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Functional process capability indices for a simple linear profile in fuzzy environment

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Abstract

In some practical applications, the quality of a process or a product is characterized by a relationship, namely a profile between a response variable and one or more explanatory variables. Assuring the capability of a profile to meet the process requirements is particularly important. Process capability indices (PCIs) are widely used to measure whether a process is capable of reproducing product items within the specification limits (SLs). Background literature on the PCIs for profiles mostly take crisp values for process data. However, in practice, the outcomes of a measurement are often imprecise. So, the basic assumption of crisp data for process capability analysis (PCA) in profiles is not valid. Hence, fuzzy methods are developed to analyze the capability of a fuzzy profile with fuzzy response data. To this end, we extend the functional approach based on fuzzy set theory for the situations in which the SLs and target values of the response variable are imprecise. The performance of the proposed indices is investigated through simulation studies. The simulation results confirm that the proposed method performs well regarding $D_{p,q}$-distance between the estimated value and the true value of fuzzy PCIs. Furthermore, a case study shows the applicability of the proposed method.

Keywords: Fuzzy simple linear profile, fuzzy process capability index, functional approach, fuzzy random variable (FRV)

1- Introduction

Profile monitoring is the use of control charts in situations in which the quality of a process or product can be characterized by a functional relationship between a response variable and one or more explanatory variables. This relationship is referred to as a profile. According to the applications, there are various types of profiles including linear, nonlinear, generalized linear model (GLM), nonparametric, and wave-shaped profiles. Most of the works on profile monitoring are conducted on a linear profile (Ebadi and Shahriari, 2013; Wang, 2014a). Generally, there are two kinds of response variable including univariate and multivariate. The univariate response is the case that the response variable is related to only one explanatory variable, while in the multivariate response, multiple response variables are considered simultaneously.

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On the other hand, in some real-world applications, the measured values or expression of response data has a kind of uncertainty that lead to fuzzy response data. The uncertainty usually comes from several sources such as vague estimation, observations, measurement error, human judgment and linguistic definitions and so forth. Note that these data represent two kinds of uncertainty including vagueness and randomness. The first one can be modeled by the theory of fuzzy sets which initially proposed by Zadeh (1965), and the latter is related to variability and could be modeled by probability distributions and stochastic models (Viertl, 2011). In the presence of both types of uncertainties (vagueness and randomness), fuzzy random variables (FRVs) are used to model status of the process. So, in these situations, the response data is denoted by a fuzzy random variable (FRV) and a fuzzy regression model can be used to construct the fuzzy profile model (Maleki, Amiri and Castagliola, 2018).

Process capability analysis (PCA) is a statistical method that has been used for decades with a purpose to reduce the variability in industrial processes and products. The major outputs of PCA are process capability indices (PCIs). For an in-control process, PCIs are used to provide a numerical measure to find out how well a process produces products that conform to engineering specification limits (SLs). In the literature, capability analysis of process with single and multivariate product quality characteristics have been conducted based on univariate and multivariate PCIs, respectively. However, as mentioned in Woodall's (2007) review article on profile monitoring, there has been no research on the capability analysis of profile processes up to the year 2007. So, because of the importance of assuring the process capability in profiles to meet the requirements, he offered researchers to investigate PCIs for profile processes. After that, evaluating the capability of profile processes has been studied by some researchers. Generally, there are two different approaches in capability assessment of the profile including capability assessment using response or profile parameters. Here, a comprehensive review of the literature study on PCIs for profiles during the period 2009-2019 is discussed and summarized in table 1. The main information about the profile type, the type of the explanatory and response variable, process assessment approach, and the status of SLs (constant, functional, one-sided, and two-sided) is presented in table 1. Research in the area of PCIs for profiles started by study on simple linear profile (SLP). The first attempt was done by Shahriari and Sarafian (2009). They presented the PCI of response variable based on prediction interval and used the C_{pk} index in n fixed levels of the explanatory variable. Finally, they introduced the minimum value of them as a SLP process index indicator. This may result in underestimation of the process capability (PC); hence Ebadi and Shahriari (2013) presented two methods based on observed response and the value of the predictive response variable to resolve this problem. In another effort, Hosseinifard and Abbasi (2012a) utilized the proportion of nonconforming items to estimate the PC of SLPs. They considered both fixed and random explanatory variables with constant and functional SLs to assess the effectiveness of their method. Furthermore, Hosseinifard and Abbasi (2012b) studied the PCI for non-normal linear profiles. Ebadi and Amiri (2012) presented three methods for measuring PC in multivariate SLPs. Wang (2014a) proposed an index for measuring the process yield for SLPs and also, he developed two new indices for SLPs with one-sided specification (Wang, 2014b). Wang and Tamirat (2014, 2015) studied process yield analysis for autocorrelation between and within linear profiles, respectively. Subsequently, Wang (2016) and Wang and Tamirat (2016) studied several process yield indices for multivariate linear profiles. While the profile parameters are monitored to control profile processes, Karimi Ghartemani, Noorossana, and Niaki (2016) and Chiang, Lio, and Tsai (2017) introduced methods to determine PCI for SLP based on profile intercept and slope. Since all mentioned methods mainly focused on the response variable only in some levels of the explanatory variable and ignore all ranges of X-values, Nemati et al. (2014a) introduced a functional approach to evaluate the PC of SLP considering the entire domain of the explanatory variables of the profile.

