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Demand-oriented timetable design for urban rail transit under stochastic demand

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Abstract

In the context of public transportation system, improving the service quality and robustness through minimizing the average passengers waiting time is a real challenge. This study provides robust stochastic programming models for train timetabling problem in urban rail transit systems. The objective is the minimization of the weighted summation of the expected cost of passenger waiting time, its variance and the penalty function including the capacity violation due to overcrowding. In the proposed formulations, the dynamic and uncertain travel demand is represented by the scenario-based multi-period arrival rates of passenger. Two versions of the robust stochastic programming models are developed and a comparative analysis is conducted to testify the tractability of the models. The effectiveness of the proposed stochastic programming model was demonstrated through the application to Tehran underground urban railway. The outcomes show the reductions in expected passenger waiting time of22%, and cost variance drop of 60% compared with the baseline plans using the proposed robust optimization approach.

Keywords: Train timetabling; urban rail; uncertain demand, robust stochastic programming

1-Introduction

A rapid transit metro is a homogeneous railway system with a high sensitivity against different type of disturbances (Lin and Sheu, 2010). Here, the homogeneity of the system refers to the similarities, particularly the same capacity and average speed on track segments(Salido et al. 2008).

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In order to cope with uncertainty, the rail managers must schedule the train operations effectively at public transit nodes. One of the main types of disturbances is stochastic variations of travel demand that affects the headway regularity and results in passenger congestion and additional waiting time. For the disturbed situation, passengers may not be allowed to board the first train and have to wait until the next train arrives due to the maximum vehicle capacity. Particularly, during the peak hours, a planned timetable may become infeasible with respect to the capacity constraints simply due to an unpredicted increase of passenger demand. An important consequence of demand disturbances is overcrowded vehicles. The on-board discomfort is likely to be influenced by congestion effects and in-vehicle overcrowding (Trozzi et al. 2013). The in-vehicle overcrowding occurs when the number of on-board passengers exceeds the normal loading capacity of a vehicle. Niu and Zhang (2012) used the in-vehicle overcrowded cost in the objective function of the train timetable problem. An optimization approach was developed to minimize both the passenger waiting time and the in-train overcrowding cost. However, to handle demand disruptions, a robust timetable is required. More specifically, the design and optimization of such a timetable that guarantees acceptable passenger waiting time is an important research topic.

From the operational point of view, the quality of the train timetables for metro service is extremely demand sensitive(Sun et al., 2014). Usually, passenger demand fluctuates on a daily basis, so that the expected demand may not reflect the real daily passenger demand, where stochastic disturbances may occur. Hence, the precise estimation of the demand flows is an important issue in timetable design and particularly for the congested metropolitan areas. Fortunately, with the emergence of automated vehicle positioning and passenger counter systems, a rich historical data is now available for the public transportation sector to manage the operations effectively and to examine the reliability of the transit system(Carrel et al. 2013).Corman and Meng (2015) provided a comprehensive survey of the online railway traffic control approaches under dynamic and stochastic environments. It was pointed out that collecting and incorporating the realistic data of passenger flows is essential to improve the timetable robustness. With the intention of avoiding passenger congestion due to train capacity and demand uncertainty, the robust approach to train timetabling problem is a practical solution.

In the following, the most significant contributions in the literature, with a particular attention to the demand-driven train timetabling problem are highlighted: The past scientific publications in the field of urban rail planning can be categorized to nominal scheduling (Barrena et al., 2014a; Chierici et al., 2004; Scholz et al., 2003), stochastic optimization (Li and Yang 2013), online control (Assis and Milani 2004; Carrel et al. 2010; Eberlein 1997; Eberlein et al. 1998; Eberlein et al. 1999; Lin and Sheu 2011; Sáez et al. 2012; Sheu and Lin 2012) and disruption management models (O'Dell and Wilson 1999; Shen 2000; Shen and Wilson 2001). To the best of our knowledge, Cury et al. (1980) presented the first methodological method for generating the optimal schedules for metro services by taking into account the variation of the passenger flow, maintaining an satisfactory level of service and providing the ability to recover from disturbed situations. A hierarchical multi-level optimization techniques based on the Lagrangian relaxation framework was proposed in order to generate optimal schedules. Albrecht (2009) presented a two-level approach for the demand-oriented train timetabling problem in suburban railways. Carrel et al. (2010) proposed a comprehensive framework to investigate decision factors and major considerations in service control on high-frequency metro line. Niu and Zhou (2013) developed a deterministic nonlinear optimization model for demand-adapted timetable design subject to the resource constraints. Wang et al. (2013) presented a real-time train scheduling model in order to minimize the total passenger time. Canca et al. (2013) developed a nonlinear integer programming model for generating train timetable which considers a dynamic behaviour of passenger demand.

Barrena et al. (2014a) developed linear time-indexed formulations for the train scheduling problem with dynamic demand in order to minimize passenger average waiting time by the use of branch-and-cut algorithm. Sun et al. (2014) developed three linear mathematical formulation of train timetabling problem to capture the period-dependent passenger demand. A sensitivity analysis was conducted to evaluate the performance of these proposed models on a metro line in Singapore. The result demonstrated that the capacitated and dynamic timetable was the most practical model. Barrena et al. (2014b) proposed general mathematical programming formulations for the train timetabling problem on a unidirectional transit line under a dynamic demand.Saharidis et al. (2014) presented a mixed-integer linear programming model in order to minimize passenger waiting times at transfer nodes in the bus network. The model takes into account the increased passenger demand at a given time period and aims to reschedule the baseline timetables, with the purpose of constructing a more efficient schedule.

As reviewed above, the most of the contributions in the field of transit scheduling assumed a deterministic travel demand. Taking the travel demand randomness and uncertainty into modeling has been captured by a limited number of studies.Smith and Sheffi (1989) studied locomotive scheduling problem where the power demand for each train is uncertain. They presented a multi-commodity flow formulation to minimize expected cost under uncertainty by penalizing trip arcs likely to have too little power. List et al. (2003) studied the robust optimization of the fleet sizing problem under the uncertainty of future demands. Yan et al. (2006) studied the inter-city bus routing and timetabling problem under stochastic demands. Yang et al. (2009) studied train scheduling problem with fuzzy passenger demand on a single-track railway. It was assumed that trains have sufficient capacity to transport all waiting passengers. A fuzzy goal-programming model with two objectives, namely, total passengers' time and total delay time was proposed.Islam and Vandebona (2010) conducted a simulation experiment to illustrate that the variability of departure headways directly influences the waiting time variations at the bus stops.

Meng and Zhou (2011) developed a multi-stage stochastic programming model to produce robust train schedule where a partial line blockage with uncertain duration happened on a single-track rail line. A branch and bound (B&B) algorithm was proposed to obtain optimal train schedule with minimal expected delays. Shafia et al. (2012) addressed the robust train timetabling problem in a single-track railway network. In order to solve large-scale instances, a beam search heuristic was suggested.

Dou et al. (2013) developed a bi-objective train dispatching model based on fuzzy passenger demand forecast to minimize the total operation cost and unserved passenger demand during holidays. Yan et al. (2013) studied the courier routing and scheduling problem in an urban area with uncertain demands and stochastic travel times. Xu et al. (2014) assumed that passenger arrival rates and alighting ratios are triangular fuzzy variables. They presented several operational strategies to reduce energy consumption and congestion degree. Canca et al. (2014) proposed mixed-integer non-linear optimization models to determine optimal short-turning strategies for the management of demand disruptions in rapid rail transit systems. The pattern of short-turn services and the train timetable were determined through the proposed optimization model with the objective of minimizing the passenger waiting time while maintaining certain level of quality of service. Hassannayebi et al. (2014) addressed the timetable optimization problem for entire daily operations of the metro services. They proposed a two-stage simulation-optimization approach to minimize the expected waiting times, subject to the capacity constraints, as well as dwell time and travel time variability and the demand randomness. Xu et al. (2015) developed a multi-objective timetable optimization approach for subway system to minimize the passenger time and energy consumption. The variation on the passenger flow at stations was analyzed. A speed-profile-generation method was proposed to search for the energy-efficient speed profile. Yin et al. (2015) addressed the timetable optimization model in subway system considering uncertain passenger demand at each stop and random running times. To deal with uncertain passenger demands, a real-time train operation model was proposed. The train operation problem then was converted into a Markov decision process with nondeterministic state transition probabilities. The objective was to minimize the penalty for both the total time delay and energy consumption in a railway line. Wales and Marinov (2015) developed a discrete-event simulation model in order to analyze the system's performance and delay responsiveness. Different delay mitigation strategies were introduced and measured to evaluate their potential in moderating delays in the system.

The present work is motivated by the lack of methodological framework to investigate how demand uncertainties affect the feasibility and optimality of the timetabling decisions in metro services. The research application is required for the transit operations in Tehran metropolitan network due to the high variability and randomness in daily transportation demand. This article contributes by developing robust stochastic programming models for train timetabling under demand uncertainty. To the best of our knowledge, this study is the first attempt to apply robust stochastic programming model to rapid rail transit timetabling problem under demand uncertainty. Most of the existing timetabling models for urban rail management systems were implemented in a deterministic and static setting. Therefore, dynamic and stochastic models such as those presented here have considerable practical application.

The remainder of the study is as follows. In Section 2, the problem is described. Robust stochastic programming formulations are provided in Section 3. The researchers organize and interpret the results of real test experiments in Section 4, which followed by conclusion in Section 5.

2-Problem statement

In the metro rail systems, the time between two consecutive departures of the trains is known as headway time. In this study, the aim is to optimize the headway time of train services at rail station with the objective of minimizing the expected and variance of the passenger waiting times as well as the expected overloading under stochastic demand. In what follows, we explain why we utilize the robust optimization methodology and its advantages: The number of passengers arriving at the station is an uncertain variable during the planning horizon. Consequently, using uncertain variables to characterize passenger demand is an essential assumption (Xu et al., 2014). Fortunately, in modern transportation system, the historical data of the travel demand is gathered and maintained in automatic passenger counter (APC) systems. The passenger arrival rate can be estimated and represented by demand scenarios where each one corresponds to an arrival rate profile. With this explanation, the objective is to develop a robust stochastic optimization model to construct robust train timetables. Here, the robustness of a timetable is related to its stability against stochastic variations in arrival rate of passengers. The robust timetable will be constructed subject to the maximum number of train services, fleet size and capacity, and the minimum and maximum headways. In the next section, the basic concept of the robust mathematical programming will be discussed.

3-Robust mathematical programming

Approaches to optimization under uncertainty have conducted on a variety of methods, including expectation minimization, goal programming, mini-max approach, stochastic programming including recourse models, robust stochastic programming, and fuzzy programming (Sahinidis 2004).The concept of robust stochastic programming was first introduced by Mulvey et al. (1995) to solve multiple-scenario stochastic optimization problems. This robust optimization approach is a kind of adjustable robust optimization (Goerigk and Schöbel 2015). The flexibility of the implementation and its practical solutions for decision-makers made the robust stochastic programming approach a valuable technique for uncertainty management. Robust stochastic programming is capable of handling soft constraint where the data uncertainty is represented by scenarios and the associated probability of occurrence. It also assumes that the probability distribution of the uncertain variable is given. Hopefully, the demand data of passengers can be estimated through implementation of the high-tech data recording system. Thus, the probability distribution of the arrival rates can be derived and implemented in the model.

