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A novel robustness measure for multi-objective optimization problems under interval uncertainty

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Abstract

In this paper, a novel robustness index is introduced to provide a measure of the robustness of a solution against variations in decision variables and parameters. Most of the proposed robustness measures in the literature consider only magnitude of variations in the objectives space and don't take into account the direction, or in the other words, the type of variations. In this paper, two types of variation named dominating and Pareto variations are introduced and argued that the Pareto variations are more robust than the other one. An index is also proposed here to help measuring the proportion of dominating variations. We proved that this index is independent of magnitude of variations. A robustness index is developed based on these two measures. The robustness index is then used as an additional objective and constraint function so that the uncertain multi-objective optimization problem is transformed to a deterministic one. The resulting deterministic multi-objective optimization problem is solved by NSGA-III. Moreover, Mont Carlo simulation is used to evaluate solutions during the algorithm and compute the robustness index. Two test problems from the context of engineering design optimization are used to illustrate the applicability and efficiency of our proposed robustness index.

Keywords: Multi-objective optimization, uncertainty, robustness measure, engineering design

1-Introduction

Multi-objective optimization (or multi-criteria decision making) has a vast application in the real world decision making problems, such as problems arising in the domain of economy, social and behavioral sciences, biology, management and engineering (Bertsimas & Thiele, 2004; Chen et al., 2012; Chi, Xu, & Zhang, 2020; Disser, Müller–Hannemann, & Schnee, 2008; Elyasi, Roudbari, & Hajipourzadeh, 2020; Fliege & Werner, 2014; Gilani & Sahebi, 2020; Hamarat, Kwakkel, Pruyt, & Loonen, 2014; Ju et al., 2019; Najafi, Eshghi, & Dullaert, 2013; Nikjoo & Javadian, 2019; Yu & Liu, 2013; Zhou et al., 2018). Mathematical programming is a powerful tool in modeling and formulation of real-world decision-making problems. In the following, some contributions of the mathematical programming to the decision-making process are listed:

- Giving an optimal solution to the decision maker,
- Cooperating with the decision maker in an interactive process to determine the optimal solution,
- Giving a limited set of excellent solutions to the decision maker, and
- Giving an insight about the problem to the decision maker through classifying the solution space.

Although Mathematical programming is capable of facilitating the decision making process in real world problems, but its application is naturally limited (Fliedner & Liesiö, 2016).

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Data uncertainties as a common property of real-world problems can considerably reduce the efficiency of Mathematical programming methods and limit their application area. To deal with this uncertainty, different approaches have been proposed by researchers. Among the existing approaches, robust optimization, as a conservative manner of dealing with uncertainty, has attracted more attention, especially from the practitioners. Robust optimization methods search for solutions that are less sensitive to uncertainties and variations of variables and parameters. These methods are simpler and more practical than other methods, such as stochastic programming (Abdelaziz, 2012; Gutjahr & Pichler, 2016). Hence, an extensive study has been done on single objective robust optimization (Ben-Tal, Ghaoui, & Nemirovski, 2009; Bertsimas, Brown, & Caramanis, 2011; Goerigk & Schöbel, 2016). Unlike single objective optimization, application of robust optimization approaches to multi-objective context is limited to few publications. In fact, due to the complex nature of multi-objective concept, translation of single objective robustness concepts to the multi-objective context is not straightforward. Nonetheless, some works have been done to introduce robustness concept in multi-objective context. We classify these robust multi-objective optimization approaches into four categories: *probabilistic dominance*; *mean value replacement*; *variation measure*; and *worst case analysis*. In the first category, the concept of domination is extended to probabilistic context. Teich (2001) introduced concept of probabilistic dominance and used it as a probabilistic counterpart of Pareto dominance concept in the deterministic problems. In addition, Khosravi, Borst, and Teich (2018) applied a histogram-based approach to compare candidate solutions with arbitrarily distributed uncertain objectives.

The second category comprises methods that are based on the simple idea of Branke (1998) that replaces objective functions value with their mean value. Kalyanmoy Deb and Gupta (2006) used this idea in their work. They proposed two procedures to find robust Pareto frontier. These procedures are based on two definitions of multi-objective robust solutions called type I and type II. In the first procedure, values in the objective vector of a solution are replaced by the mean of objective function values in the neighborhood of the solution. In second procedure, a robustness constraint is added to the model. This constraint requires that the deviations of objective values from the nominal values (distance of mean point from nominal point) don't exceed a predetermined value. They used evolutionary Multi-objective genetic algorithm based on NSGA-II (developed by K. Deb, Pratap, Agarwal, and Meyarivan (2002)) to generate robust Pareto solutions. In this algorithm, random points are generated via simulation to compute the robustness of a solution during the algorithm.

In the third category methods, a measure of variation size for each solution is considered as robustness index for that solution. Degree of robustness concept proposed by Barrico and Antunes (2006) measures the percentage of a solution x neighboring points whose objective function values belong to a predefined neighborhood of $f(x)$. Hence, the degree of robustness can be regarded as a measure of variation. Augusto, Bennis, and Caro (2012) used sensitivity analysis to derive robustness index. They used the global sensitivity Jacobian matrix to compute the sensitivity of objective functions to variation of variables. It should be mentioned that in order to use their approach some assumptions about the objective functions must be satisfied. In fact, the objective functions must belong to the C^2 class. Works of Ferreira, Fonseca, Covas, and Gaspar-Cunha (2008); Saha, Ray, and Smith (2011); Zhou et al. (2018) Daryani, Omran, and Makui (2020) and Kusch and Gauger (2021) can be classified in this category.

The fourth category includes methods that only consider the worst cases. As robustness concept, intrinsically, is a conservative approach to deal with uncertainty, the worst-case methods for measuring robustness may be more compatible with its concept, but this is not always in accordance with the decision maker's preferences. However, there are some reasons to use worst case approaches: they don't need any information about the distribution of uncertain variables, there is a strong rationale (conservatism) behind them, and finally, definition of them is straightforward. Li, Azarm, and Aute (2005) considered two types of robustness called performance robustness (objective functions robustness) and feasibility robustness (constraint functions robustness). In performance robustness, diameter of sensitivity region (maximum distance from nominal point) is used as a worst-case robustness measure. In feasibility robustness, a solution is regarded as a robust solution if and only if the sensitivity region of that solution is completely inside the feasible space. To compute the robustness index of a solution, a single objective optimization problem is solved via a meta-heuristic algorithm and then, the computed robustness index is regarded as an additional objective function in the original multi-objective optimization. Also, a similar approach is used in the Gunawan and Azarm (2005) to generate robust Pareto solutions. In recent years, researches on Multi-objective robustness have focused on worst case approaches (Goberna, Jeyakumar, Li, & Vicente-Pérez, 2018; Ide & Schöbel, 2016). Initial works in this regard include the works of Avigad and Branke (2008); Soares, Parreiras, Jaulin, Vasconcelos, and Maia (2009) and Kuroiwa and Lee (2012). More recently, some researchers applied the worst case Pareto frontier to generate robust solutions (Bokrantz & Fredriksson, 2017; Ehrgott, Ide, & Schöbel, 2014; Fakhar, Mahyarinia, & Zafarani, 2018; Goberna, Jeyakumar, Li, & Vicente-Pérez, 2015; Schmidt, Schöbel, & Thom, 2019). We have summarized above-mentioned classification in the table (1).

Table 1. classification of robust multi-objective optimization methods

Probabilistic dominance	**Mean value replacement**	**Variation measure**	**Worst case analysis**
Teich (2001); Khosravi et al. (2018)	Kalyanmoy Deb and Gupta (2006)	Barrico and Antunes (2006); Augusto et al. (2012); Ferreira et al. (2008); Saha et al. (2011); Daryani et al. (2020); Kusch and Gauger (2021); Zhou et al. (2018)	Li et al. (2005); Gunawan and Azarm (2005); Avigad and Branke (2008); Soares et al. (2009); Kuroiwa and Lee (2012); Ehrgott et al. (2014); Goberna et al. (2015); Bokrantz and Fredriksson (2017); Fakhar et al. (2018); Schmidt et al. (2019);.

Mavrotas, Pechak, Siskos, Doukas, and Psarras (2015) present a new interpretation of robustness concept. Previous works suppose that the uncertainty is about the decision variables and parameters, but they considered the uncertainty about decision maker’s preferences. To do that, they used an additive weighting approach to aggregate objective functions. they call a Pareto optimal solution as a robust solution if its performance is less sensitive to the perturbations within weights of the objective functions. They used Mont Carlo simulation to compute robustness of Pareto optimal solutions. A survey of simulation based methods of robust Multi-objective optimization is provided by Steponavice and Miettinen (2012).

Some researchers have applied the existing concepts and presented new algorithms for generating robust solutions (Kuhn, Raith, Schmidt, & Schöbel, 2016; Meneghini, Guimaraes, & Gaspar-Cunha, 2016; Sun, Zhang, Fang, Li, & Li, 2018; Xie et al., 2018). Some scholars have also applied existing robustness concepts in various applications (Doolittle, Kerivin, & Wiecek, 2018; Habibi-Kouchaksaraei, Paydar, & Asadi-Gangraj, 2018; Peng, Hou, Che, Xu, & Li, 2019; Sun et al., 2018; Xidonas, Mavrotas, Hassapis, & Zopounidis, 2017).