Table 1. A literature review on PCIs for profiles

ID	Ref	Profile Type	Type of explanatory variables		Type of response variable		Approach		Status of SLs			
			Constant	Variable	Univariate	Multivariate	Response	Profile parameters	Constant	Functional	One- sided	Two- sided
1	Shahriari and Sarafian (2009)	SLP	✓		✓		✓		✓	✓		✓
2	Hosseinifard and Abbasi (2012a)	SLP	✓	✓	✓		✓		✓	✓	✓	✓
3	Hosseinifard and Abbasi (2012b)	SLP	✓		✓		✓		✓		✓	✓
4	Ebadi and Amiri (2012)	SLP	✓			✓	✓			✓		✓
5	Ebadi and Shahriari (2013)	SLP	✓		✓		✓			✓		✓
6	Wang (2014a)	SLP	✓		✓		✓			✓		✓
7	Wang (2014b)	SLP	✓		✓		✓			✓	✓	
8	Wang and Tamirat (2014)	SLP	✓		✓		✓			✓		✓
9	Nemati et al. (2014a)	SLP	✓		✓		✓			✓		✓
10	Nemati et al. (2014b)	Circular profile	✓		✓		✓			✓		✓
11	Wang and Guo (2014)	Nonlinear profile	✓		✓		✓			✓		✓
12	Rezaye Abbasi Charkhi et al. (2015)	Logistic profile	✓		✓		✓			✓		✓
13	Guevara and Vargas (2015a)	Nonlinear profile	✓		✓		✓			✓		✓
14	Guevara and Vargas (2015b)	Nonlinear profile	✓			✓	✓			✓		✓
15	Wang (2015)	Circular profile	✓		✓		✓			✓		✓
16	Wang and Tamirat (2015)	SLP	✓		✓		✓			✓		✓
17	Wang and Tamirat (2016)	SLP	✓			✓	✓			✓	✓	
18	Guevara et al. (2016)	Nonlinear profile	✓		✓		✓			✓		✓
19	Rezaye Abbasi Charkhi et al. (2016)	Logistic profile	✓		✓		✓			✓		✓
20	Wang (2016)	SLP	✓			✓	✓			✓		✓
21	Karimi Ghartemani et al. (2016)	SLP	✓		✓			✓		✓		✓
22	Chiang et al. (2017)	SLP	✓		✓			✓		✓		✓
23	Alevizakos et al. (2018)	Poisson profile	✓		✓		✓			✓		✓
24	Pour Larimi et al. (2019)	Nonlinear profile	✓		✓		✓			✓		✓
25	Alevizakos et al. (2019)	Logistic profile	✓		✓		✓			✓		✓
26	Abbasi Ganji and Sadeghpour Gildeh (2019)	SLP	✓		✓		✓			✓		✓

As seen in table 1, during the period between 2009 and 2014, most of the PCIs for profile data have been provided for linear regression profiles. After that, Nemati et al. (2014b), Wang (2015) and Pour Larimi, Nemati, and Safaei (2019) utilized the functional approach to introduce PCIs for more complicated profiles such as circular and nonlinear profile. Wang and Guo (2014) evaluated the process yield for processes characterized by nonlinear profiles. Guevara and Vargas (2015a) and Guevara, Vargas, and Castagliola

(2016) proposed some methods to measure the capability of nonlinear profiles and afterwards, Guevara and Vargas (2015b) extended the approach of Guevara and Vargas (2015a) to evaluate the capability of processes characterized by multivariate nonlinear profiles. Considering logistic regression profiles, Rezaye Abbasi Charkhi, Aminnayeri, and Amiri (2015, 2016) introduced methods to measure capability of process. Alevizakos, Koukouvinos, and Lappa (2019) conducted a comparative study on the presented PCIs in Rezaye Abbasi Charkhi, Aminnayeri, and Amiri (2015, 2016) for logistic regression profile. In another study, Alevizakos, Koukouvinos, and Castagliola (2018) introduced a PCI for the Poisson regression profile.