Here, the robust optimization model attempts to find a solution that is near to optimal in all possible scenarios while ensures the feasibility of the solution in almost all scenarios by means of penalty functions. These two conflicting measures of robustness are weighted in accordance with the decision-makers' objectives. The proposed formulation aims to minimize expected and variance of passenger waiting times and at the same time the expected overloading, taking into consideration the stochastic and dynamic variations of the passenger demand.

3-1-Robust stochastic programming framework

The concept of robust stochastic programming was first introduced by Mulvey et al. (1995) to stochastic optimization problems. The methodology suggests a general robust scenario-based stochastic programming framework which combines the goal programming concept to model the uncertainty. Numerous applications of this robust stochastic programming approach have been reported in the literature including the capacity expansion problems (Malcolm and Zenios, 1994), power dispatch (Beraldi et al., 1998), chemical process planning (Ahmed and Sahinidis 1998), network design (Bai et al. 1997), parallel machine scheduling (Laguna et al., 2000), production planning (Leung and Wu, 2004), supply chain management (Bozorgi-Amiri et al., 2013) and (Saffari et al., 2015), water resource systems planning (Ray et al., 2013) and generalized assignment problem (Fu et al., 2014). The aforementioned applications prove the ability of the robust mathematical programming models to solve the optimization problems under uncertainty. The robust optimization approach uses two robustness concepts including solution robustness and model robustness. The solution robustness means the solution is close to optimal in all possible scenarios. Alternatively, the model robustness states the case when the solution is almost feasible in all situations. LetA, bdenote deterministic input data; but B, C, e are the uncertain part of the model. The robust optimization model involves two types of decisions. The first set of decision variables (design variables) denoted by $x \in \mathbb{R}^{n_1}$and their optimal values are not conditioned on the realization of the uncertain data. Oppositely, $y \in \mathbb{R}^{n_2}$ represent the set of control variables where their optimal values depend both on the realization of uncertain data as well as the optimal values of the design variables. In the present formulation, each realization of an uncertain parameter is referred to a scenario ($\omega \in \Omega$) associated with an occurrence probabilityp_ω. Let Ω denotes the finite set of scenarios$\{1,2,\dots,\omega\}$. Consequently, each scenario is associated with the subset of realized input data $\{d_\omega, B_\omega, C_\omega, e_\omega\}$. Furthermore, ξ_ω denotes the cost or benefit function associated with scenario $\omega \in \Omega$. Here, the aim is to find a trade-off between the solution and model robustness. The set of control variables for each scenario $\omega \in \Omega$ are denoted by the set $\{y_1, y_2, \dots, y_\omega\}$. The feasibility of the solutions is measured by a set of error vectors $\{\delta_1, \delta_2, \dots, \delta_\omega\}$ are introduced. Thefinal formulation of the scenario-based robust optimization program is as follows:

$$Minimize \quad \sigma(x, y_1, y_2, \dots, y_\omega) + \gamma . \pi(\delta_1, \delta_2, \dots, \delta_\omega) \tag{1}$$

$$\text{Subject to } Ax = b, \tag{2}$$

$$B_\omega x + C_\omega y_\omega + \delta_\omega = e_\omega, \omega \in \Omega \tag{3}$$

$$x, y_\omega \geq 0, \quad \omega \in \Omega \tag{4}$$

The equation (1)characterizes the objective function of the robust optimization model. The first term of this function measures the solution robustness, while the second term denotes the model robustness, penalizing infeasible solutions by a weight parameter γ. Mulvey et al. (1995)proposed quadratic and absolute penalty functions. We use the linear penalty function proposed by Yu and Li (2000). At this point, ϕrefers to the weight of cost variance. In conclusion, the robust stochastic programming model is written as follows:

$$Minimize \sum_{\omega \in \Omega} p_\omega . \xi_\omega + \phi \sum_{\omega \in \Omega} p_\omega \left\{ \left(\xi_\omega - \sum_{\omega' \in \Omega} p_{\omega'} . \xi_{\omega'} \right) + 2\tau_\omega \right\} + \gamma \sum_{\omega \in \Omega} p_\omega . \delta_\omega \tag{5}$$

$$s.t. \, \xi_\omega - \sum_{\omega' \in \Omega} p_{\omega'} . \xi_{\omega'} + \tau_\omega \geq 0 \quad \omega \in \Omega \tag{6}$$

$$\tau_\omega, \delta_\omega, \xi_\omega \geq 0 \quad \omega \in \Omega \tag{7}$$

The model robustness is related to the expected in-vehicle over crowding. The overcrowding results in passenger's dissatisfaction. Therefore, one could consider a penalty term into the objective function. This approach improves the timetable robustness against demand randomness and simultaneously it takes the advantages of maintaining the desirable level of service.

In the next sections, two versions of the above-mentioned robust stochastic optimization approach are presented. We propose alternative formulations of the train timetabling problem. The main objectives are the comparative analysis of the tractability and suitability of the models dealing with large-scaled instances. Therefore, the modelling approach benefits from different mathematical formulations. In the first model, a time-expanded linear formulation is developed. The second formulation presents a nonlinear objective function and linear constraints. The interesting feature in the second formulation is that it requires fewer binary variables than the linear model. Both formulations are developed to improve the robustness of train timetable against the stochastic variations in arrival rate of passengers.

3-2-Time-indexed linear model

This section provides a time-expanded formulation of the train timetabling model in accordance to the robust stochastic programming framework presented in the previous section. A set of train services ($i \in I$) are given to be scheduled during the period of service ($[0, \mathrm{T}]$). Let H_i be the headway between i-th and $(i+1)$-th train departures. In the proposed time-indexed stochastic programming formulation, the index $t \in \mathcal{T}$ is referred to the departure time slots with equal length α. The passenger arrival rate at interval t under scenario $\omega \in \Omega$ is denoted by$\lambda_{t\omega}$. The binary variables x_{it} correspond to the timetabling decisions where the value of 1 means the departure of i-th train at the start of the interval $[t, t+1]$. The flow variables include $b_{t\omega}$and $w_{t\omega}$which refer to the number of boarding passengers on the departing train and the number of waiting passengers at the beginning of the interval $[t, t+1]$ under scenario $\omega \in \Omega$, respectively.

The mixed-integer linear formulation is given by equations (8)-(17). The objective function has three parts. The first part is the expected value of the average waiting time per passenger (AWT) which is similar to the traditional formulation of the stochastic programming models. The second term is the variance of the cost, weighted by the parameter ϕ. The third part represents the infeasibility term due to the capacity violation, weighted by the parameter γ.

The expected value of the total waiting time is$\sum_{\omega\in\Omega}\sum_{t\in\mathcal{T}}\alpha p_{\omega}\left(w_{t\omega}+\frac{1}{2}\lambda_{t\omega}\right)$. Using the flow-oriented variables, the total waiting time of passengers is written as a linear function. It should be noticed that the terms α and $\lambda_{t\omega}$ are constants and they can be removed from the objective function. Accordingly, the scenario-dependent cost function (ξ_{ω})is formulated in equation (10). The third term of the objective function reflects the expected capacity overload, which is weighted by γ.The goal programming variables ($g_{t\omega}^{+}$and$g_{t\omega}^{-}$) are putted into formulation in order to linearize the term $\max\{0, b_{t\omega}-\sum_{i\in I}x_{it}\,.\,C\}$ in the objective function. For this purpose, equation (11)considers the positive and negative deviations from the target values. Constraint (12) states that if no train leaves at time$t$, the number of passengers boarding the train is zero. On the other hand, if a train departs at time$t$, constraint (12) establishes an upper limit on the number of passengers getting on the train. The minimum and maximum allowed headways are stated in equation (13). Constraints (14) and (15) refer to the possible departure times for each train. Equation (16) states the passenger flow preservation constraint.

$$[Model1]:\ \ Minimize\ \sum_{\omega\in\Omega} p_\omega.\xi_\omega + \phi\sum_{\omega\in\Omega} p_\omega\left\{\left(\xi_\omega - \sum_{\omega'\in\Omega} p_{\omega'}.\xi_{\omega'}\right) + 2\tau_\omega\right\} + \gamma\sum_{\omega\in\Omega}\sum_{t\in\mathcal{T}} p_\omega.g^+_{t\omega} \tag{8}$$

$s.t.$

$$\xi_\omega - \sum_{\omega'\in\Omega} p_{\omega'}.\xi_{\omega'} + \tau_\omega \geq 0 \quad \omega\in\Omega \tag{9}$$

$$\xi_\omega = \frac{\sum_{t\in\mathcal{T}}(w_{t\omega} + \frac{1}{2}\lambda_{t\omega})}{\sum_{t\in\mathcal{T}}\lambda_{t\omega}} \quad \omega\in\Omega \tag{10}$$

$$b_{t\omega} - \sum_{i\in I} x_{it}\ .C = g^+_{t\omega} - g^-_{t\omega}\ t\in\mathcal{T},\ \ \omega\in\Omega \tag{11}$$

$$b_{t\omega} \leq \sum_{i\in I}\left(x_{it} * \sum_{t'=1}^{t-1}\lambda_{t'\omega}\right) \quad t\in\mathcal{T},\ \ \omega\in\Omega \tag{12}$$

$$h_{min} \leq \sum_{t\in\mathcal{T}} \alpha.(t-1).x_{i+1,t} - \sum_{t\in\mathcal{T}} \alpha.(t-1).x_{it} \leq h_{max} i\in I \tag{13}$$

$$\sum_{t\in\mathcal{T}} x_{it} \leq 1 \qquad i\in I \tag{14}$$

$$\sum_{i\in I} x_{it} \leq 1 \qquad t\in\mathcal{T} \tag{15}$$

$$w_{t\omega} = w_{t-1,\omega} + \lambda_{t-1,\omega} - b_{t\omega} t\in\mathcal{T}, \quad \omega\in\Omega \tag{16}$$

$$b_{t\omega}, w_{t\omega}, \tau_\omega, g^+_{t\omega}, g^-_{t\omega} \quad \in\mathcal{R}^+ x_{it} \in\{0,1\} \tag{17}$$

The above robust stochastic programming model can be transformed into the deterministic equivalent formulation of the classic stochastic programming model. Here, the objective is minimizing the expected value of the average waiting time per passenger. The linear stochastic programming model is written as follows:

$$[SP1]:\ \ Minimize\ \sum_{\omega\in\Omega} p_\omega.\xi_\omega \tag{18}$$

$s.t.$

Constraints (10), (13)-(16), and

$$b_{t\omega} \leq \sum_{i\in I} x_{it}\ .C\ t\in\mathcal{T},\ \ \omega\in\Omega \tag{19}$$

$$b_{t\omega}, w_{t\omega} \quad \in\mathcal{R}^+ x_{it} \in\{0,1\} \tag{20}$$

Constraints (19) ensurethat the number of boarding don't exceed the maximum capacity of the trains. The value of stochastic solution generated by the above formulation will be analyzed in the result section.