The existing methods in the third category measure only the magnitude of variations. In other words, these methods don’t consider the type of variations (direction of variations). To deal with this issue, we distinguish between two types of variations and discuss that one type is more robust than the other one. Since we use the mean of objective functions and the variation measure as objectives to be minimized, our proposed robust optimization approach lay in the second and third category.

The reminder of this paper is as follows: in section 2, we give an illustration of problem. Also, some definitions and preliminaries are stated. In section 3, we present our proposed robustness index. In section 4, two well-studied test problems from engineering design optimization are solved by our proposed algorithm and results are discussed. finally, in section 5, we give some concluding remarks.

2- Definitions and preliminaries

Consider a multi-objective optimization model as follows:

$$\begin{aligned} \min \quad & f_1(\tilde{x}^c, \tilde{x}^d, \tilde{p}),\ f_2(\tilde{x}^c, \tilde{x}^d, \tilde{p}), \ldots,\ f_n(\tilde{x}^c, \tilde{x}^d, \tilde{p}) \\ s.t. \quad & g_j(\tilde{x}^c, \tilde{x}^d, \tilde{p}) \le 0 \quad for\ 1 \le j \le m \\ & l_i \le \tilde{x}_i^c \le u_i \quad for\ 1 \le i \le N_c \\ & \tilde{x}_k^d \in M_k \quad for\ 1 \le k \le N_d \end{aligned} \tag{1}$$

Where, f_i denotes the ith objective function, g_j is the jth constraint function. x^c, x^d and P are respectively the vector of continuous and discrete (combinatorial) decision variables and decision parameters. It is supposed that the decision variables and the parameters (x^c, x^d and P) are random variables. Where the probability distribution of a variable is not known (as interval uncertainty), uniform distribution is supposed.

In this paper, the tolerance region and sensitivity region concept (introduced by Gunawan and Azarm (2005)) is used to represent the uncertainty. Let $x^0 = (x_1^0, x_2^0)$ be an arbitrary point in the decision space (as shown in figure 1.a). If a variable has interval uncertainty, then, the tolerance region is in form of a rectangle. The mapping of tolerance region into the objectives space is called sensitivity region (as shown in figure 1.b).

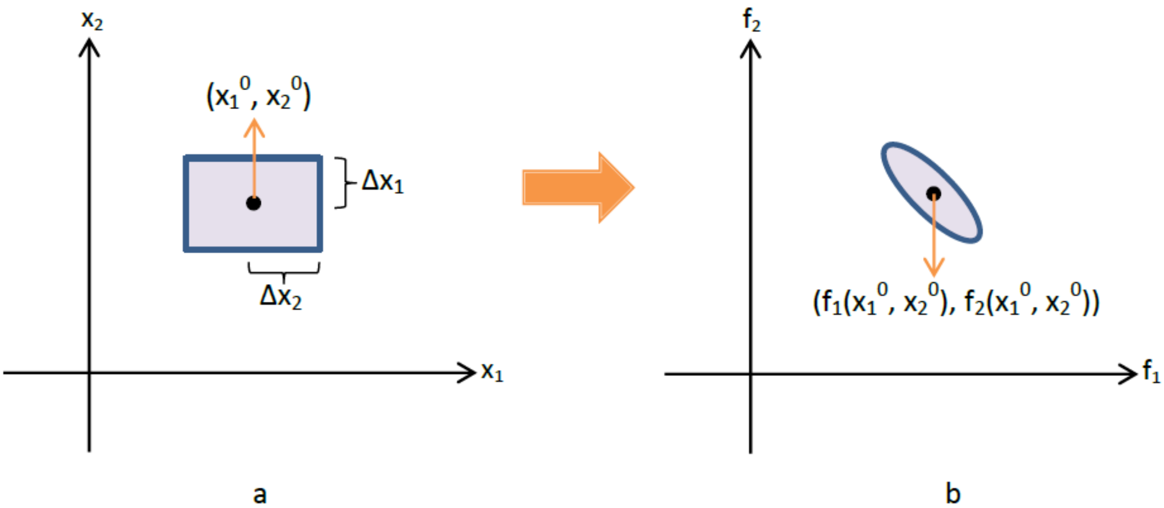


Fig 1.a. Tolerance region of a solution, **b.** Sensitivity region

The sensitivity region of a solution represents the response of the solution to the uncertainty. So, studying the features of the sensitivity region is the only way to measure robustness of a solution. Li et al. (2005) used the diameter of the sensitivity region as robustness index. Figure 2 represents two different sensitivity regions with the same diameter. According to this robustness index, two solutions are equivalent, but the question is now whether these two solutions are really equivalent? To answer to this question, more detailed analysis of the sensitivity regions is required.

First of all, we must define the term *robustness* in mathematical terms:

Definition 2.1. let $U(x)$ denote the total utility function of solution x. We say solution x is more robust than solution y if and only if $Variance[U(x)] \leq Variance[U(y)]$.

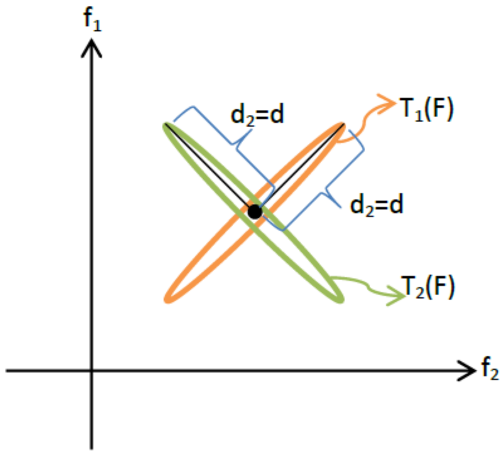


Fig 2. Two sensitivity regions with the same diameter

By studying the two above-mentioned sensitivity regions we introduce two different types of variation:

Definition 2.2. *Dominating variation.* A variation (deviation) between point A and point B in a sensitivity region is called dominating variation if and only if:

$$\forall i, j \in \{1,\dots,n\}, \left(f_i(A) - f_i(B)\right) \times \left(f_j(A) - f_j(B)\right) \geq 0 \tag{2}$$

Definition 2.3. *Pareto variation*. A variation between point A and point B in a sensitivity region is called Pareto variation if and only if:

$$\exists i, j \in \{1,\dots,n\}, \left(f_i(A) - f_i(B)\right) \times \left(f_j(A) - f_j(B)\right) < 0 \tag{3}$$

In other words, a variation from point A to point B is dominating variation if and only if A dominates B or vice versa, otherwise, the variation is a Pareto variation. Dominating and Pareto variations are illustrated in figure 3. It's evident that a Pareto variation is more robust than a dominating variation in the sense that in a Pareto variation a decrease in an objective function can be considered as a compensation to an increase in some other objective function while in a dominating variation, all objective functions increase or decrease simultaneously. We give a mathematical proof for this result as follows:

Claim 1. Pareto variation is more robust than dominating variation.

Proof. Let A, B, C and D be four points in the objectives space (as illustrated in figure 3). Suppose the two pairs (A, B) and (C, D) respectively define Pareto and dominating variations with equal magnitude of variation: $\left|\Delta f_i^{(A,B)}\right| = \left|\Delta f_i^{(C,D)}\right|, 1 \leq i \leq n,$ where $\Delta f_i^{(A,B)} = f_i(A) - f_i(B).$ Then, for any vector of objectives weights *w* (a representative for decision maker preferences) we have:

$$\begin{aligned} \Delta U^{(A,B)} &= \left|U(A) - U(B)\right| = \left|\sum_{i=1}^{n} w_i f_i(A) - \sum_{i=1}^{n} w_i f_i(B)\right| \\ &= \left|\sum_{i=1}^{n} w_i \left[f_i(A) - f_i(B)\right]\right| = \left|\sum_{i=1}^{n} w_i \Delta f_i^{(A,B)}\right| \end{aligned} \tag{4}$$

As (A,B) is a Pareto variation we know by definition 2.3 that for some i, the term $\Delta f_i^{(A,B)}$ is negative. Also, by definition 2.2 for all $1 \leq i \leq n, \Delta f_i^{(C,D)} \geq 0$, then, we have:

$$\left|\sum_{i=1}^{n} w_i \Delta f_i^{(A,B)}\right| < \sum_{i=1}^{n} w_i \left|\Delta f_i^{(A,B)}\right| = \sum_{i=1}^{n} w_i \Delta f_i^{(C,D)} = \Delta U^{(C,D)} \tag{5}$$

$$\Rightarrow \Delta U^{(A,B)} < \Delta U^{(C,D)} \tag{6}$$

Thus, by definition 2.1 we conclude that variation (A,B) is more robust than variation (C,D) □.

One can intuitively see that in the sensitivity region of figure 3.a, Pareto variations are more dominant than dominating variation while, in the sensitivity region of figure 3.b dominating variations overcome Pareto variations. Therefore, we conclude that the sensitivity region of figure 3.a is more robust than the sensitivity region of figure 3.b.