In dealing with real engineering problems, it is a frequent occurrence that due to the measurement uncertainty such as the appraiser's subjective judgement based on their experience, unstable measurement instrument, and change of measurement environment or the nature of quality characteristics, the data can't be recorded or measured precisely. Therefore, in the presence of these situations, the traditional PCIs with crisp data may mislead to know the capability of manufacturing process. On the other hand, in some applications, there are processes with linguistics or imprecise SLs (Parchami et al., 2005). In this regard, many studies aggregate PCIs with the fuzzy set theory. Literatures on PCIs in fuzzy environments could be clustered into four groups in terms of their methods in estimating fuzzy PCIs. The first group is related to PCIs based on fuzzy quality concept. Bradshaw (1983) used the fuzzy set theory as a basis for a graded degree of product conformance based on a graded representation of the SLs in the field of fuzzy process control. An initial study by Yongting (1996) introduced the concept of fuzzy quality in PCA and presented a PCI as a crisp number. Then, it is extended by Parchami and Mashinchi (2010), Parchami et al. (2013) and Sadeghpour Gildeh and Moradi (2012) on fuzzy SLs, fuzzy data and SLs, and general multivariate PCIs for crisp data, respectively. The second group includes the methods based on estimating the membership function of fuzzy PCIs, which was first introduced by Lee (2001) and developed by others in univariate and multivariate cases (see Hong, 2004; Tsai and Chen, 2006a, 2006b; Shu and Wu, 2009, 2012; Chen, Lai, and Nien, 2010; Abdolshah et al., 2011; Abbasi Ganji and Sadeghpour Gildeh, 2016b, 2016c). The third group consists of the methods based on the extension principle. Parchami et al. (2005) suggested a fuzzy PCI based on the extension principle for the cases in which SLs are fuzzy numbers. Then, Moeti, Parchami, and Mashinchi (2006) discussed a generalized version of PCIs introduced in (Parchami et al., 2005), based on L–R fuzzy number SLs. Also, similar arguments and findings can be seen in (Kahraman and Kaya, 2009; Kaya and Kahraman, 2011a). The last group include the methods that estimate fuzzy PCIs using Buckley's (2004) approach. Parchami and Mashinchi (2007) proposed an algorithm for fuzzy estimation of PCIs based on Buckley's approach. Hsu and Shu (2008) estimated fuzzy PCIs when observations and SLs are all real numbers using Buckley's approach. Also, the Buckley's fuzzy estimation method is used to obtain the membership function of several PCIs by Kaya and Kahraman (2010, 2011b), Moradi and Sadeghpour Gildeh (2013) and Abbasi Ganji and Sadeghpour Gildeh (2016a).

In the case of profile capability assessment, there is a recent paper in which fuzzy logic is applied to measure the capability of SLP when SLs are imprecise (Abbasi Ganji and Sadeghpour Gildeh, 2019). On the other hand, in some real-world applications, it is likely to face with imprecise or linguistic response variables. Hence, some researchers have conducted studies on fuzzy profiles analysis (Noghondarian and Ghobadi, 2012; Ghobadi et al., 2014; Moghadam, Ardali and Amirzadeh, 2015, 2018). All of these studies have focused on monitoring of fuzzy profiles and there is yet no method for analyzing the capability of profile process with fuzzy response data. So, taking into account the fuzzy response data to propose novel process capability indices is needed. This fact is well supported by the recent review by Maleki, Amiri, and Castagliola (2018). Since a crisp process capability indices are not adequate for the case that the response variable in each level of the explanatory variable is fuzzy quantities, the present study aims to extend the functional approach proposed by Nemati et al. (2014a) based on the fuzzy set theory to suggest novel fuzzy functional PCIs for fuzzy simple linear profile. The remainder of this paper is organized as follows. In section 2 a brief overview of functional PCIs for SLP is presented. Section 3 includes some basic fuzzy concepts. In Section 4, fuzzy simple linear profiles are described. The proposed methods for evaluating the PC of fuzzy simple linear profiles are presented in section 5. In section 6, a simulation study is carried out to investigate the performance of the proposed method. Section 7 presents the application of the proposed method through real case study and finally, conclusions are drawn and suggestions are made for future research.