3-3-Nonlinear model

A drawback of the linear model [$Model1$] noticeably is the relatively large number of binary variables. In contrast to the formulations using the variables indexed by time, here a new formulation is proposed that use variables indexed by train service. Using new assignment variables, a new nonlinear formulation is derived. Thus, the main difference between the linear and non-linear stochastic programming models lies in the definition of the decision variables. The non-linear model is formulated using less number of binary variables. Whereas the constraints of the nonlinear model are similar to the linear model their structure is to some extent more complicated. The objective function as presented in equation (22) includes three terms. The first and second terms are as they are defined in the linear model. The infeasibility term (third part) is used to penalize violations of the capacity constraints, capable of modifying the solution in response to variations in data under different demand scenarios. Equation (23) represents the total waiting time of passengers under scenario $\omega \in \Omega$. The goal programming variables ($g_{i\omega}^{+}$and$g_{i\omega}^{-}$) are defined to linearize the term $\max\{0, \delta_{i\omega} - C\}$ in the objective function. The positive deviation from the target values is of concern and thus the equation (24) is presented in the formulation. Constraint (25) establishes an upper limit on the number of passengers boarding a train. Constraint (26) represents the minimum and maximum headway times between train services. The planning horizon is divided into a number of periods ($p \in P$) with length θ_p that correspond to different scenario-dependent arrival rates ($\lambda_{\omega}^{(p)}$). The p-th period starts at time $t = t_p$.

The binary variables ($y_i^{(p)}$) are used to assign train services to the demand periods. These variables are associated with the departure times (d_i) through constraint (27). Equation (28) guarantees that each train service must be assigned to a specific time period. The services frequency variables (F_p) are defined as auxiliary variables. The relation between the binary variables ($y_i^{(p)}$) and the integer variable (F_p) is presented in equation (29). Let $\delta_{i\omega}$ denotes the number of passengers arriving between i-th and $i+1$-th train services under scenario $\omega \in \Omega$. An important part of the formulation is the calculation of$\delta_{i\omega}$. For this purpose, the accumulative number of passengers arrived before the departure time of i-th train under scenario $\omega \in \Omega$ is denoted by$\Delta_{i\omega}$. Simply, the relationship between accumulative demand ($\Delta_{i\omega}$) and inter-departure flows ($\delta_{i\omega}$) is expressed through equation (30). It is required to compute the number of passengers according to the arrival rates in the periods. In this regard, suppose i-th service is assigned to the period p. Consequently, the following equation computes the accumulative input flow under scenario ω:

$$\Delta_{i\omega} = \sum_{t' \in P} \left\{ \left(1 - \sum_{t=1}^{t'} y_i^{(t)} \right) . \theta_{t'} . \lambda_{\omega}^{(t')} \right\} + \left(d_i - t_{p-1} \right) . \lambda_{\omega}^{(p)} \qquad i \in I, \quad \omega \in \Omega \tag{21}$$

The flow conservation is written in equation (31). Inequalities (32) and (33) are written to linearize the above equation. Constrains (32) and (33) calculate the cumulative flow of passengers arriving to the station until the i-th departure. Likewise, equations (34)-(35) calculate the number of passengers arrived to station after the last train services, respectively. Finally, the robust train timetabling model under period-depended and uncertain demand is written as a mixed-integer non-linear programming (MINLP) model with the following equations:

$$[Model2]:\ Minimize \sum\nolimits_{\omega\in\Omega} p_\omega.\xi_\omega + \phi\sum\nolimits_{\omega\in\Omega} p_\omega\left\{\left(\xi_\omega - \sum\nolimits_{\omega'\in\Omega} p_{\omega'}.\xi_{\omega'}\right) + 2\tau_\omega\right\} + \gamma\sum\nolimits_{\omega\in\Omega}\sum\nolimits_{i\in I} p_\omega.g_{i\omega}^{+} \tag{22}$$

$s.t.$

Constraint (9),

$$\xi_\omega = \frac{\sum_{i\in I\setminus\{n\}}\left(w_{i\omega} + \frac{1}{2}\delta_{i\omega}\right).\left(d_{i+1} - d_i\right) + \frac{1}{2}\Delta_{1,\omega}.d_1 + \frac{1}{2}\delta_{n,\omega}.\left(T - d_n\right)}{\sum_{p\in P}\theta_p.\lambda_\omega^{(p)}} \quad \omega\in\Omega \tag{23}$$

$$b_{i\omega} - C = g_{i\omega}^{+} - g_{i\omega}^{-}\ i\in I,\ \ \omega\in\Omega \tag{24}$$

$$b_{i\omega} \le \Delta_{i\omega}\ i\in I,\ \ \omega\in\Omega \tag{25}$$

$$h_{min} \le d_{i+1} - d_i \le h_{max} i\in I \tag{26}$$

$$t_{p-1} - M.\left(1 - y_i^{(p)}\right) \le d_i < t_p + M.\left(1 - y_i^{(p)}\right) \quad i\in I,\ \ p\in P \tag{27}$$

$$\sum\nolimits_{p\in P} y_i^{(p)} = 1 \qquad i\in I \tag{28}$$

$$\sum\nolimits_{i\in I} y_i^{(p)} = F_p\ \ p\in P \tag{29}$$

$$\delta_{i\omega} = \Delta_{i+1,\omega} - \Delta_{i,\omega} \qquad i\in I\setminus\{n\},\ \ \omega\in\Omega \tag{30}$$

$$w_{i\omega} = w_{i-1,\omega} + \delta_{i-1,\omega} - b_{i\omega} \qquad i\in I,\ \ \omega\in\Omega \tag{31}$$

$$\Delta_{i\omega} \le \sum\nolimits_{t'\in P}\left\{\left(1 - \sum\nolimits_{t=1}^{t'} y_i^{(t)}\right).\theta_{t'}.\lambda_\omega^{(t')}\right\} + \left(d_i - t_{p-1}\right).\lambda_\omega^{(p)} + M.\left(1 - y_i^{(p)}\right) i\in I,\ \ p\in P,\ \ \omega\in\Omega \tag{32}$$

$$\Delta_{i\omega} \ge \sum\nolimits_{t'\in P}\left\{\left(1 - \sum\nolimits_{t=1}^{t'} y_i^{(t)}\right).\theta_{t'}.\lambda_\omega^{(t')}\right\} + \left(d_i - t_{p-1}\right).\lambda_\omega^{(p)} + M.\left(1 - y_i^{(p)}\right) \quad i\in I,\ \ p\in P,\ \ \omega\in\Omega \tag{33}$$

$$\delta_{n\omega} \le \sum\nolimits_{t'\in P}\left\{\left(1 - \sum\nolimits_{t=t'}^{NT} y_n^{(t)}\right).\theta_{t'}.\lambda_\omega^{(t')}\right\} + \left(t_p - d_n\right).\lambda_\omega^{(p)} + M.\left(1 - y_n^{(p)}\right)\ p\in P,\ \ \omega\in\Omega \tag{34}$$

$$\delta_{n\omega} \ge \sum\nolimits_{t'\in P}\left\{\left(1 - \sum\nolimits_{t=t'}^{NT} y_n^{(t)}\right).\theta_{t'}.\lambda_\omega^{(t')}\right\} + \left(t_p - d_n\right).\lambda_\omega^{(p)} + M.\left(1 - y_n^{(p)}\right)\ p\in P,\ \ \omega\in\Omega \tag{35}$$

$$d_i\ ,\delta_{i\omega}, b_{i\omega}, w_{i\omega}, \Delta_{i\omega}, \tau_\omega, g_{i\omega}^{+}, g_{i\omega}^{-} \in \mathcal{R}^{+} y_i^{(p)} \in \{0,1\} \tag{36}$$

One may want to optimize train schedules subject to strict capacity constraints in order to generate feasible solutions in all scenarios. Thus, the nonlinear stochastic programming model with expected value objective function is formulated as follows:

$$[SP2]:\ Minimize\ \ z = \sum_{\omega \in \Omega} p_\omega . \xi_\omega \tag{37}$$

$s.t.$

Constraints (23), (26)-(35), and

$$b_{i\omega} \le C\ i \in I,\ \ \omega \in \Omega \tag{38}$$

$$d_i\ , \delta_{i\omega}, b_{i\omega}, w_{i\omega}, \Delta_{i\omega} \in \mathcal{R}^+ y_i^{(p)} \in \{0,1\} \tag{39}$$

Constraints (38) impose an upper limit on the number of boarding. As concluding note, the complexity of the proposed robust stochastic programming formulations is not only depends on the number of variables and constraints but also the inherent complexity of the train scheduling problem. In what follows, the size of the proposed mathematical models is formally quantified. The linear model requires $\|\mathcal{T}\| * \|I\|$ number of binary variables and $2.(\|\Omega\| + \|I\|) + 3.\|\mathcal{T}\|.\|\Omega\| + \|\mathcal{T}\|$constraints. On the other hand, the non-linear model requires $\|P\| * \|I\|$ binary variables and $3.\|\Omega\| + 4.(\|I\| + \|P\|).\|\Omega\| + 2.\|I\| + \|P\|.\|I\| + \|P\| + 2.\|I\|.\|\Omega\|.\|P\|$number of constraints. The complexity of the proposed models depends on number of train services ($\|I\|$), length of the planning horizon and the number of scenarios ($\|\Omega\|$). The number of train services and the discretization level (α) have great impacts on the complexity of the models. Regularly, the integer linear programming model requires more binary variables compared with the non-linear model. This indicates the difficulty of solving the problem via the presented linear stochastic integer programming model for the large-sized instances. On the other hand, the nonlinear integer programming models are inherently more difficult to solve. It should be noted that the train timetabling problem under dynamic demand was proved as a NP-hard problem (Sun et al. 2014). In the next section, we conduct computational experiments on robust stochastic programming models that provide useful insight on the performance and tractability of the models.

3-4-Computational experiments on illustrative examples

In this section, the objective is to quantify the potential benefits of the robust optimization approach using the analytical result obtained under the mathematical model developed above, on a hypothetical rail system. Further, we investigate the efficiency and flexibility of the proposed robust and pure stochastic programming models via numerical examples. In the first case, the detail data of demand scenarios and their summarized information are provided in Table 1 and Table 2, respectively. The arrival profiles are generated randomly with a non-convex function including two demand peaks which is close to real-world situation. The minimum (h_{min}) and maximum (h_{max}) headways are 1 and 5 minutes, respectively. Random patterns for travel demand are defined in three optimistic, pessimistic and most likely scenarios ($\|\Omega\|$=3). In every scenario the travel demand consisting of two peaks. In this example, it is assumed that in worse scenarios (2 and 3) the rate of arrival is more than the optimistic scenario (see Figure 1). The goal is to construct a robust timetable that minimizes the expected and variance of average waiting time per passenger. The computational experiments are performed on a personal computer with 2.5 GHz Intel core processor and 2GB memory which running on Windows 7 platform.