Now, the problem is how to assess a sensitivity region based on dominating and Pareto variations. To do that, we propose a measure and call it *domination ratio*:

Definition 2.4. *Domination ratio*. Let x be a solution. Then, the domination ratio of the sensitivity region of the solution, denoted by $r_d(x)$, is calculated as follows:

$$r_d(x) = \iint_{A,B\in S(x)} \psi(A,B)\times\xi(A)\times\xi(B)\times dA\times dB$$
$$\psi(A,B) = \begin{cases} 1 & \text{if } A \prec B \\ 0 & Otherwise \end{cases} \tag{7}$$

Where, $\xi(A)$ and $\xi(B)$ are respectively probability distribution functions of points A and B. $S(x)$ is the sensitivity region of solution x. If the probability with which, point A is dominated by any other point in the sensitivity region is denoted by $P_d(A)$, then, we have:

$$r_d(x) = \int_{A\in S(x)} P_d(A)\times\xi(A)\times dA \tag{8}$$

In fact, the domination ratio of a sensitivity region is the probability with which a solution in the sensitivity region dominates any other solution.

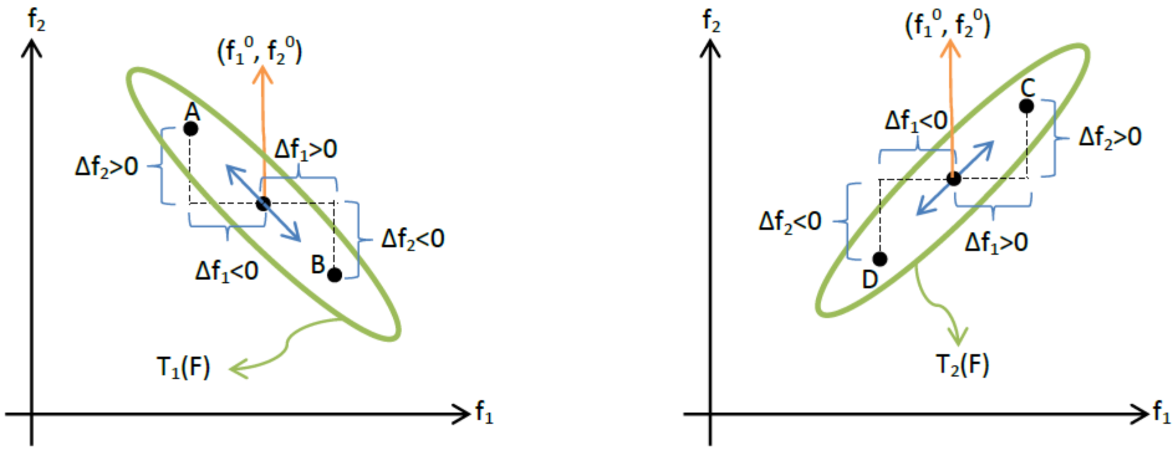


Fig 3.a. Illustration of a Pareto variation, **b.** illustration of a dominating variation

Definition 2.5. *Pareto optimal solution.* A feasible solution x^* is called Pareto optimal solution, if and only if, for any feasible solution x, $\exists i, 1\le i\le n,\ f_i(x^*) < f_i(x)$.

3-Robustness index

In this section we introduce a novel robustness index. At first, two theorems will be proved.

Theorem 3.1. let $S(x)$ be the sensitivity region of solution *x*. then, the domination ratio of $S(x)$ is between 0 and 0.5. In the other words: $0 \le r_d(x) \le \frac{1}{2}$.

Proof. We define three functions for any pair of points (A, B) in the sensitivity region:

$$\psi(A,B) = \begin{cases} 1 & \text{if } A \prec B \\ 0 & Otherwise \end{cases}, \tag{9}$$

$$\phi(A,B) = \begin{cases} 1 & \text{if } A \succ B \\ 0 & Otherwise \end{cases}, \tag{10}$$

$$\delta(A,B) = \begin{cases} 1 & \text{if } A \sim B \\ 0 & Otherwise \end{cases}, \tag{11}$$

Since for any pair of points (A, B) we have three statuses: $A \prec B$ or $A \succ B$ or $A \sim B$, and only one of these three statuses can occur, then we have: $\psi(A,B) + \phi(A,B) + \delta(A,B) = 1$.

since $A \prec B$ and $A \succ B$ are equivalent, then, $\phi(A,B) = \psi(B,A)$. Thus, we have: $\psi(A,B) + \psi(B,A) + \delta(A,B) = 1$. From the definition 2.3 we have:

$$\iint_{A,B\in S(x)} [\psi(A,B) + \phi(A,B) + \delta(A,B)] \times \xi(A) \times \xi(B) \times dA \times dB = \iint_{A,B\in S(x)} \xi(A) \times \xi(B) \times dA \times dB = 1 \tag{12}$$

$$\Rightarrow 1 - \iint_{A,B\in S(x)} \delta(A,B) \times \xi(A) \times \xi(B) \times dA \times dB = \iint_{A,B\in S(x)} \psi(A,B) \times \xi(A) \times \xi(B) \times dA \times dB + \iint_{A,B\in S(x)} \psi(B,A) \times \xi(A) \times \xi(B) \times dA \times dB \tag{13}$$

$$\Delta = \iint_{A,B\in S(x)} \delta(A,B) \times \xi(A) \times \xi(B) \times dA \times dB \Rightarrow \tag{14}$$

$$1 - \Delta = \iint_{A,B\in S(x)} \psi(A,B) \times \xi(A) \times \xi(B) \times dA \times dB + \iint_{A,B\in S(x)} \psi(B,A) \times \xi(A) \times \xi(B) \times dA \times dB \tag{15}$$

$$1 - \Delta = 2 \times r_d, \tag{16}$$

$$\Delta \ge 0 \Rightarrow 0 \le r_d \le \frac{1}{2}. \tag{17}$$

It's essential to note that when the sensitivity region is a line segment, the extreme cases in the theorem 3.1 ($r_d = 0$ and $r_d = 0.5$) will happen. When slope of the line is positive, then all variations in the sensitivity region are in the form of dominating variation and then $\Delta = 0 \Rightarrow r_d = 0.5$. If slope of the line is negative, then all variations in the sensitivity region are in the form of Pareto variation and then, $\Delta = 0 \Rightarrow r_d = 0$.

Theorem 3.2. The domination ratio is insensitive to the size of sensitivity region. In mathematical terms, for a scalar value $k > 0$, we define $S' = \{A' | A' = k \times A, A \in S\}$, then, $r_d(S') = r_d(S)$.

Proof. For any pair of pints, *A* and *B* we have $\psi(A,B) = \psi(A',B')$ because:

$$A \prec B \Rightarrow \forall i, 1 \le i \le n,\ f_i(A) \le f_i(B),\ \exists j,\ f_j(A) < f_j(B) \Rightarrow \tag{18}$$
$$\forall i, 1 \le i \le n, k \times f_i(A) \le k \times f_i(B), \exists j, k \times f_j(A) < k \times f_j(B) \Rightarrow \tag{19}$$
$$k \times A \prec k \times B \Rightarrow A' \prec B' \Rightarrow \psi(A,B) = \psi(A',B'). \tag{20}$$
Also, it's obvious that $\forall A \in S, \xi(A) = \xi(A'), A' \in S'$. Then, we have:

$$r_d(S') = \iint_{A',B' \in S'} \psi(A',B') \times \xi(A') \times \xi(B') \times dA' \times dB' = \iint_{A,B \in S} \psi(A,B) \times \xi(A) \times \xi(B) \times dA \times dB \tag{21}$$

This theorem says that the domination ratio is independent of the size of variation (or size of sensitivity region). In fact, domination ratio only determines the type of variation. Therefore, we should introduce an index that represents the size of variation for a solution (sensitivity region). We call the index *magnitude of variation* and display it by M_v. It is calculated as follows:

$$M_v(S) = \int_{A \in S} d(A, \bar{S})\, \xi(A)\, dA, \qquad \bar{S} = (\bar{f}_1, \ldots, \bar{f}_n). \tag{22}$$

Where $\bar{S}$ is the center of sensitivity region, and $d(A, \bar{S})$ is distance of point A from the center. Therefore, M_v is expected value of distance from the center. $d(A, \bar{S}) = \|A - \bar{S}\|_p, p = 1, \ldots, \infty.$ but we use $p = \infty$. As the objective functions may be in different scales, normalized objective values must be used in measuring the distance between two points. Then the distance is calculated as follows:

$$d(A, \bar{S}) = \|A - \bar{S}\|_\infty = \max\left\{\left|f_1^{norm}(A) - \bar{f}_1\right|, \ldots, \left|f_n^{norm}(A) - \bar{f}_n\right|\right\}, \qquad f_i^{norm}(A) = \frac{f_i(A) - f_i^{\min}}{f_i^{\max} - f_i^{\min}},\ \bar{f}_i = \int_{A \in S} f_i^{norm}(A)\, \xi(A)\, dA,\ i = 1, \ldots, n. \tag{23}$$

Now, we are ready to introduce our proposed robustness index. The robustness index of solution x is denoted by $\eta(x)$ and is calculated as follows:

$$\eta(x) = [r_d(x) + 0.5] \times M_v(x) \tag{24}$$

For some solutions, $r_d(x)$ may equal to zero and then, the robustness index becomes zero. This means that the effect of $M_v(x)$ in equation (24) is destroyed, therefore, we add $r_d(x)$ by 0.5, then $0.5 \le r_d(x) + 0.5 \le 1$. The term $[r_d(x) + 0.5]$ in equation (24) is, in fact, a coefficient that modifies the magnitude of variation.