2- Functional PCIs for SLP; a brief overview

A simple linear profile is usually defined by a simple linear regression model. Considering m random samples of size n are taken from the process, the model relating a single explanatory variable to the response when the process is in statistical control can be defined as:

$$Y_{ij} = A_0 + A_1 X_i + \epsilon_{ij}, \quad i = 1,2,\dots,n, \qquad j = 1,2,\dots,m. \tag{1}$$

where ϵ_{ij} are normally and independently distributed with the mean zero and variance σ^2 and X_i is explanatory variable that is assumed to has fix values for each sample. The profile parameters including A_0, A_1 and σ^2 are unknown and estimated by $a_0 = \frac{\sum_{j=1}^{m} a_{0j}}{m}$, $a_1 = \frac{\sum_{j=1}^{m} a_{1j}}{m}$, and $\sigma^2 = \frac{\sum_{j=1}^{m} \sigma_j^2}{m}$, respectively. Where, the values of A_0, A_1 and, σ^2 for profile sample j could be estimated using maximum likelihood method, which can be seen in (Noorossana, Saghaei, and Amiri, 2011).

According to table 1, there are several PCIs for evaluating the capability of SLPs. The emphasis on using the functional approach, is due to its advantages in measuring PC in the entire domain of the explanatory variables. The indices $C_p(Profile)$ and $C_{pk}(Profile)$ proposed by Nemati et al. (2014a) are established as equations (2) and (3).

$$C_p(Profile) = \frac{\int_{x_l}^{x_u}\left[USL_y(x) - LSL_y(x)\right]dx}{\int_{x_l}^{x_u}\left[UNTL_y(x) - LNTL_y(x)\right]dx} \tag{2}$$

$$C_{pk}(Profile) = min\left\{\frac{\int_{x_l}^{x_u}\left[USL_y(x) - \mu_y(x)\right]dx}{\int_{x_l}^{x_u}\left[UNTL_y(x) - \mu_y(x)\right]dx}, \frac{\int_{x_l}^{x_u}\left[\mu_y(x) - LSL_y(x)\right]dx}{\int_{x_l}^{x_u}\left[\mu_y(x) - LNTL_y(x)\right]dx}\right\} \tag{3}$$

where $USL_y(x)$, $LSL_y(x)$, $UNTL_y(x)$, $LNTL_y(x)$ and $\mu_y(x)$, are functional form of upper SL, lower SL, upper natural tolerance limit, lower natural tolerance limit, and mean of response variable, respectively. All of them are functions of explanatory variable $x \in [x_l, x_u]$, that x_l and x_u are the minimum and maximum value of the explanatory variable, respectively. The functional SLs are obtained by linear regression and the functional form of μ, UNTL, and LNTL are as $\mu_y(x) = a_0 + a_1 x$, $UNTL_y(x) = a_0 + a_1 x + 3\sigma$, and $LNTL_y(x) = a_0 + a_1 x - 3\sigma$, respectively, where σ is the process standard deviation. It should be noted that, the interpretation of these two functional PCIs is conducted similar to the traditional univariate cases, by comparing the values of them with 1 (Nemati et al., 2014a).

3- Some fuzzy concepts

Here, some definitions, which will be needed throughout the paper are reviewed.

Definition 1- $\tilde{A} = (a, \lambda, \beta)$ is called the asymmetric triangular fuzzy number with center a, a right spread β and a left spread λ if its membership function can be defined by equation (4).

$$\mu_{\tilde{A}}(x) = \begin{cases} \frac{x - (a - \lambda)}{\lambda} & a - \lambda < x \le a \\ \frac{(a + \beta) - x}{\beta} & a \le x \le a + \beta \\ 0 & Otherwise \end{cases} \tag{4}$$

If right and left spreads of the triangular fuzzy number are equal to S, then a triangular fuzzy number called symmetric and denoted by $\tilde{A} = (a, S)$ (Moghadam, Ardali and Amirzadeh, 2018).