Table 1. The arrival rate scenarios, λ_t (passenger per minutes)

Period (t)	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**	**14**	**15**
Scenario 1	3.5	8.8	11.4	13.1	20.5	25.1	33.5	34.6	32.4	31.3	30.6	28.1	25	25	21.5
Scenario 2	11.1	10.4	23.7	21.8	25.6	30.9	34.7	38.9	35.2	41.9	36.2	33.6	29.1	29.8	31.5
Scenario 3	14.6	24.5	37.5	35.1	36.3	31.9	36.4	44.1	37.8	53.5	46.5	44.5	36.4	41.1	42.4
Period (t)	**16**	**17**	**18**	**19**	**20**	**21**	**22**	**23**	**24**	**25**	**26**	**27**	**28**	**29**	**30**
Scenario 1	13	15.8	21.3	22.3	27.4	31	35.5	37.4	37	35.1	31.5	26.1	22	7.9	5.5
Scenario 2	25.2	27.8	33.1	27.6	38.8	40.9	40.2	44.9	39.1	41.9	39.2	35.8	26	11	18.7
Scenario 3	29.2	42.4	33.4	33.1	38.9	52.6	54	59.4	45.2	55.2	53.7	48.1	33.6	12.5	21.3

Table 2. The summarized information of demand scenarios

Scenarios	Likelihood (p_ω)	Demand (number of passengers)	Minimum arrival rate (passenger per minutes)	Maximum arrival rate (passenger per minutes)	Average arrival rate (passenger per minutes)
1	0.2	713.2	3.5	37.4	23.8
2	0.5	924.6	10.4	44.9	30.8
3	0.3	1175	12.5	59.4	39.2

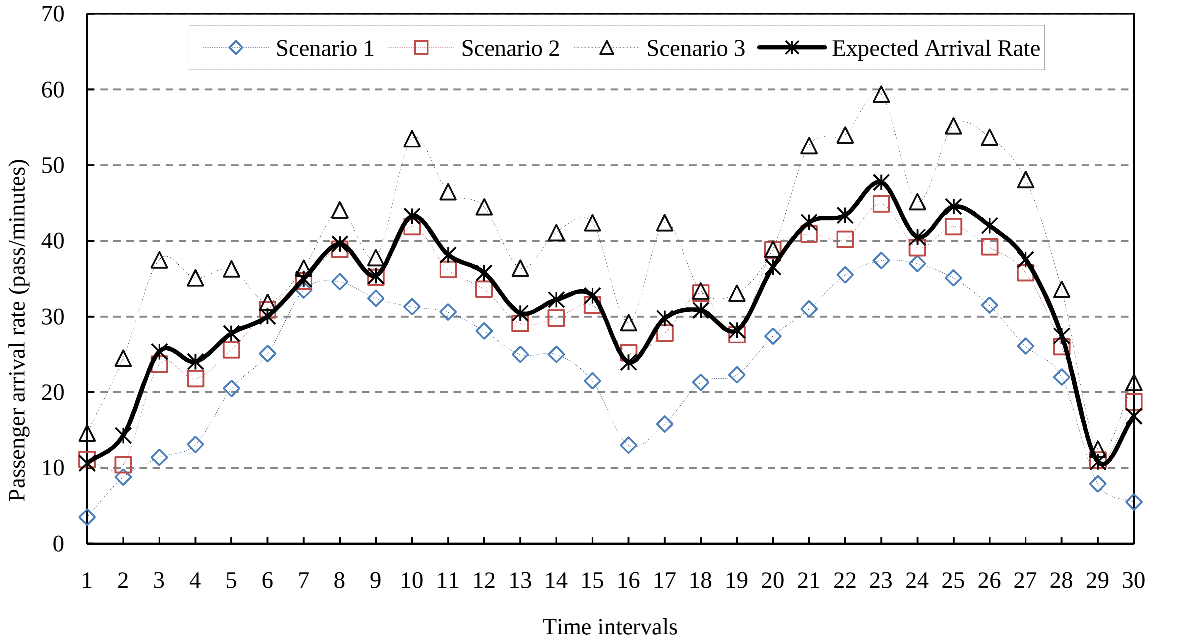

Fig.1. The demand scenarios in the numerical example

3-4-1-Expected value minimization

In this section, the objective is to find the solution of stochastic programming models with the aim of minimizing the expected value of the average waiting time per passenger. The linear and non-linear stochastic programming models ([$SP1$]and[$SP2$]) are solved using CPLEX and DICOPT/GAMS solvers, respectively. The computational results of the linear and non-linear models are given in Table 3, where the both expected value and variances of the cost are reported. As expected, the average cost reduces as the number of train services (n) increases. However, the variance of cost increases first and then decreases with the increase of the number of services (Figure 2). Overall, the non-linear model outperforms the linear model in terms of the mean and variance of cost. In the present example, the nonlinear model generates efficient and robust solutions where the relative reduction in expected waiting time and cost variance are 13.38% and 90.7% on average, respectively. On the other hand, more computational effort is required to solve the nonlinear model compared to the linear model. The average computational time of the nonlinear model is less than 4 minutes. The important observation is the highest CPU time belongs to the case with n=17 trains. As can be seen, the CPU time decreases by scheduling more or less train services.

Table 3. The computational results of the pure stochastic programming models in example 1 (C=40)

n	**Linear model** $[\boldsymbol{SP1}]$			**Nonlinear model** $[\boldsymbol{SP2}]$		
	$\mathbb{E}(\boldsymbol{AWT})$	**S.D.**	**CPU time (sec)**	$\mathbb{E}(\boldsymbol{AWT})$	**S.D.**	**CPU time (sec)**
10	2.8459	0.96	0.43	2.5517	0.17	20.4
11	2.6155	1.13	0.32	2.3410	0.19	32.15
12	2.4608	1.17	0.42	2.1797	0.18	29.53
13	2.1940	1.14	0.54	2.0880	0.14	30.04
14	2.0443	1.21	0.57	1.9221	0.14	46.67
15	1.9337	1.16	0.54	1.6568	0.32	236.60
16	1.7698	1.21	0.46	1.5379	0.30	601.63
17	1.6693	1.18	0.67	1.4234	0.31	804.83
18	1.5232	1.22	0.68	1.3151	0.30	322.87
19	1.4274	1.26	0.65	1.2110	0.27	209.89
20	1.3834	0.87	0.51	1.1094	0.23	299.91
21	1.3392	0.90	0.59	1.0149	0.19	194.61
22	1.1639	0.65	0.62	0.9318	0.18	82.08
23	1.0425	0.45	0.54	0.8601	0.21	48.57
24	0.9360	0.47	0.51	0.8180	0.18	33.21
25	0.7992	0.28	0.20	0.7512	0.20	51.90
Average	1.69	0.95	0.52	1.48	0.22	190.30

S.D.: Standard deviation of cost (minutes)

3-4-2-Convergence analysis

For convergence test, many numbers of scenarios is generated randomly, and the corresponding models are solved optimally. It is assumed that the arrival rates follow uniform probability distribution. The parameters of the probability distributions are given for different time-intervals (Table 4). The performance of the solutions is simulated via sampling techniques and the optimality gap is estimated. The mean, minimum, and maximum values of the linear stochastic programming model are provided in Table 5. Figure 3 illustrates the gap between the maximum and minimum values which decreases progressively by increasing the number of scenarios ($|\Omega|$). It shows the convergence of the solution of the stochastic programming model as the sample size is large enough. Expectedly, the results demonstrate that the average CPU time grows exponentially by increasing the number of scenarios. Furthermore, the variance of the cost decreases by increasing the sample size.

Table 4. The probability distribution of the arrival rate (passenger per minutes)

Interval	$\mathbf{1 \le t \le 5}$	$\mathbf{6 \le t \le 10}$	$\mathbf{11 \le t \le 15}$	$\mathbf{16 \le t \le 20}$	$\mathbf{21 \le t \le 25}$	$\mathbf{26 \le t \le 30}$
Uniform distribution	U[0,10]	U[10,30]	U[30,50]	U[10,30]	U[35,50]	U[5,15]

Table 5. Computational results of test instances under different numbers of scenarios (n=15)

Number of scenarios (|Ω|)	Mean value	Standard deviation	Minimum value	Maximum value	Relative gap (max–min)/min	Average CPU time (s)
5	1.680	0.124	1.514	1.851	22.26%	0.066
10	1.761	0.109	1.603	1.915	19.46%	0.270
20	1.803	0.071	1.675	1.912	14.15%	0.432
50	1.794	0.040	1.719	1.881	9.42%	2.451
75	1.795	0.030	1.732	1.872	8.08%	3.482
100	1.798	0.031	1.730	1.894	9.48%	5.598
150	1.802	0.023	1.728	1.856	7.41%	12.890
200	1.801	0.020	1.762	1.855	5.28%	25.515
250	1.799	0.020	1.753	1.857	5.93%	47.506
500	1.799	0.013	1.754	1.840	4.90%	195.603

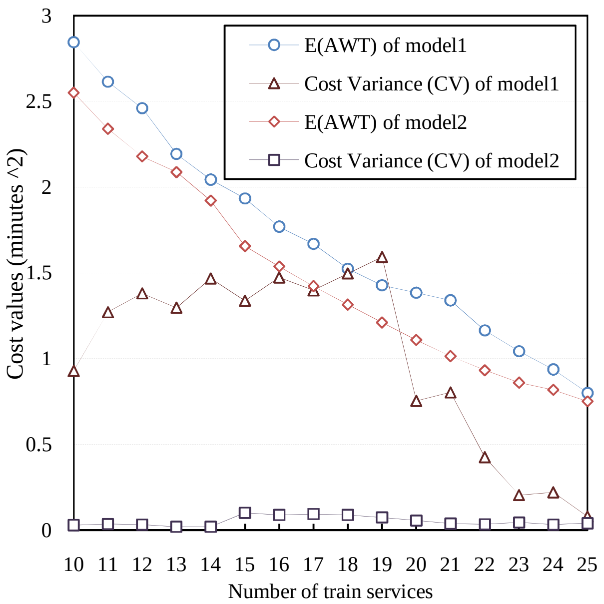

Fig. 2. Sensitivity analysis of the stochastic programming models regarding the number of train services

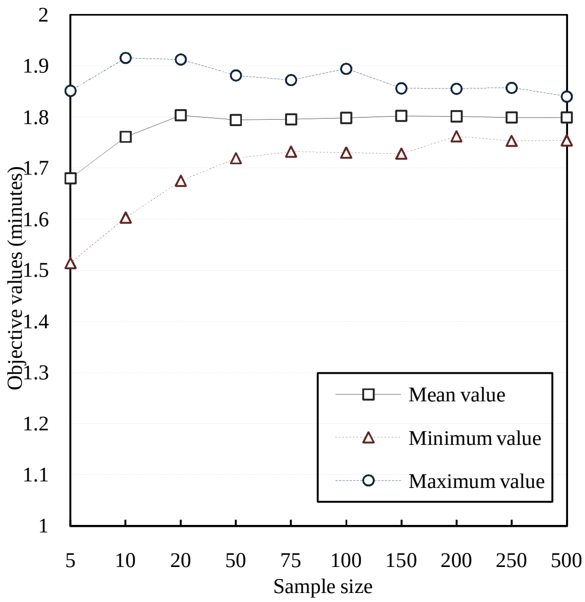

Fig. 3. Convergence of the objective values with the number of scenarios increasing