We have used another robustness index to generate robust Pareto frontier. Concept of the robustness index is introduced by Li et al. (2005). The robustness index is in fact the diameter of sensitivity region and calculated as follows:

$$D(x) = \max_{A \in S(x)} d(A, \bar{S}) \tag{25}$$

In section 4, we compare the results of these two robustness indexes based on two test problems. Since variation of variables and parameters can lead to infeasible solutions, feasibility robustness of each solution should also be considered. In this regard, we use feasibility robustness index that is denoted by η_F and calculated as follows:

$$\eta_F = \Pr(A\, is\ feasible),\ A \in S(x) \tag{26}$$

Using the performance and feasibility robustness indexes, the uncertain multi-objective optimization model (1) can be transformed to deterministic multi-objective optimization model:

$$\begin{aligned} \min \quad & \vec{\bar{S}}(x) = \left[\bar{f}_1(x), \ldots, \bar{f}_n(x)\right] \\ \min \quad & \eta(x) \\ s.t. \quad & \eta(x) \le u_\eta \\ & \eta_F(x) = 1 \end{aligned} \tag{27}$$

Where, u_η is a threshold for performance robustness index. We solve the resulting deterministic multi-objective optimization model with NSGA-III , developed by Kalyanmoy Deb and Jain (2013). During this algorithm, each solution is evaluated with the aid of Mont Carlo simulation. In fact, for any solution, number of N_{sim} points in the sensitivity region is randomly generated (based on probability distribution function). Then, the robustness indexes are estimated as follows:

$$\hat{\eta}(x) = \frac{\sum_{i=1}^{N_{sim}} n(A_i)}{N_{sim}^{\ 2}}, A_i \in S(x), \tag{28}$$

$$\hat{\eta}_F(x) = \frac{n_F}{N_{sim}} \times 100. \tag{29}$$

Where, $n(A_i)$ is the number of generated points in the sensitivity region that are dominated by A, and n_F is the number of feasible points.

To Compare the performance of the two robustness indexes $\eta(x)$ and $D(x)$, we have used them as robustness index in model (27). The resulting Pareto frontiers for $\eta(x)$ and $D(x)$ are called respectively, DRR (Domination Ratio based Robustness) and MDR (Max Distance based Robustness).

4- Results and discussion

In this section, we will apply the proposed robustness index to two test problems from engineering design optimization context. We have taken these problems from Li et al. (2005). So, description of problems is extracted textually from the reference paper.

We have implemented the robust multi-objective genetic algorithm in MATLAB 2009 and run it in a system with Intel CORE i3 CPU and 2 GB RAM. The population size in the genetic algorithm was set to 100 and number of Mont Carlo simulations for each solution in the population (N_{sim}) was set to 100. Run time of the algorithm for test problem 1 and test problem 2 became respectively 266 and 290 seconds. It’s obvious that by increasing N_{sim} accuracy of results will be improved but also, the run time of the algorithm increases.

4-1- Test Problem 1

The first test problem, that is called two-bar truss design problem, is a popular test problem from the engineering design optimization literature. Figure 4 shows a two-bar truss.

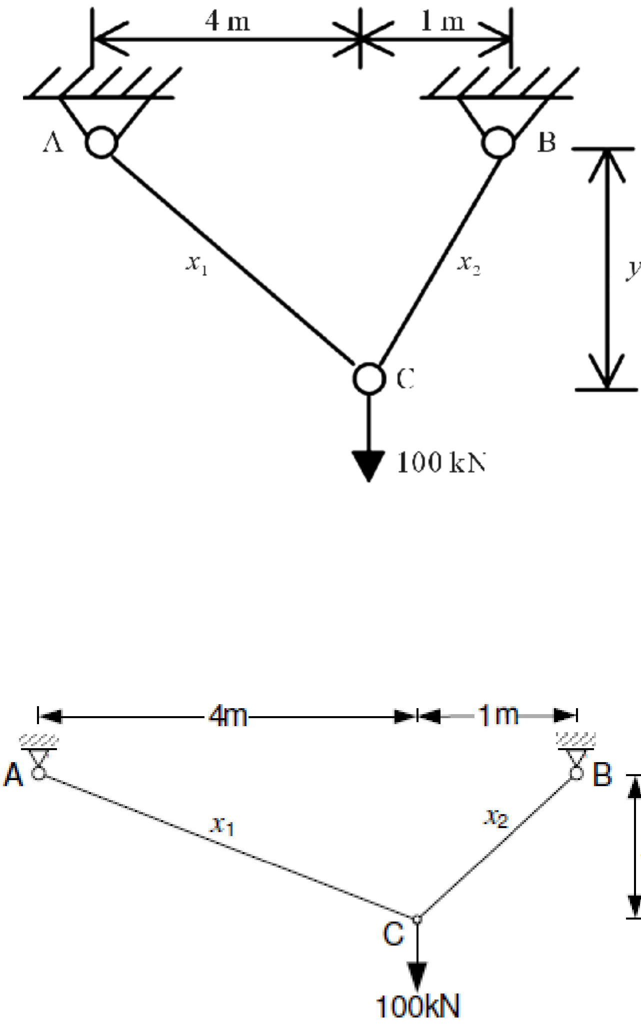


Fig 4. Schematic for two-bar truss design problem

The problem is to design a two-bar truss that can carry a single vertical load of 100kN at joint *C*. The truss comprises of two links as shown in the figure 4. The objectives are to minimize the volume of the two links and to minimize the stress in them as well. The variables are the cross-sectional areas of the links, x_1, x_2, and the vertical drop of the joint *y*. The constraints are: an upper limit of 100,000 kN/m^2 for the stress, the range 1.0-3.0 m for *y*, and a non-negative value for the cross-sectional areas. The problem is formulated as follows:

$$
\begin{aligned}
&\min \; f_1(x_1, x_2, y) = x_1\sqrt{16+y^2} + x_2\sqrt{1+y^2} \\
&\min \; f_2(x_1, x_2, y) = \frac{20\sqrt{16+y^2}}{x_1 y} \\
&s.t. \\
&\qquad 20(16+y^2)^{1/2} - 100000x_1 y \le 0 \\
&\qquad 80(16+y^2)^{1/2} - 100000x_2 y \le 0 \\
&\qquad 1 \le y \le 3 \\
&\qquad 0 \le x_1, x_2 \le 0.01
\end{aligned}
\tag{30}
$$

The known variation in the design variables was set as $\Delta x_1 = \Delta x_2 = 0.0001$ and $\Delta y = 0.05$. Figure 5 shows DRR Pareto frontier obtained by solving problem (30). Figure 6 compares robust Pareto solutions (DRR) with nominal Pareto solutions (problem (30) without any uncertainty, in other words, all variables are deterministic and can only take their nominal values). As we have anticipated, performances of robust solutions are slightly worse than those of nominal solutions. Also, robust solutions are more compact than nominal solutions, that is, robust solutions are distributed over a smaller range of objective values. In fact, to reach to more robust solutions, quality of solutions (performance with respect to objective functions) must be sacrificed. The trade-off between robustness and quality can be attained by regulating the parameter u_η in model (27). Figure 7 shows the feasibility robustness values $\hat{\eta}_F$ for nominal Pareto solutions. The figure represents that the feasibility robustness value for most of solutions is less than 70%, and only one solution has feasibility robustness. In figure 8, the effects of components of proposed robustness index $\eta(x)$ is analyzed. To do that, DRR robust Pareto solutions are sorted based on the robustness index $\eta(x)$. Then values of M_v and $(r_d(x)+0.5)$ (components of $\eta(x)$ in equation (24)) for sorted solutions are depicted in the figure. The figure shows that the trends of magnitude of variation and domination ratio (M_v and $r_d(x)$) are different, and as claimed in theorem 3.2, independent.