Definition 2- Let $\tilde{A} = (a, \lambda_a, \beta_a)$ and $\tilde{B} = (b, \lambda_b, \beta_b)$ be two asymmetric triangular fuzzy numbers. The operators of fuzzy summation, fuzzy subtraction, and multiplication of a fuzzy number by a scalar (K) are calculated as follows (Moghadam, Ardali and Amirzadeh, 2015).

$$\tilde{A} \oplus \tilde{B} = (a + b, \lambda_a + \lambda_b, \beta_a + \beta_b\ ) \tag{5}$$

$$\tilde{A} \ominus \tilde{B} = (a - b, \lambda_a + \beta_b, \beta_a + \lambda_b\ ) \tag{6}$$

$$K \otimes \tilde{A} = \begin{cases} (Ka, K\lambda_a, K\beta_a) & \text{if } K > 0 \\ (Ka, -K\beta_a, -K\lambda_a) & \text{if } K < 0 \end{cases} \tag{7}$$

Definition 3- The α-cut of a fuzzy number $\tilde{A}$ $(\alpha \in [0,1])$ is a crisp set denoted by $\tilde{A}(\alpha) = \{x \in \mathbb{R}\, | \mu_{\tilde{A}}(x) \geq \alpha\, \}$. Hence, $\tilde{A}(\alpha) = [\tilde{A}_l(\alpha), \tilde{A}_r(\alpha)]$, where left and right side of α-cut set of fuzzy number $\tilde{A}$ are $\tilde{A}_l(\alpha) = inf\{x \in \mathbb{R}\ | \mu_{\tilde{A}}(x) \geq \alpha\, \}$ and $\tilde{A}_r(\alpha) = sup\{x \in \mathbb{R}\ | \mu_{\tilde{A}}(x) \geq \alpha\, \}$, respectively (Lee, 2005). If $\tilde{A}$ is an asymmetric triangular fuzzy number, α-cut sets of $\tilde{A}$ is as equation (8).

$$\tilde{A}(\alpha) = [(a - \lambda) + \alpha\lambda, (a + \beta) - \alpha\beta] \tag{8}$$

Definition 4- The $D_{p,q}$-distance between two fuzzy quantity $\tilde{A}$ and $\tilde{B}$ is determined by equation (9).

$$D_{p,q}(\tilde{A}, \tilde{B}) = \left[(1 - q)\int_0^1 [\tilde{A}_l(\alpha) - \tilde{B}_l(\alpha)]^p d\alpha + q\int_0^1 [\tilde{A}_r(\alpha) - \tilde{B}_r(\alpha)]^p d\alpha\right]^{\frac{1}{p}} \tag{9}$$

Where often, p and q are equal to 2 and 0.5, respectively. For triangular fuzzy numbers $\tilde{A} = (a, \lambda_a, \beta_a)$ and $\tilde{B} = (b, \lambda_b, \beta_b)$, the $D_{2,\frac{1}{2}}(\tilde{A}, \tilde{B})$ is shown in equation (10) (Sadeghpour Gildeh, 2011).

$$D_{2,\frac{1}{2}}(\tilde{A}, \tilde{B}) = \sqrt{(a - b)^2 + \frac{1}{6}[(\lambda_a - \lambda_b)^2 + (\beta_a - \beta_b)^2] + \frac{1}{2}(a - b)[(\beta_a - \lambda_a)^2 + (\beta_b - \lambda_b)^2]} \tag{10}$$

Definition 5- To rank two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, one can calculate their $D_{p,q}$-distance from fuzzy number $\tilde{C}$. Where, $\tilde{C}$ is an arbitrary value which is greater (less) than $\tilde{A}$ and $\tilde{B}$. So, for the case that $\tilde{C}$ is greater than both of $\tilde{A}$ and $\tilde{B}$, the fuzzy number $\tilde{A}$ is greater than or equal to the fuzzy number $\tilde{B}$, if and only if $D_{p,q}(\tilde{A}, \tilde{C}) \leq D_{p,q}(\tilde{B}, \tilde{C})$. Similarly, when $\tilde{C}$ is less than both of $\tilde{A}$ and $\tilde{B}$, the fuzzy number $\tilde{A}$ is greater than or equal to the fuzzy number $\tilde{B}$, if and only if $D_{p,q}(\tilde{A}, \tilde{C}) \geq D_{p,q}(\tilde{B}, \tilde{C})$.

