\ 0 & 1-P(Y=1) & 0 \\ -P(Y=0) & 0 & 1 \end{bmatrix}^{-1}} = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\overset{\begin{matrix} 2 & \quad 3 & \quad 4 \end{matrix}}{\begin{bmatrix} 1.07 & 0 & 0.13 \\ 0 & 1.33 & 0 \\ 0.53 & 0 & 1.07 \end{bmatrix}}$$

With the long-run absorption probability matrix of

$$\mathbf{F} = \mathbf{M}\times\mathbf{R} = \begin{matrix} 2 \\ 3 \\ 4 \end{matrix}\overset{\begin{matrix} 1 & \quad 5 \end{matrix}}{\begin{bmatrix} 0.8 & 0.2 \\ 0.67 & 0.33 \\ 0.4 & 0.6 \end{bmatrix}}$$

In this case, the probability of accepting the batch and the expected number of inspected items are

$$\text{Probability of accepting the batch} = \sum_{j=2}^{4} f_{j5} P(Y = j-2) + P(Y \geq 3) = 0.38$$

$$\text{Expected number of inspected items} = \sum_{j=2}^{4} (j-2) m_{jj} = 1.33 + 2(1.07) = 3.57$$

Now since $0.38 \leq \varepsilon_2 = 0.4$, the proposed acceptance sampling plan satisfies the constraint on the Type-II error as well. As a result, the plan fulfils the two constraints of the first and the second type errors and can be applied in real case situations.

The numerical illustration shows that depending on the batch proportion of nonconformities, the magnitudes of the probabilities of Type-I and Type-II errors, AQL, and LTPD the values of the lower and the upper thresholds can be numerically determined using the proposed procedure. As a result, the desired acceptance-sampling plan can be designed in real-world quality control environments.

4. A COMPARISON STUDY

To compare the performances of the proposed method with the ones of a traditional acceptance-sampling plan, a comparison study is performed in this Section. Assume a single stage sampling policy that is defined as follows; *n* items from a large batch are inspected without replacement. If the number of nonconforming items is below a lower control threshold c_1, the batch is accepted. If this number is above a control threshold c_2, the batch is rejected, and if it falls within the thresholds c_1 and c_2, the process of inspecting *n* more items continues. Assuming *n*=3, Table (1) shows 9 different alternative combination values of c_1 and c_2 together with their probability of rejecting or accepting the batch, of which the ones in bold are feasible.

Table 1 The probabilities of rejecting and accepting the batch in a traditional sampling plan

Expected Number of Inspected Items when $\delta_2 = 0.5$	Expected Number of Inspected Items when $\delta_1 = 0.2$	Probability of rejecting the batch when $\delta_2 = 0.5$	Probability of accepting the batch when $\delta_1 = 0.2$	c_2	c_1
3.00	3.00	0.88	0.51	0	0
4.80	**4.87**	**0.80**	**0.83**	**1**	**0**
12.00	5.77	0.50	0.98	2	0
24.00	5.86	0.00	1.00	3	0
3.00	3.00	0.50	0.90	1	1
4.80	3.32	0.20	0.99	2	1
6.00	3.35	0.00	1.00	3	1
3.00	3.00	0.13	0.99	2	2
3.43	3.02	0.00	1.00	3	2

Based on the results in Table (1), the only feasible case in the traditional acceptance-sampling plan is *n*=3, $c_1 = 0$ and $c_2 = 1$, where the probability of accepting the batch when $p = \delta_1 = 0.2$ is 0.83 and the probability of rejecting the batch when $p = \delta_2 = 0.5$ is 0.8. Thus, these acceptance sapling plans fulfill the constraints of Type-I and Type-II errors. However, as can be seen from Table 1, the expected number of inspected items when $p = \delta_1 = 0.2$ is 4.87 where the expected number of inspected items in the proposed sampling plan is 3.25. Moreover, the expected number of inspected items when $p = \delta_2 = 0.5$ is 4.8 where the expected number of inspected items in the proposed sampling plan is 3.57. Consequently, it is inferred that the proposed sampling plan can perform better than the traditional sampling plan. This conclusion is also affirmed by Bourke (2003).

5. CONCLUSION

In this paper, a new methodology based on Markov chain was developed to design proper lot acceptance sampling plans. In the proposed procedure, the sum of two successive numbers of nonconforming items was monitored using lower and upper thresholds, where the proper values of these thresholds were determined numerically using a Markovian approach based on the two points on OC curve. The probabilities of accepting the batch and the expected number of inspected items were determined using the properties of absorbing Markov chains. A numerical example was given to demonstrate the applicability of the proposed method. Further, a comparison of the proposed method with a traditional acceptance sampling plan in a numerical example showed the advantages of applying the proposed method in real-world problems.

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