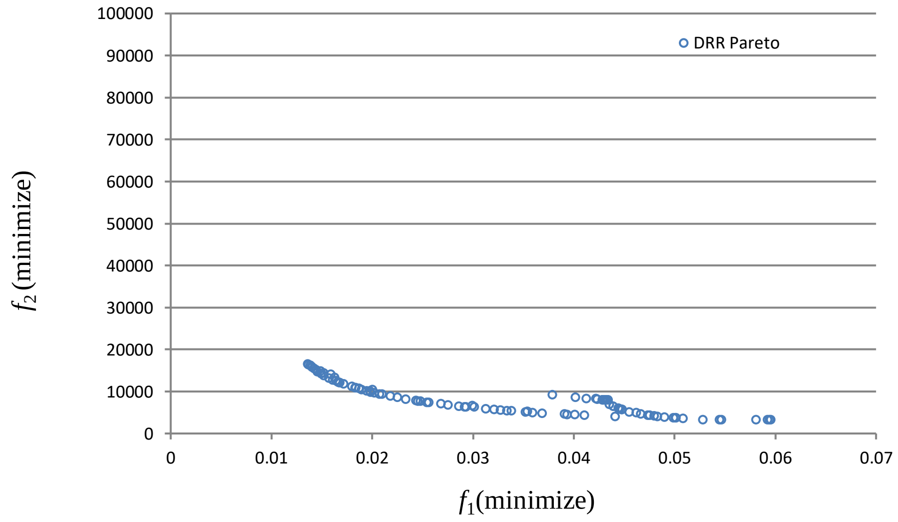


Fig 5. Robust Pareto frontier for the two-bar truss design problem

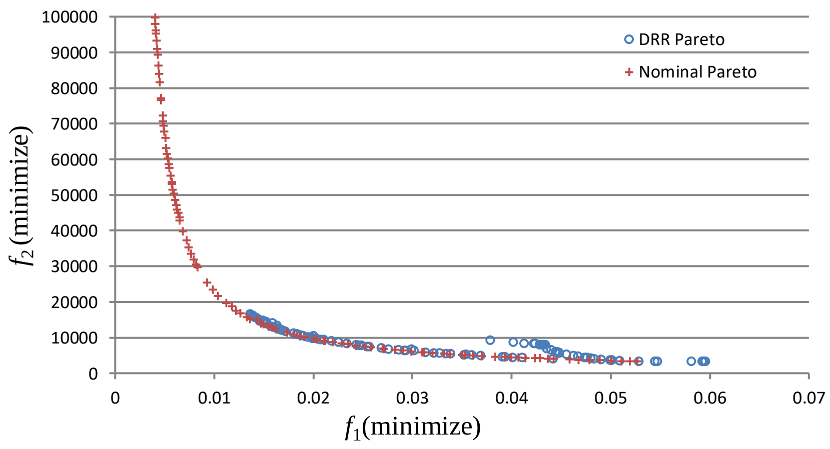


Fig 6. Comparison of robust Pareto frontier with nominal Pareto frontier for the two-bar truss design problem

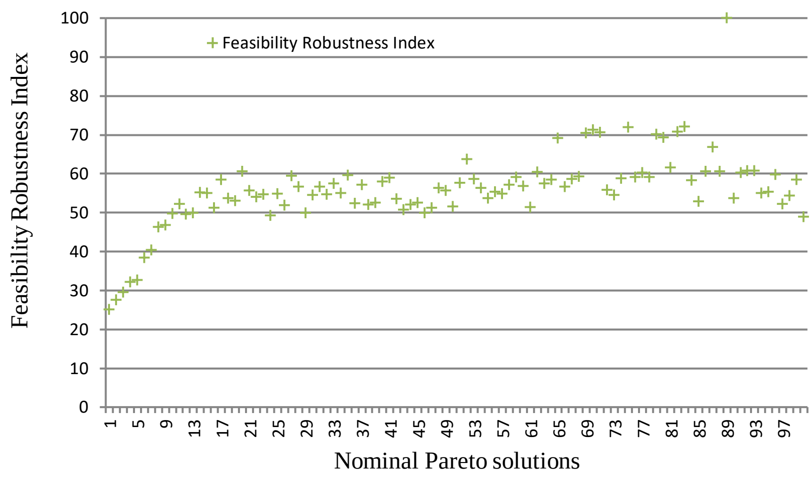


Fig 7. Feasibility robustness of nominal Pareto solutions for the two-bar truss design problem

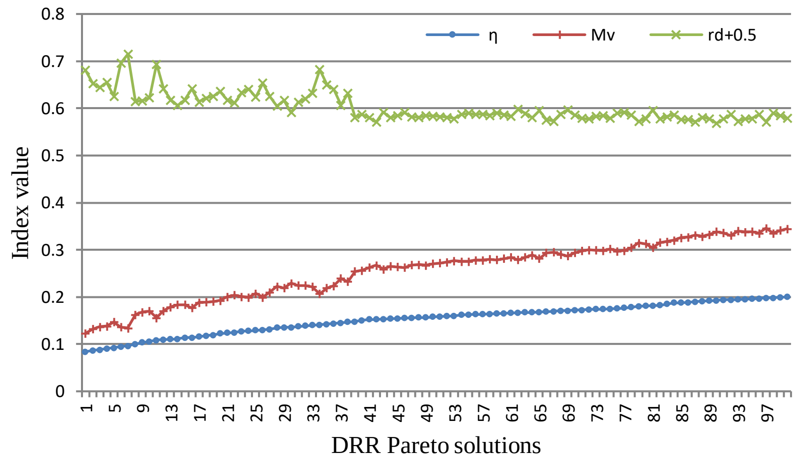


Fig 8. Comparison of trends of three indexes η, M_v, and r_d for robust solutions of the robust Pareto solutions

Figure 9 compares DRR and MDR Pareto frontiers. As represented in the figure, Pareto frontiers obtained by using two robustness indexes $\eta(x)$ and $D(x)$ are different. This was predictable, as the two robustness indexes have different philosophy. The upper limits for $\eta(x)$ and $D(x)$ in model (9) were set respectively to $u_\eta = 0.2$ and $u_D = 1$. We use DRR and MDR Pareto sets to compare performances of the two robustness indexes. For this purpose, we sorted solutions in DRR and MDR Pareto sets with respect to their respective robustness index. Then, we computed the other robustness index for the sorted solutions in each Pareto set ($\eta(x)$ for MDR Pareto set and $D(x)$ for DRR Pareto set). Figures 10 and 11 give the results. In these figures, the upper limit for robustness index is represented by a horizontal line. One can see from figure 10 that the DRR robust solutions (robust solutions based on $\eta(x)$ robustness index) are also robust with respect to the other robustness index ($D(x)$). In fact, only one solution exceeds the upper limit (but its value is also very close to the upper limit). Figure 11 shows values of the two robustness index for MDR Pareto solutions. As can be seen, 14 solutions lie above the threshold line, and therefore, are not robust with respect to $\eta(x)$ robustness index. Based on these observations, we conclude that out proposed robustness index $\eta(x)$ is more efficient than the robustness index $D(x)$. Another important matter with regard to figures 10 and 11 is that, trends of the two robustness indexes on the same set of solutions are different.

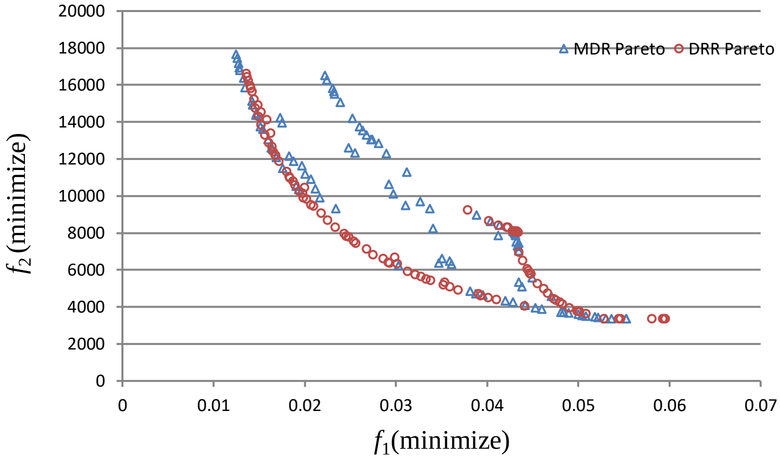


Fig 9. Comparison of the two robust Pareto frontiers related to the two robustness indexes (η, D) for the two-bar truss design problem

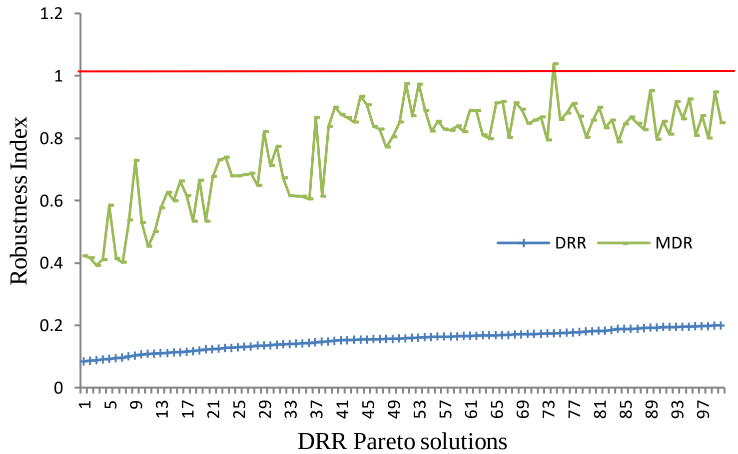


Fig 10. Comparison of performance of the two robustness indexes (η, D) on the DRR Pareto solutions for the two-bar truss design problem

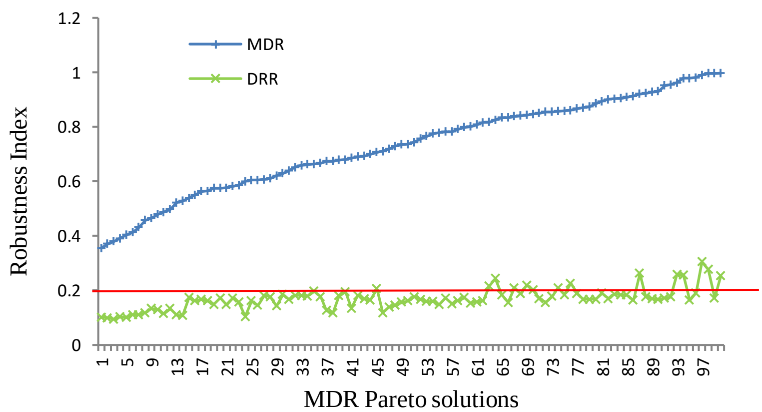


Fig 11. Comparison of performance of the two robustness indexes (η, D) on the MDR Pareto solutions for the two-bar truss design problem

4-2- Test Problem 2

The second test problem is to robustly design a simple speed reducer that might be used in a light airplane between the engine and the propeller. A schematic of the speed reducer to be optimized is shown in figure 12. The first design objective is to minimize the volume of the speed reducer and the second objective is to minimize the stress in the first gear shaft. The problem has seven design variables: gear face width (x_1), teeth module (x_2), number of teeth in the pinion (x_3, integer), distance between bearings 1 (x_4), distance between bearings 2 (x_5), diameter of shaft 1 (x_6) and diameter of shaft 2 (x_7).