3-4-3-Value of stochastic solution

In order to evaluate the efficiency of the proposed stochastic programming models, their result are compared with those of a deterministic model, which replaces the stochastic parameters ($\lambda_{\omega}^{(p)}$) by using their expected values ($\bar{\lambda}^{(p)}$) or the nominal scenario. More specifically, the expected value models work with a single expected scenario. According to the notation given in Section 3.2, the average arrival rate is calculated according to the scenario probabilities:

$$\bar{\lambda}_t = \sum_{\omega \in \Omega} p_{\omega} . \lambda_{t\omega} \; t \in \mathcal{T} \tag{40}$$

Similarly, the expected arrival rate of passenger is obtained for $[Model2]$ with the following equation:

$$\bar{\lambda}^{(p)} = \sum_{\omega \in \Omega} p_{\omega} . \lambda_{\omega}^{(p)} \; p \in P \tag{41}$$

Here, the deterministic equivalent models are referred to as **[EV_MODEL1]**and**[EV_MODEL2]**, which are written as follows:

$$[\textbf{EV\_MODEL1}]: \; Minimize \;\; z = \frac{\sum_{t \in \mathcal{T}} \left(w_t + \frac{1}{2} \bar{\lambda}_t \right)}{\sum_{t \in \mathcal{T}} \bar{\lambda}_t} + \gamma \sum_{t \in \mathcal{T}} g_t^{+} \tag{42}$$

$s.t.$

Constraints (13)-(15), and

$$b_t - \sum_{i \in I} x_{it} . C = g_t^{+} - g_t^{-} \; t \in \mathcal{T} \tag{43}$$

$$b_t \leq \sum_{i \in I} \left(x_{it} * \sum_{t'=1}^{t-1} \bar{\lambda}_{t'} \right) \tag{44}$$

$$w_t = w_{t-1} + \bar{\lambda}_{t-1} - b_t \; t \in \mathcal{T} \tag{45}$$

$$b_t , w_t , g_t^{+}, g_t^{-} \in \mathcal{R}^{+} x_{it} \in \{0,1\} \tag{46}$$

$$[\textbf{EV\_MODEL2}]: \; Minimize \;\; z = \frac{\sum_{i \in I \setminus \{n\}} \left(w_i + \frac{1}{2} \delta_i \right) . \left(d_{i+1} - d_i \right) + \frac{1}{2} \Delta_1 . d_1 + \frac{1}{2} \delta_n . \left(T - d_n \right)}{\sum_{p \in P} \theta_p . \bar{\lambda}^{(p)}} + \gamma \sum_{i \in I} g_i^{+} \tag{47}$$

$s.t.$

Constraints (26)-(29), and

$$\delta_i - C = g_i^{+} - g_i^{-} \; i \in I \tag{48}$$

$$\delta_i = \Delta_{i+1} - \Delta_i \qquad i \in I \setminus \{n\} \tag{49}$$

$$w_i = w_{i-1} + \delta_{i-1} - b_i \; i \in I \setminus \{1\}$$

$$\Delta_i \leq \sum_{t' \in P} \left\{ \left(1 - \sum_{t=1}^{t'} y_i^{(t)} \right) . \theta_{t'} . \bar{\lambda}^{(t')} \right\} + \left(d_i - t_{p-1} \right) . \bar{\lambda}^{(p)} + M . \left(1 - y_i^{(p)} \right) i \in I, \; p \in P \tag{50}$$

$$\Delta_i \geq \sum_{t' \in P} \left\{ \left(1 - \sum_{t=1}^{t'} y_i^{(t)} \right) . \theta_{t'} . \bar{\lambda}^{(t')} \right\} + \left(d_i - t_{p-1} \right) . \bar{\lambda}^{(p)} + M . \left(1 - y_i^{(p)} \right) i \in I, \; p \in P \tag{51}$$

$$\delta_n \leq \sum_{t' \in P} \left\{ \left(1 - \sum_{t=t'}^{NT} y_n^{(t)} \right) . \theta_{t'} . \bar{\lambda}^{(t')} \right\} + \left(t_p - d_n \right) . \bar{\lambda}^{(p)} + M . \left(1 - y_n^{(p)} \right) \; p \in P \tag{52}$$

$$\delta_n \geq \sum_{t' \in P} \left\{ \left(1 - \sum_{t=t'}^{NT} y_n^{(t)} \right) . \theta_{t'} . \bar{\lambda}^{(t')} \right\} + \left(t_p - d_n \right) . \bar{\lambda}^{(p)} + M . \left(1 - y_n^{(p)} \right) \; p \in P \tag{53}$$

$$d_i , \delta_i , b_i , \Delta_i, g_i^{+}, g_i^{-} \in \mathcal{R}^{+} y_i^{(p)} \in \{0,1\} \tag{54}$$

An optimal solution to above-mentioned deterministic equivalent models may result in overloaded trains when the uncertain demands are realized. However, the obtained solution can be regarded as a basis to evaluate the value of the stochastic solution versus the determinist solution. The value of the stochastic solution (VSS) is defined as difference between the objective function value of the solution chosen based on the decision regarding the expected value of the uncertain variables over all scenarios and the optimal value of the stochastic model. It measures the advantage gained if a stochastic model is utilized instead of a deterministic one (Birge 1982). In our implementation, the optimal solution of the deterministic model including the optimal departure times (d_{EV}^*) is an input to the objective function of the main stochastic programming models, namely$Z_{SP}(d_{EV}^*)$. It means that the uncertainty of arrival rates is disregarded however the expected values are utilized for decision making. In this regard, the **EV_MODEL1** is used to find optimal departure times (d_{EV}^*).

The objective value $Z_{SP}(d_{EV}^*)$is regarded as a basis to compare with the optimal objective value of the stochastic programming models, i.e. $Z_{SP}(d_{SP}^*)$ which assesses the performance of the decision in a realistic uncertain environment. Formally, the value of stochastic solution refers to the cost associated with the ignorance of uncertainty when making a decision (Avriel and Williams 1970). It calculates the difference between $Z_{SP}(d_{SP}^*)$ and$Z_{SP}(d_{EV}^*)$ as follows:

$$VSS = Z_{SP}(d_{EV}^*) - Z_{SP}(d_{SP}^*) \tag{55}$$

In what follows, the computational results are provided to show the advantage of the proposed stochastic programming model and examine the value of stochastic solution under different level of uncertainty and model parameters. Consider the case$\phi = 0$. First, we examine the value of stochastic solution under different size of the problem. The results are summarized in Table 6, where the objective values of the solutions with expected demand are provided. The outcomes indicate that the VSS decreases with increasing the number of train services (Figure 4).

Table 6.The value of stochastic solution of the linear SP model regarding the number of train services (C=40)

$\boldsymbol{n}$	$\boldsymbol{\gamma}$	$\boldsymbol{Z_{SP}(d_{EV}^*)}$	$\boldsymbol{Z_{SP}(d_{SP}^*)}$	$\boldsymbol{VSS}$
10	0.0282	6.5943	2.8459	3.7484
11	0.0282	5.8458	2.6155	3.2303
12	0.0179	5.1931	2.4608	2.7323
13	0.0177	4.5399	2.1940	2.3459
14	0.0158	4.0764	2.0443	2.0321
15	0.0155	3.5694	1.9337	1.6357
16	0.0151	3.1857	1.7698	1.4159
17	0.0149	2.8209	1.6693	1.1516
18	0.0144	2.4865	1.5232	0.9633
19	0.0140	2.1902	1.4274	0.7628
20	0.0136	1.9438	1.3834	0.5604
21	0.0130	1.7811	1.3392	0.4419
22	0.0103	1.5291	1.1639	0.3652
23	0.0098	1.4116	1.0425	0.3691
24	0.0095	1.1164	0.9360	0.1804
25	0.0094	1.0534	0.7992	0.2542

A set of experiments are also accomplished using the linear model to examine the different level of uncertainty, which are represented by the average range of uncertain arrival rate, i.e.$(\lambda_t^{max} + \lambda_t^{min})/2$. All computations are performed on the test instances of example 1 with $n = 10$ train services and 25 scenarios. The stochastic parameters ($\lambda_{t\omega}$) are randomly generated following uniform probability distributions $U[\lambda_t^{max}, \lambda_t^{min}]$. The results indicate that the proposed stochastic programming model can find decisions with high value of stochastic solution when the uncertainty in the arrival rate increases (Table 7). The result gives evidence that the higher is the degree of uncertainty, the more advantage could be gained by using the proposed stochastic programming model (Figure 5).

Table 7. The value of stochastic solution regarding the average range of uncertain arrival rate ($n = 10, |\Omega| = 25$)

$\boldsymbol{Uncertaintydegree}$	$\boldsymbol{Z_{SP}(d^*_{EV})}$	$\boldsymbol{Z_{SP}(d^*_{SP})}$	$\boldsymbol{VSS = Z_{SP}(d^*_{EV}) - Z_{SP}(d^*_{SP})}$	$\boldsymbol{VSS\%} = \frac{\left(\boldsymbol{Z_{SP}(d^*_{EV}) - Z_{SP}(d^*_{SP})}\right)}{\boldsymbol{Z_{SP}(d^*_{EV})}}\%$
12	1.6065	1.5716	0.035	2.22%
12.5	1.5212	1.4958	0.025	1.70%
13	1.7723	1.6921	0.080	4.74%
13.5	1.8526	1.6911	0.162	9.55%
14	2.0045	1.7573	0.247	14.07%
14.5	2.1266	1.8728	0.254	13.55%
15	2.2397	1.9847	0.255	12.85%

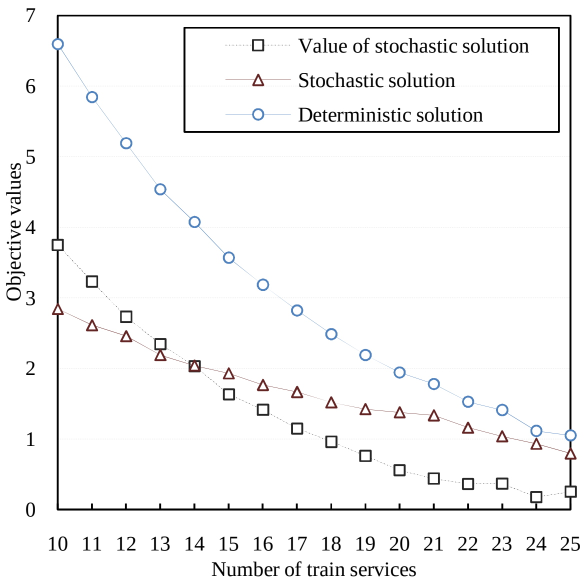

Fig. 4. The value of stochastic solution regarding the number of train services

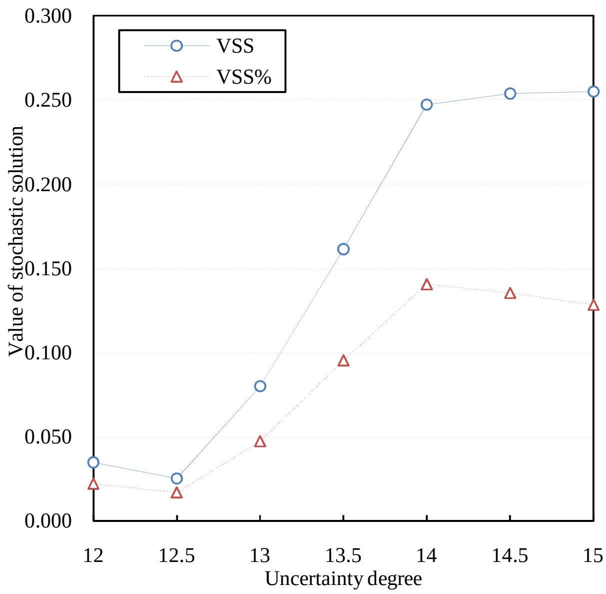

Fig. 5. The value of stochastic solution regarding the uncertainty degree

3-4-4-Trade off between solution robustness and model robustness

It is valuable to find trade-offs between the expected waiting time of passenger, variation of passenger waiting time and the expected capacity overload. As mentioned earlier, the role of weight γ in the objective function; equations (8) and (22), is to find a trade-off between solution robustness and model (or quality) robustness for train timetables. The empirical approach to determine the value of trade-off parameters is quite common in robust optimization method(Liao et al. 2013). Thus, we shall let γ vary within the sensitivity range limits ($0.00 \leq \gamma \leq 0.05$) and observe the performance of the robust solution.

As discussed in subsection 3.1, parameter ϕ specifies the weight considered on the solution variance wherein the solution is less sensitive to change in data under all scenarios as ϕ increases.A sensitivity analysis is conducted to evaluate the solution and model robustness with respect to the weight of gamma. The solution robustness of the linear model is computed according to the different value of parameter ϕ (Figure 6). The value of expected cost increases first and then stays relatively constant at higher value of weight γ. However, the solution robustness of the model does not differ noticeably regarding the different value of weight ϕ.The outcomes illustrates that the nonlinear model generates solutions with lower expected cost thus indicating the nonlinear model delivers a high solution robustness compared to the linear model (Fig. **7**).As described by Mulvey et al. (1995), the variance of cost (CV) is measured with the following equation:

$$CV = \sum_{\omega \in \Omega} p_\omega \left(\xi_\omega - \sum_{\omega' \in \Omega} p_{\omega'} . \xi_{\omega'} \right)^2 \tag{56}$$

Figure 8 and Figure 9 show the standard deviation of cost versus values of weight γ. As the weight of infeasibility norm increases the cost variance decreases. The cost variance is zero for the cases when $\phi \geq 2$. The result indicates the lower variability for the solutions obtained from the non-linear model than the linear model with the same value of trade-off parameters. Figure 10 demonstrates that the penalty cost gradually decreases to zero with an increase in the value of weight γ. The outcomes direct the decision maker towards a robust solution through choosing the appropriate value for the weight γ.

Overall, as the weightγincreases, the expected cost increase (or solution robustness decreases), and on the other hand the model robustness increases. In other words, for larger values of γ, the generated solution is nearly feasible for any realization of the scenarios with the cost of additional waiting time. Therefore, the outcomes are consistent with the robust optimization perspectives. The outcomes indicates that the nonlinear model generates solutions with less expected cost of waiting time and consequently higher solution robustness compared to the linear model. The solution obtained from the nonlinear model also exhibit less variance of cost.

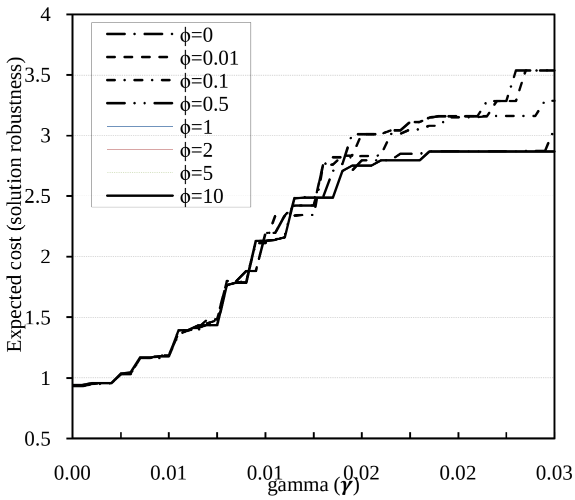

Fig. 6. Solution robustness with respect to γ values (linear model)

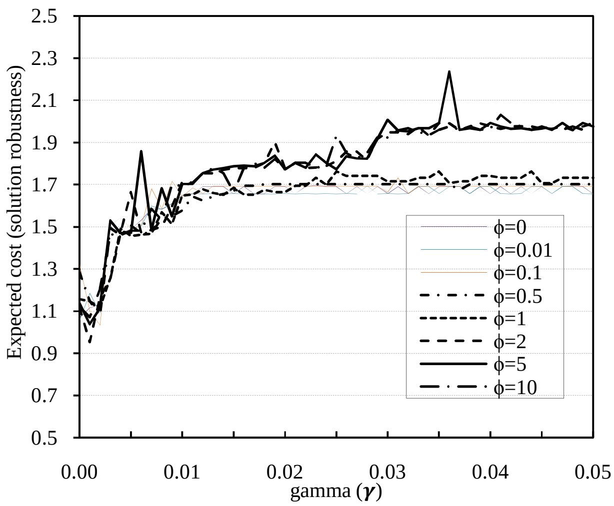

Fig. 7. Solution robustness with respect to γ values (non-linear model)

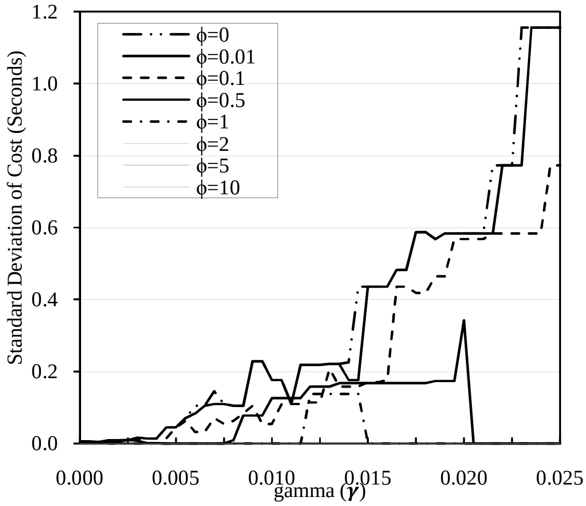

Fig. 8. Standard deviation of cost with respect to γ values (linear model)

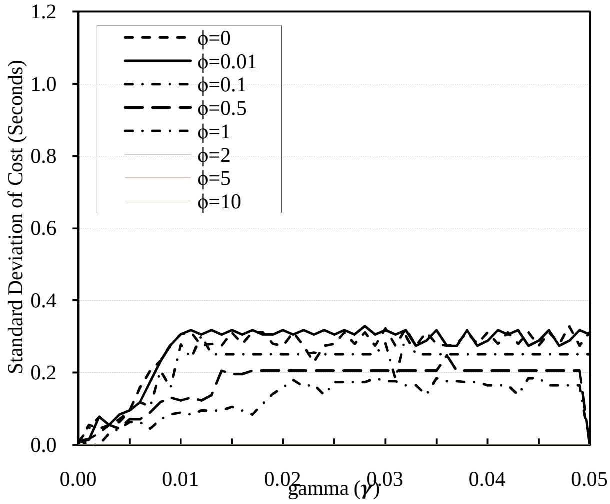

Fig. 9. Standard deviation of cost with respect to γ values (non-linear model)

Robust stochastic programming models are also used to test model robustness with respect to parameterγ (Fig. **10** and Fig. **11**). As can be seen, higher value for weight γ is less likely to capacity overload. The model robustness that measures the infeasibility of the generated solutions is decreased by increasing the weight of error term. While, the model robustness was improved more rapidly using the nonlinear model which is indicative for high accuracy of the model. Furthermore, the weight of cost variance (ϕ) influences the model robustness. Technically speaking, the model robustness increases by decreasing the importance of cost variance. Results from the numerical experimentation show that the nonlinear stochastic programming model has superior performance in terms of both the solution and model robustness.

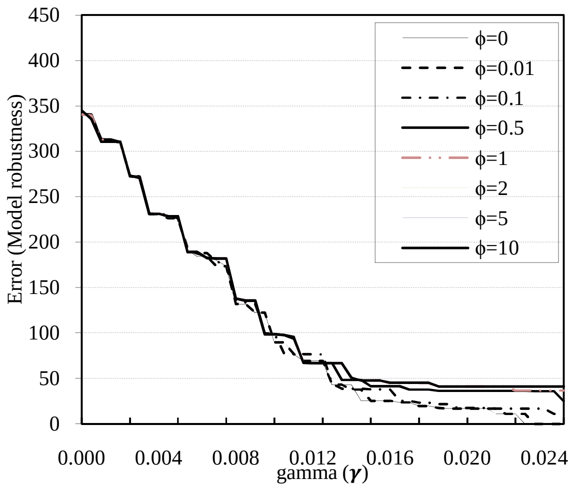

Fig. 10. Model robustness with respect to γ values (linear model)

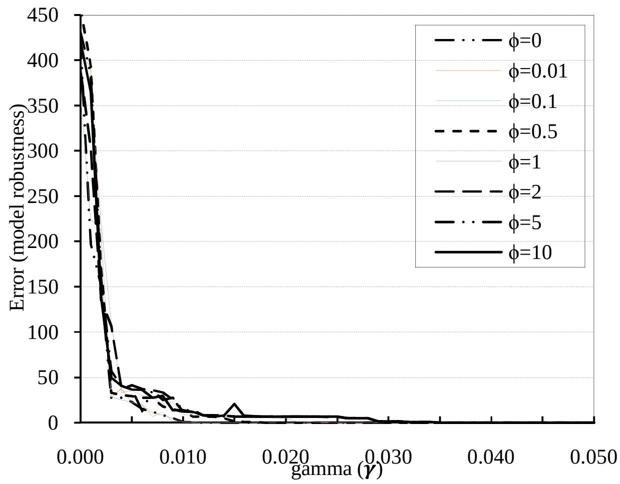

Fig. 11. Model robustness with respect to γ values (nonlinear model)

4-Real world implementation

In this section, a discussion on two different applications are presented where demand uncertainty is dominant and it is handled using the stochastic programming models introduced in Section 3. The computational experiments are conducted for realistic cases drawn from metropolitan network in Tehran. The objectives are to determine robust solutions for the realistically sized problem instances. The stochastic programming models are solved to find the optimal headway times under stochastic arrival rate of passengers. Two important public transit terminals of Tehran-Karaj subway line are considered for timetable optimization problem. For the numerical analysis, the normal train capacity equals to $C = 500$ passengers and the scheduled minimum and maximum train headways are $h_{min} = 7$ and $h_{max} = 25$ minutes, respectively. The required data for the experimentation are collected from the APC system obtained from Tehran suburban railway. The demand profiles in the case study were collected during the month October 2014. The study period is between 5:00AM to 12:00AMconsisting of7 hourly intervals. The current headways were planned manually by the rail experts (Figure 12).