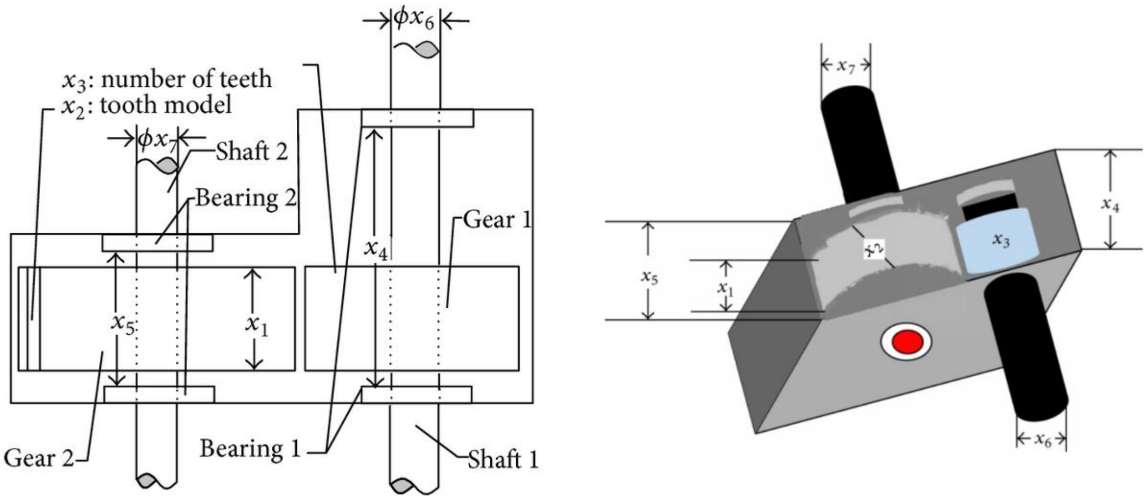


Fig 12. Schematic for speed reducer design problem

The design is subject to a number of constraints imposed by gear and shaft design practices. The seven design variables are subject to an upper and a lower bound. There are eleven inequality constraints that take into consideration: stresses, deflections, space restrictions and design requirements. The units for all the design variables are in cm except for x_3 and those of the objectives f_1 and f_2 are cm^3 and kPa, respectively. The formulation of the problem is as follows:

$$
\begin{aligned}
&\min\ f_1 = 0.7854 x_1 x_2^{\,2}\left(\frac{10x_3^{\,2}}{3} + 14.933x_3 - 43.0934\right) - \\
&1.508x_1(x_6^{\,2} + x_7^{\,2}) + 7.477(x_6^{\,3} + x_7^{\,3}) + 0.7854(x_4 x_6^{\,2} + x_5 x_7^{\,2}) \\
&\min\ f_2 = \frac{\sqrt{\left(\frac{745x_4}{x_2 x_3}\right)^2 + 1.69\times 10^7}}{0.1x_6^{\,3}} \\
&s.t. \\
&g_1 : \frac{1}{x_1 x_2^{\,2} x_3} - \frac{1}{27} \le 0, \quad g_2 : \frac{1}{x_1 x_2^{\,2} x_3^{\,2}} - \frac{1}{397.5} \le 0, \\
&g_3 : \frac{x_4^{\,3}}{x_2 x_3 x_6^{\,4}} - \frac{1}{1.93} \le 0, \quad g_4 : \frac{x_5^{\,3}}{x_2 x_3 x_7^{\,4}} - \frac{1}{1.93} \le 0, \\
&g_5 : x_2 x_3 - 40 \le 0, \quad g_6 : \frac{x_1}{x_2} - 12 \le 0, \\
&g_7 : 5 - \frac{x_1}{x_2} \le 0, \quad g_8 : 1.9 - x_4 + 1.5x_6 \le 0, \\
&g_9 : 1.9 - x_5 + 1.1x_7 \le 0, \quad g_{10} : f_2 \le 1800, \\
&g_{11} : \frac{\sqrt{\left(\frac{745x_5}{x_2 x_3}\right)^2 + 1.575\times 10^8}}{0.1x_7^{\,3}} \le 1100, \\
&2.6 \le x_1 \le 3.6, \quad 0.7 \le x_2 \le 0.8, \\
&17 \le x_3 \le 28, \quad 7.3 \le x_4, x_5 \le 8.3, \\
&2.9 \le x_6 \le 3.9, \quad 5.0 \le x_7 \le 5.5.
\end{aligned} \tag{31}
$$

The known variation in the design variables was set as $\Delta x_2 = 0.01$ and $\Delta x_6 = 0.1$. Figure 13 shows DRR Pareto frontier obtained by solving problem (31). Figure 14 compares robust Pareto solutions (DRR) with nominal Pareto solutions. Similar to the results obtained for test problem 1, quality of robust solutions is slightly worse than nominal solutions and they are more compact. Figure 15 shows the feasibility robustness values $\hat{\eta}_F$ for nominal Pareto solutions. Feasibility robustness of nominal Pareto solutions for this problem is less than that of test problem 1. For this problem, feasibility robustness of 53% of nominal Pareto solutions is less than 10%. In figure 16, the effects of components of proposed robustness index $\eta(x)$ is analyzed. Like the results obtained for the test problem 1, the trends of magnitude of variation and domination ratio (M_v and $r_d(x)$) are different and independent.

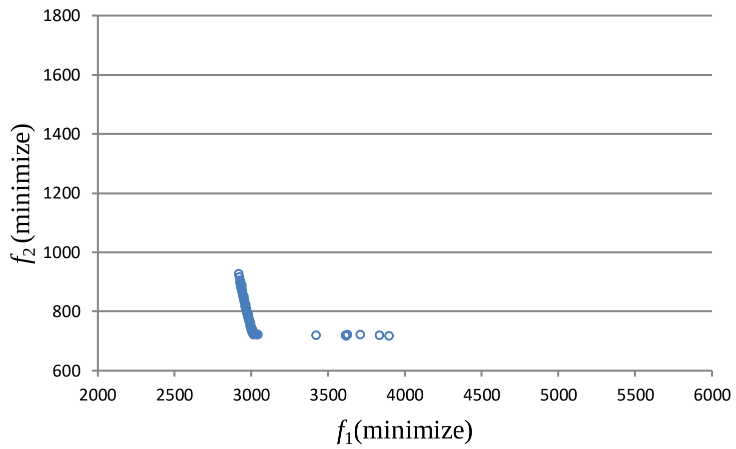


Fig 13. Robust Pareto frontier for the speed reducer design problem

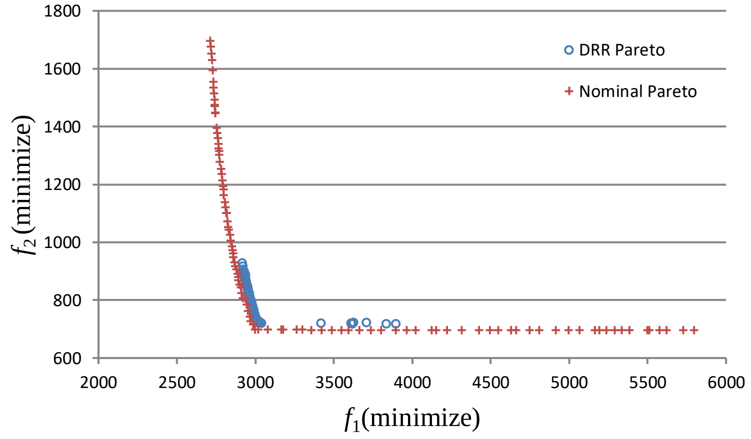


Fig 14. Comparison of robust Pareto frontier with nominal Pareto frontier for the speed reducer design problem

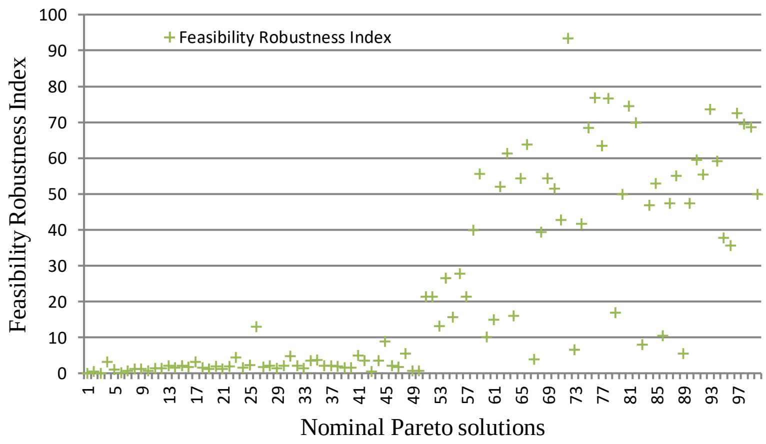


Fig 15. Feasibility robustness of nominal Pareto solutions for the speed reducer design problem

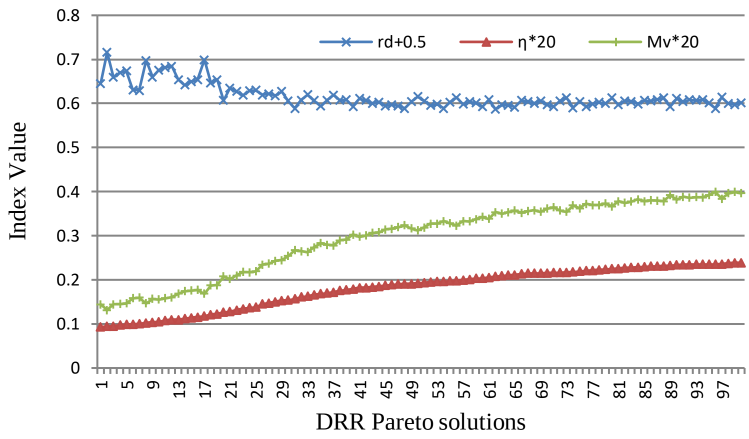


Fig 16. Comparison of trends of three indexes η, M_v, and r_d for robust solutions of the speed reducer design problem

Figure 17 compares DRR and MDR Pareto frontiers. As represented in the figure, the Pareto frontier obtained by using robustness index $\eta(x)$ is more compact than the Pareto frontier obtained by using robustness index $D(x)$. The upper limits for $\eta(x)$ and $D(x)$ in model (27) were set respectively to $u_\eta = 0.012$ and $u_D = 0.05$. As we did for the test problem 1, we use DRR and MDR Pareto sets to compare performances of the two robustness indexes. The results are represented in figures 18 and 19. It is obvious from figure 18 that all of the DRR robust solutions (robust solutions based on $\eta(x)$ robustness index) are also robust with respect to the other robustness index ($D(x)$). Figure 19 shows values of the two robustness index for MDR Pareto solutions. As can be seen, 25 solutions lie above the threshold line, and therefore, are not robust with respect to $\eta(x)$ robustness index. Based on this result, we conclude more strongly than we did from test problem 1 that our proposed robustness index $\eta(x)$ is more efficient than the robustness index $D(x)$. Figures 18 and 19 also show that trends of the two robustness indexes on the same set of solutions are different.

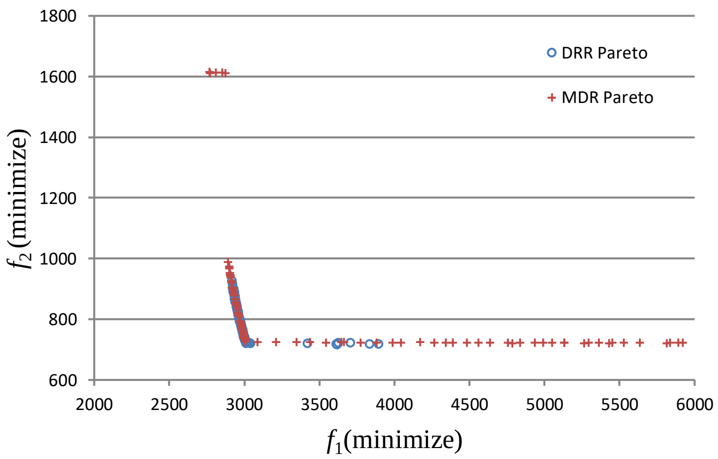


Fig 17. Comparison of two robust Pareto frontiers related to the two robustness indexes (η, D) for the speed reducer design problem

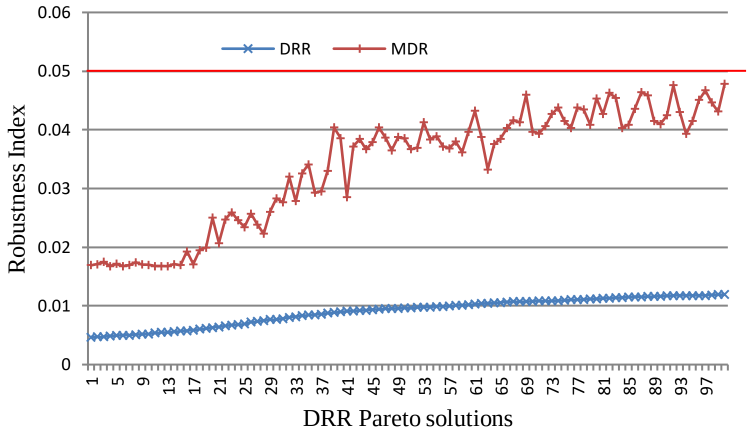


Fig 18. Comparison of performance of the two robustness indexes (η, D) on the DRR Pareto solutions for the speed reducer design problem

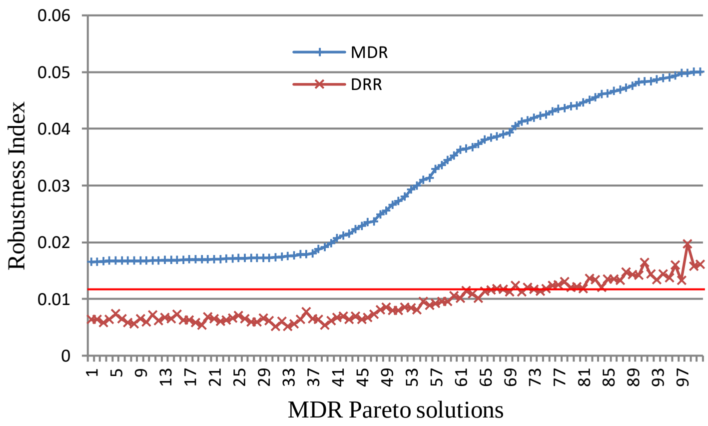


Fig 19. Comparison of performance of the two robustness indexes (η, D) on the MDR Pareto solutions for the speed reducer design problem

4-3- Comparative analysis

Kalyanmoy Deb and Gupta (2006) extended the concept of "robust solution type II" to multi-objective context. They classify a solution x as robust solution type II, if it is a global minimum solution of the following problem:

$$\begin{aligned} &\min \quad f(x) \\ &\textit{subject to} \quad \frac{\left\| \tilde{f}(x) - f(x) \right\|_p}{\left\| f(x) \right\|_p} \leq 1 - \eta \\ &\qquad\qquad\quad x \in D \end{aligned} \tag{32}$$

Where $f = (f_1, \ldots, f_M)$ and η is a threshold determined by decision makers to exert their direct control to the extent of desired robustness. Actually, they have used the *p-norm* function to directly extend the concept of robust solution type II from single-objective to multi-objective context.

In this section, performance of our proposed domination-based robustness concept (DBRC) in generating true robust solutions is compared with MORST2 concept (multi-objective robust solution type II) for a numerical example:

$$\begin{aligned}
&max \;\; \tilde{c}_{11}x_1 + \tilde{c}_{12}x_2 \\
&max \;\; \tilde{c}_{21}x_1 + \tilde{c}_{22}x_2 \\
&subject \; to: \\
&0 \le x_1 \le 1. \\
&0 \le x_2 \le 1.
\end{aligned} \tag{33}$$

Where $\tilde{c}_{ij}$ $1 \le i.j \le 2.$ are 4 jointly distributed uniform random variables with the mean and correlation matrixes as follows:

$$\mu_{\tilde{C}} = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix}, \quad \rho = \begin{bmatrix} \rho_{C_{11},C_{11}} & \rho_{C_{11},C_{12}} & \rho_{C_{11},C_{21}} & \rho_{C_{11},C_{22}} \\ \rho_{C_{12},C_{11}} & \rho_{C_{12},C_{12}} & \rho_{C_{12},C_{21}} & \rho_{C_{12},C_{22}} \\ \rho_{C_{21},C_{11}} & \rho_{C_{21},C_{12}} & \rho_{C_{21},C_{21}} & \rho_{C_{21},C_{22}} \\ \rho_{C_{22},C_{11}} & \rho_{C_{22},C_{12}} & \rho_{C_{22},C_{21}} & \rho_{C_{22},C_{22}} \end{bmatrix},$$
$$\rho = \begin{bmatrix} 1 & 0 & 0.9 & 0 \\ 0 & 1 & 0 & -0.7 \\ 0.9 & 0 & 1 & 0 \\ 0 & -0.7 & 0 & 1 \end{bmatrix} \tag{34}$$

The example is designed to illustrate the conditions under which MORST2 fails to generate true robust solutions.