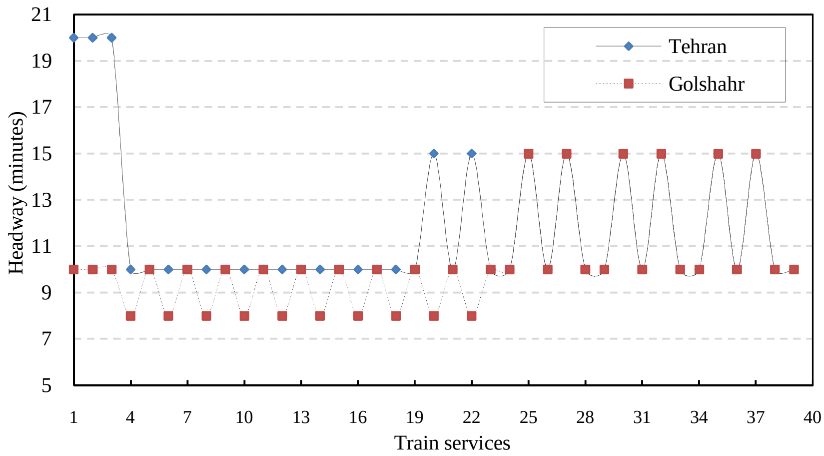

Fig.12. Baseline train headways at Tehran and Golshahr terminals

Due to the difficulties that can arise with stochastic demand, the rail planners construct the baseline headway with the expected demand and ignore its stochastic variations. Here, the stochastic programming models are implemented to improve the robustness of the timetable.

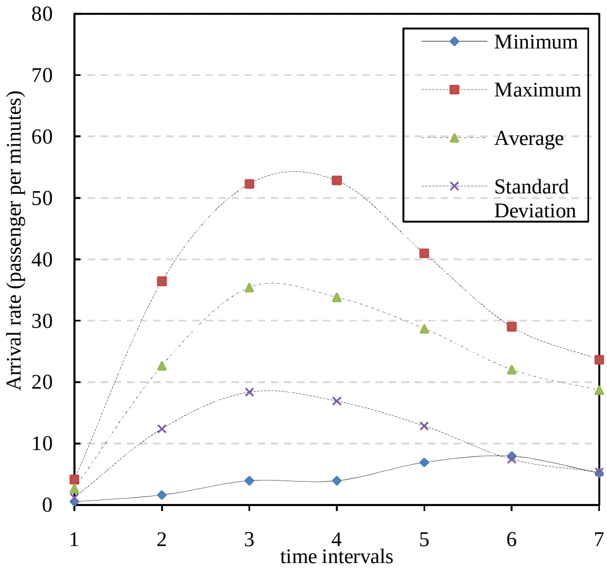

Fig. 13. Arrival rate scenarios at Tehran station (Case 1)

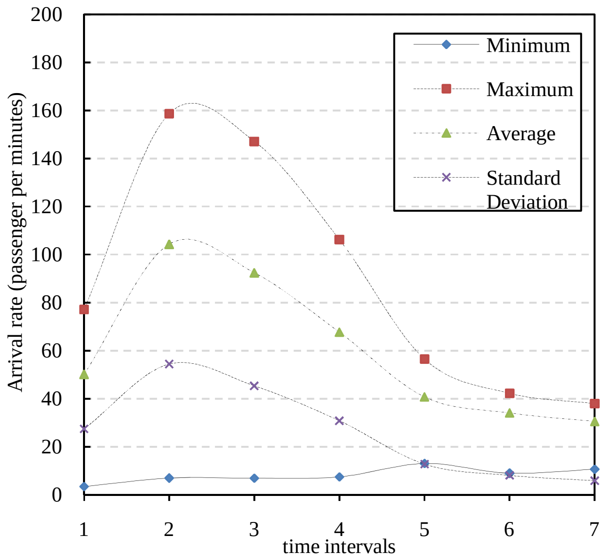

Fig. 14. Arrival rate scenarios at Golshahr station (Case 2)

Computational experiments are conducted to evaluate the solutions obtained from the classic approach of stochastic programming and its robust versions. First, we attempt to minimize the expected waiting time in which a predefined number of scenarios including the minimum, maximum and the average arrival rate patterns are available. In the computational experiments we use real data from a Tehran underground rail which exhibits a representative time-of-day demand seasonality. Figure 13 and Figure 14 show the arrival rates as a function of the time intervals in the day. Alongside the average arrival rate, two extreme samples corresponding to busy and not busy days are represented. The arrival rate at the beginning and at the end of the period of study is relatively low, demonstrates a high peak in the morning. Due to the high amount of variability in arrival rates that occurs in the peak demand hours, we are expecting to gain significant improvements after implementing the robust optimization models.

The results of expected value minimization are given in Table 8 and Table 9. We report the expected waiting time of the passenger,$\mathbb{E}(\mathrm{AWT})$,variance of cost (CV), lower and upper bounds of the expected waiting time, and the computational time. The results illustrate significant improvements of 17.7% and 93.45% in the expected value and the variance of average waiting time of passengers serving at Tehran station, respectively, compared to the baseline schedule (Table 8).This verifies that the solution of robust optimization model is less sensitive to the variation of the demand. Moreover, the expected waiting time of passengers serving at Golshahr station is reduced 23.59% (through linear model) and 26.04% (through nonlinear model), on average, compared to the baseline schedule. The quality of the solution obtained from the linear and nonlinear models are slightly different in two cases. The linear model generated solutions with fewer expected cost (0.24% on average) in case 1. On the other hand, the expected cost of the solution obtained from nonlinear model (in case 2) is reduced by 3.22% on average, compared to the linear model. The improvement obtained from the stochastic programming approach strengthens the importance of accounting for arrival rate uncertainties in the timetable design of public transportation systems. Moreover, the result indicates that the computational time of the nonlinear model was substantially lower than linear model which exhibits the intractability of the linear model dealing with large-sized instances. In fact, the convex structure of the demand function significantly influences the computational time of non-linear model. To conclude the above discussion, the outcomes demonstrate that the performance of the baseline timetable is to a certain extent poor while robust stochastic programming models generate timetables with lower average and variance of cost.

Table 8.The computational results for minimizing expected waiting time(Case1: Tehran station)

Number of train services (n)	Baseline schedule		Linear Model $[SP1]$				Nonlinear Model $[SP2]$		
	$\mathbb{E}(AWT)$	S.D.	Upper bound	Lower bound	Gap%	CPU time (sec)	Upper bound	S.D.	CPU time (sec)
30	9.4312	0.84	6.6857	4.3752	52.81	65476	6.5418	0.14	13.556
35	6.0247	0.17	5.4351	3.2401	67.74	16488	5.4576	0.04	36.023
40	5.537	0.18	4.7536	2.9465	61.33	31125	4.8018	0.05	7.473

S.D.: Standard deviation of cost (minutes)

Table 9.The computational results for minimizing expected waiting time (Case2: Golshahr station)

Number of train services (n)	Baseline schedule		Linear Model $[SP1]$			Nonlinear Model $[SP2]$		
	$\mathbb{E}(AWT)$	S.D.	Upper bound	Lower bound	CPU time (sec)	Upper bound	S.D.	CPU time (sec)
30	20.2633	2.92	12.299	12.299	50.26	11.8794	2.22	4.868
35	13.2497	2.03	10.9595	10.9595	76.26	10.6584	2.28	3.103
40	11.8402	3.88	10.1627	10.1627	620.50	9.8068	2.33	7.815

It is remarkable to study how the variance of the cost would change the solution and model robustness. Therefore, in order to gain further insight, we present Figure 16 and Figure 15 to show the trade-off between expected cost (i.e., solution robustness) and expected overload in (i.e., model robustness) for the solution obtained using the nonlinear model through increasing weight gamma in the robust optimization model. The computation experiments demonstrate that the expected cost increases gradually by increasing the value of Gamma and then converges to a maximum level (see Figure 16). Also, Fig.15 shows that, with the increase of weight Gamma, the capacity violation (expected error) decreases, gradually to zero. The performance of the robust stochastic programming model with a specific setting for variability weight ($\phi = 0.01$) indicates that the expected overloading can be reduced by over 44.58% with only a 12.19% efficiency loss or increase in the expected waiting cost. According to the experimental results, for a specific range of parameter $\gamma \leq 0.0007$, theexpected waiting cost and the expected capacity overload are almost the same overall values of ϕ. For the higher values of the parameter γ, the expected waiting cost and the expected overload can be controlled by changing the value of ϕ. Although, the observed trends demonstrate the optimal policy is moving toward the conservative direction. Also, it can be observed that as the weight of cost variability (ϕ) increases, the solution robustness improves but the model robustness degrades.