To judge between the two methods (DBRC and MORST2), a reference robustness measure (RRM) is needed. As both methods try to extend the same single-objective robustness concept (single-objective robust solution type II) to multi-objective context, RRM is calculated through weighted-sum scalarization of the multi-objective problem and then applying the concept of single-objective robust solution type II. Thus, RRM is a function of objectives weights $w = (w_1, w_2)$:

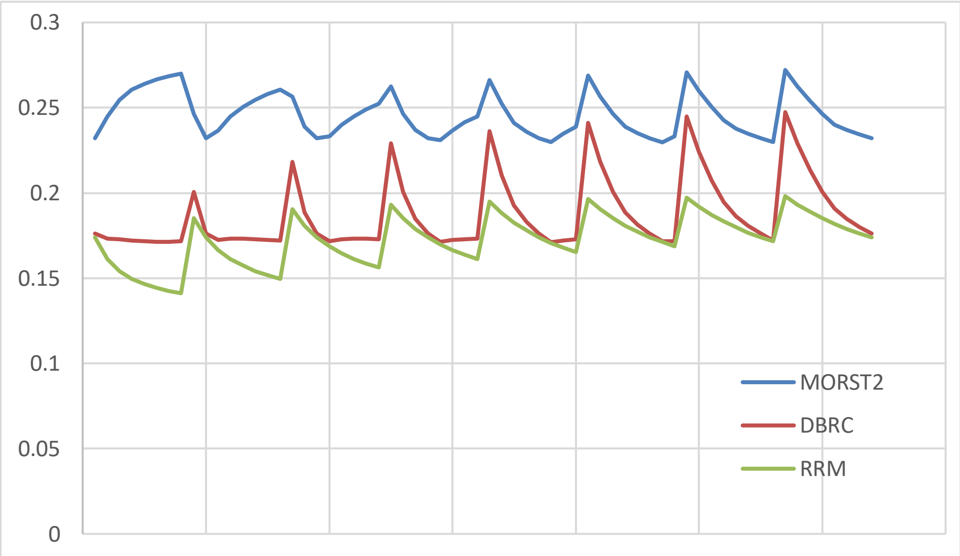
$$RRM\ (x, w) = \max_{C \varepsilon \tilde{C}} \left| \frac{w.(C.x) - w.(\mu_{\tilde{C}}.x)}{w.(\mu_{\tilde{C}}.x)} \right|$$

To compare the performance of our proposed robustness concept (DBRC) with the existing robustness concept in the literature (MORST2), the correlation between each of the two measures and RRM is calculated for points that were uniformly selected from across the decision space. So, the points $x_{ij} = \left(\frac{i}{10}, \frac{j}{10}\right), 1 \le i, j \le 10$, are selected from the decision spaces. Then, $N = 1000$ random matrices C are generated via Mont Carlo simulation. In the next step, the two robustness measures DBRC and MORST2 are calculated for each point x_{ij} $1 \le i, j \le 10$ based on the randomly generated matrices$C$. Also, the RRM is calculated for each point x_{ij} $1 \le i, j \le 10$ and each objectives weight $w_i = \left(\frac{i}{10}, 1 - \frac{i}{10}\right), 1 \le i \le 9$. Table summarizes the results. Columns of the table represent the objective weights for which RRM is calculated and the two rows represent the two robustness concepts. Values of the cells in the table represent the correlation between the corresponding robustness measure values and RRM values for the corresponding objectives weight. For example, based on the table, the correlation between the DBRC and RRM for objective weight $w = (0.1, 0.9)$ equals to 0.90. Figure 20 visualizes the results summarized in the Table. Based on the results, it can be concluded that our proposed method generates solutions that are much closer to the true robust solutions than the existing method in the literature.

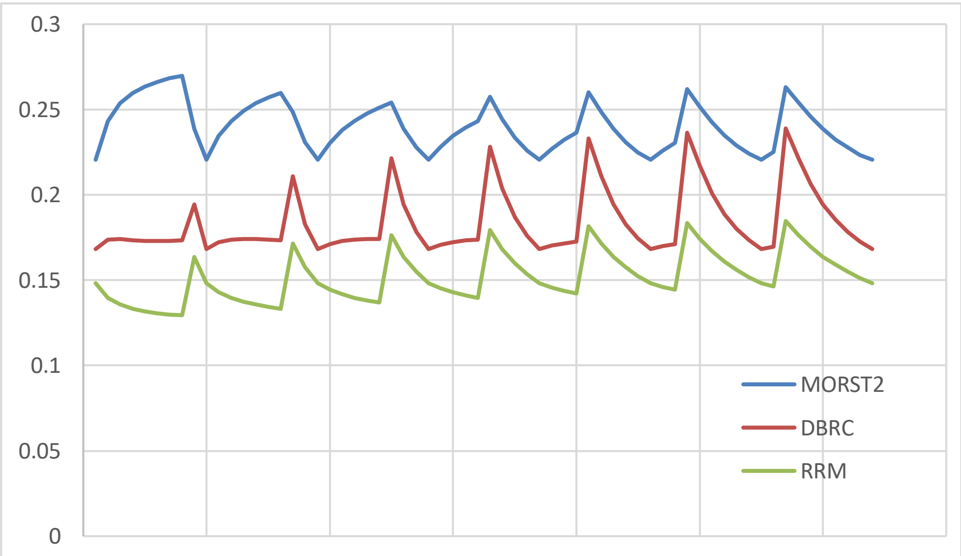
Table 2. Comparison of the correlation between the two robustness measures and RRM for different objectives weights

w / Method	(0.1,0.9)	(0.2,0.8)	(0.3,0.7)	(0.4,0.6)	(0.5,0.5)	(0.6,0.4)	(0.7,0.3)	(0.8,0.2)	(0.9,0.1)
MORST2	0.12	0.26	0.44	0.65	0.20	0.35	0.07	0.70	0.37
DBRC	0.90	0.88	0.92	0.99	0.95	0.98	0.96	0.98	0.95

a.



b.



c.

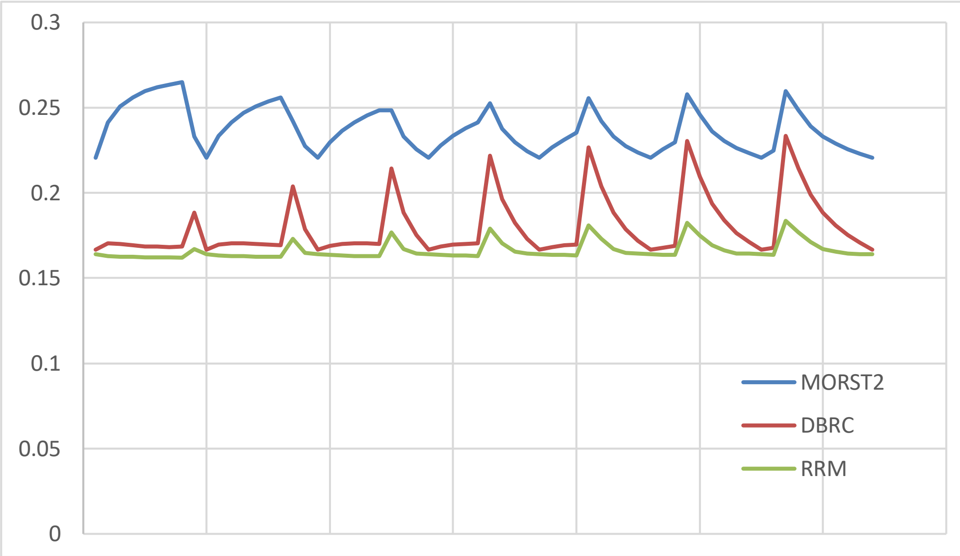


Fig 20. a,b,c. Comparison of Performance of the two robustness concepts DBRC and MORST2 with RRM for different objectives weights: a. $w = (0.3,0.7)$, b. $w = (0.5,0.5)$, c. $w = (0.7,0.3)$

5-Conclusions

A new robustness measure for multi-objective optimization problem is proposed in this paper. Literature indicates the existing proposed methods only address the magnitude of variations in the objectives space and don't take into account the direction, or in the other words, the type of variations. In this paper, we distinguished between two types of variations and called them dominating and Pareto variations. We showed that the Pareto variation is more robust than the dominating variation. Then, we developed an index that measures proportion of dominating variation in a sensitivity region. We proved that this index is always between 0 and 0.5. Also we proved that domination ratio is insensitive to the size of sensitivity region and then it is independent of variation size. Therefore, we introduced an index that measures the size of variation and called it magnitude of variation. We used these two indexes to construct our proposed robustness index.

A Multi-objective genetic algorithm is used to generate optimal Pareto solutions. Every solution is evaluated using Mont Carlo simulation. Two test problems from the engineering design optimization are used to illustrate the applicability and efficiency of proposed robustness index. Concept of sensitivity region diameter is taken from the literature and is used as another robustness index to generate robust Pareto solutions. By comparing the two robust Pareto frontiers we conclude that our proposed robustness index is efficient than the other robustness index. Results show that performance of robust Pareto solutions with respect to objective functions are slightly worse than those of nominal Pareto solutions. This means that increasing robustness leads to decreasing quality and therefore, the decision maker must make a balance. The results also show that robust solutions are compact (confined to a small range of objective functions values) while nominal solutions are dispersed over a large range of objective functions values. On the other hand, feasibility robustness values of nominal Pareto solutions are low. The results also confirm our claim that domination ratio and magnitude of variation are independent.

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