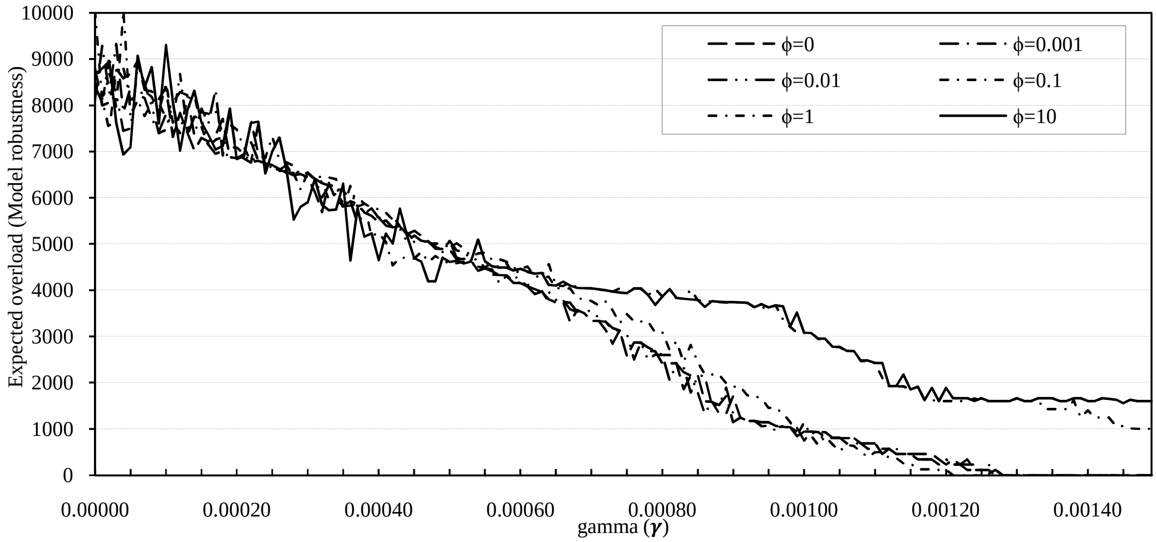

Fig.15. Model robustness with respect to γ values

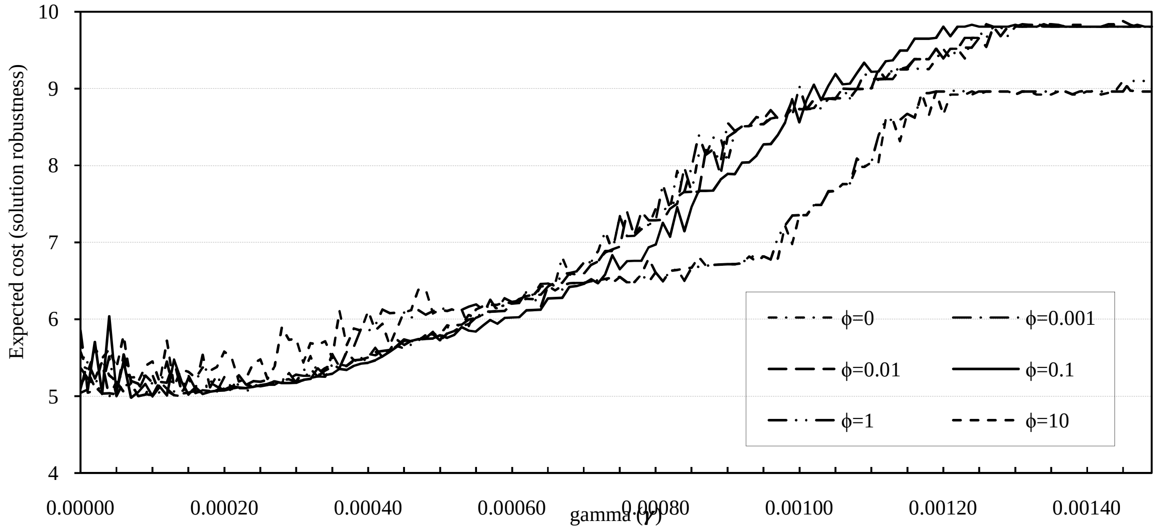

Fig.16. Solution robustness with respect to γ values

A sensitivity analysis is performed to obtain the cost variance against the multipliers of the model and solution robustness. As Figure 17 demonstrates, the cost variance grows exponentially and then converges to nearly 2.25minutes with an increase in the value of Gamma, where $\phi \in \{0, 0.001, 0.001, 0.1\}$. With the higher weight of the cost variability (e.g. $\phi = 10$), the cost variance function becomes dominant and the generated solutions have no variability. The above discussion highlight the research limitations as follows: the present robust optimization framework requires a multi-objective optimization method to find the robust Pareto optimal solutions. The multi-objective optimization method supports the decision making process by delivering alternative non-dominated solutions. Also, the computational effort reduces by eliminating the need for sensitivity analysis.

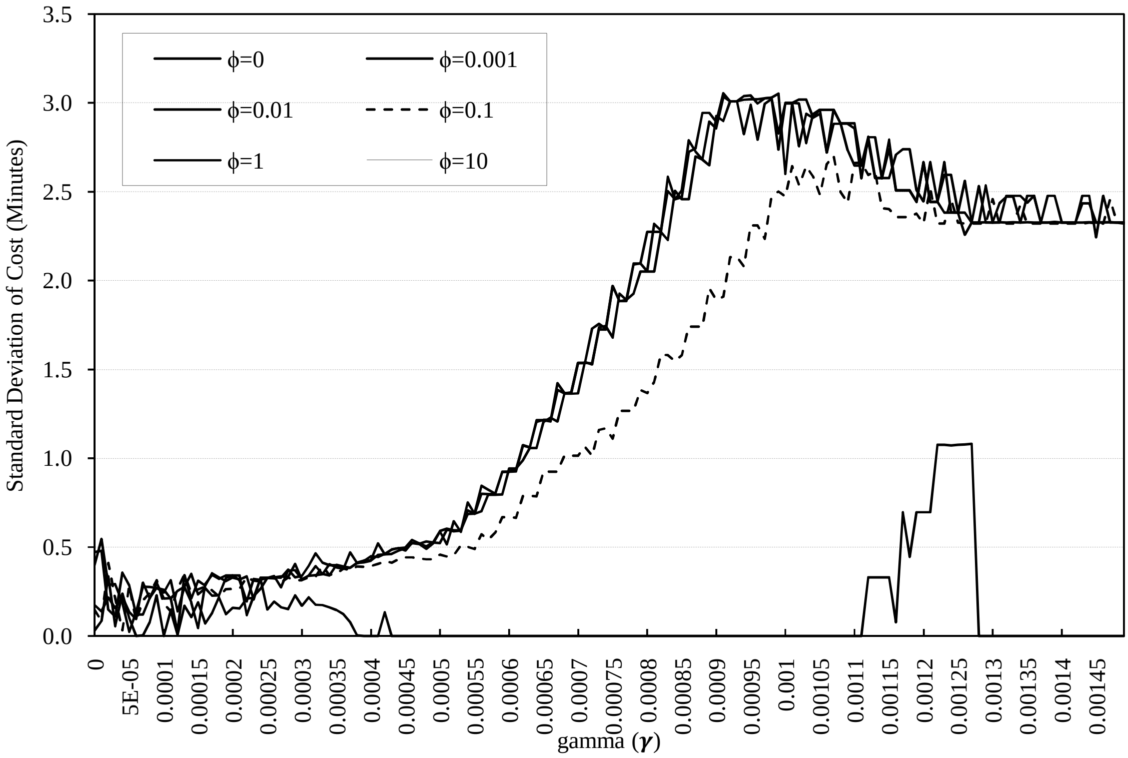

Fig.17. Standard deviation of waiting cost with different values of γ

In what follows, a convergence analysis is conducted to find the solutions with minimum expected waiting time under a realistic situation via Monte Carlo sampling approach. In order to examine the quality of the solutions, we solve the stochastic programming models on different sets of randomly generated samples. The number of passengers in each period was collected through an automatic passenger counter (APC) system. The arrival rates of passengers are fitted by a triangular probability distribution function (Table 10).The number of replications is R=10 for simulation experiments. The result of convergence analysis on the real instances is provided in Table 11 (Tehran station) and

Table **12** (Golshahr station). The results illustrate that the approximate relative gap is decreased by increasing the sample size. The estimated relative optimality gaps are 1.2% and 3% for the cases 1 and 2, respectively. The variance of cost is decreased quickly as the sample size increases. We note that the average computational times to solve the Case 1 were longer than those obtained for the second Case due to the increased uncertainty in the system. The outcomes demonstrate the benefits of including the stochastic demand in the adjustment of departure times.

Table 10. The parameters of the Triangular distribution (a: minimum, b: most likely, c: maximum) for the arrival rate of passengers (passenger per minutes)

Time intervals	[5:00,6:00]	[6:00,7:00]	[7:00,8:00]	[8:00,9:00]	[9:00,10:00]	[10:00,11:00]	[11:00,12:00]
Tehran	(0.15, 3.26, 4.52)	(1, 33.4, 37)	(3, 50.3, 53)	(3, 48, 53)	(6, 39.2, 41)	(7, 29.2, 30)	(5, 22.1, 24)
Golshahr	(3, 72.4, 78)	(7, 147, 159)	(6, 134, 148)	(7, 97, 107)	(13, 52.6, 57)	(9, 39.6, 43)	(10, 35.2, 38)

Table 11.Computational results of Case 1 under different numbers of scenarios(n=30)

Number of scenarios (|Ω|)	Mean value	Standard deviation	Minimum value	Maximum value	Relative gap (max–min)/min	Average CPU time (s)
3	6.453	0.130	6.306	6.665	5.70%	8.933
5	6.339	0.027	6.286	6.390	1.70%	31.847
10	6.341	0.029	6.295	6.396	1.60%	85.326
20	6.362	0.020	6.322	6.398	1.20%	312.944
50	6.361	0.019	6.341	6.404	1.00%	1494.109
100	6.381	0.027	6.338	6.427	1.40%	6902.183
200	6.367	0.023	6.336	6.415	1.30%	18205.22
250	6.365	0.020	6.342	6.418	1.20%	35946.95

Table 12.Computational results of Case 2 under different numbers of scenarios(n=30)

Number of scenarios (|Ω|)	Mean value	Standard deviation	Minimum value	Maximum value	Relative gap (max–min)/min	Average CPU time (s)
3	11.786	0.791	10.253	12.740	24.30%	2.43
5	12.047	0.553	11.009	12.904	17.20%	8.501
10	11.940	0.397	11.252	12.605	12.00%	16.568
20	12.012	0.282	11.539	12.605	9.20%	41.983
50	12.076	0.132	11.92	12.358	3.70%	209.546
100	12.108	0.124	11.924	12.311	3.20%	953.191
200	12.076	0.119	11.867	12.219	3.00%	3942.204
250	12.080	0.112	11.951	12.332	3.20%	6430.985

5-Conclusion

The robust train timetable design is an important problem for the public transportation systems. It aims at generating an operational schedule for a set of trains and with respect to a number of operational constraints. In this study, the objective was to incorporate the stochastic demand flows in mathematical formulation in order to construct a robust train timetable with minimum expected average waiting time as well as cost variance. The train scheduling problem was formulated as linear and nonlinear scenario-based robust stochastic programming models. The applicability of the proposed robust mathematical programming models was examined with carrying out numerical test instances. Numerical examples illustrated the computational efficiency of the proposed modeling approach and the potential benefit of solving the robust stochastic programming model compared to the deterministic models. Afterward, the robustness and effectiveness of the developed stochastic programming models were verified through numerical test instances of real-world cases, and the trade-off between solution robustness and model robustness was investigated. On the basis of computational experiments, we found that the robust stochastic optimization models can obtain almost feasible and near to optimal solutions by controlling the weight parameters. The outcomes proved that the model robustness increases by decreasing the importance of cost variance. The computation experimentations validate that the average cost rises gradually by increasing the value of Gamma and then converges to a maximum value. Significant improvements were achieved in both solution quality and robustness through the implemented stochastic optimization approach. The average reduction in expected value and the variance of passenger waiting time passengers were 22% and 60% compared to the non-robust baseline timetables. In conclusion, the outcomes showed the efficiency, robustness and the tractability of the nonlinear stochastic programming model compared to the linear model. The present study also recommends a number of fields for further research. The present formulation can be extended to consider the supply-side uncertainty. Furthermore, multi-objective optimization approach can be used to overcome the difficulties of dealing with weight parameters.

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