4- Fuzzy simple linear profiles

In a fuzzy SLP, the fuzzy response variable $\tilde{Y}$ is related to only one explanatory variable X through a fuzzy linear regression model. Assume that there are *m* sample sets for *n* levels of the explanatory variable. Therefore, we deal with samples in the form of $(x_i, \tilde{y}_{ij}), i = 1, 2, \dots, n, j = 1, 2, \dots, m$, where $\tilde{y}_{ij}$ is the value of fuzzy quality characteristic and modeled by asymmetric triangular fuzzy number in the general form of $\tilde{y}_{ij} = (y_{ij}, \lambda_{ij}, \beta_{ij})$ in which *j* and *i* are the sample number and level of the explanatory variable, respectively. For each sample, suppose the relationship between the response and the explanatory variable can be modeled as equation (11).

$$\tilde{Y}_{ij} = \tilde{A}_{0j} + \tilde{A}_{1j}X_i, \qquad i = 1, \dots, n, \qquad j = 1, \dots, m. \tag{11}$$

In equation (11), it is assumed that X_i has a fixed non-fuzzy value for all the samples. $\tilde{A}_{0j}$ and $\tilde{A}_{1j}$ are the coefficients of sample profiles and are unknown. In order to estimate the coefficients of model in equation (11), we can use a fuzzy linear regression (FLR) model based on goal programming developed by Hassanpour, Maleki and Yaghoobi (2009). This model takes into account the centers of fuzzy data as an important feature as well as their spreads. Furthermore, the model can deal with both symmetric and asymmetric triangular fuzzy numbers (for more information, see Appendix A). So, the estimators of $\tilde{A}_{rj}, r = 0, 1$ are asymmetric triangular fuzzy number $\tilde{a}_{rj} = (a_{rj}, \lambda_{rj}, \beta_{rj}), r = 0, 1$. Hence, $\hat{\tilde{Y}}_{ij} = \tilde{a}_{0j} + \tilde{a}_{1j}X_i, i = 1,2, \ldots, n, j = 1, \ldots, m$, where $\hat{\tilde{Y}}_{ij}$ denotes the estimated value of the response variable. Since the square of the deviation between the observed and estimated response variable ($\left(\tilde{Y}_{ij} \ominus \hat{\tilde{Y}}_{ij}\right) \otimes \left(\tilde{Y}_{ij} \ominus \hat{\tilde{Y}}_{ij}\right)$) equals the square of estimation errors in FLR, and $\oplus_{i=1}^{n} \left(\left(\tilde{Y}_{ij} \ominus \hat{\tilde{Y}}_{ij}\right) \otimes \left(\tilde{Y}_{ij} \ominus \hat{\tilde{Y}}_{ij}\right)\right)$ equals the sum of squared errors (SSE), the mean of squared error, which is calculated in equation (12), can be considered as estimate of error variance of j^{th} profile sample.

$$\hat{\tilde{\sigma}}_j^2 = \frac{\oplus_{i=1}^{n} \left(\left(\tilde{Y}_{ij} \ominus \hat{\tilde{Y}}_{ij}\right) \otimes \left(\tilde{Y}_{ij} \ominus \hat{\tilde{Y}}_{ij}\right)\right)}{n}, j = 1,2, \ldots, m. \tag{12}$$

After applying phase I monitoring fuzzy SLPs and achieving the in-control process, all remained samples follow a unique stable profile model, are shown by $\tilde{Y} = \tilde{A}_0 + \tilde{A}_1 X$. The stable values of the parameters $\tilde{A}_0$, $\tilde{A}_1$, and $\tilde{\sigma}^2$ are assumed to be known or can be estimated from an in-control profile samples through equation (13) for use in PCA (Ghobadi et al., 2014).

$$\hat{\tilde{A}}_o = \tilde{a}_0 = \frac{\oplus_{j=1}^{m} \tilde{a}_{0j}}{m}, \hat{\tilde{A}}_1 = \tilde{a}_1 = \frac{\oplus_{j=1}^{m} \tilde{a}_{1j}}{m}, \hat{\tilde{\sigma}}^2 = \frac{\oplus_{j=1}^{m} \hat{\tilde{\sigma}}_j^2}{m}. \tag{13}$$

5- Fuzzy functional PCIs for fuzzy simple linear profile

In this section, to evaluate the capability of fuzzy SLPs, we introduce fuzzy PCIs to take into account fuzzy environments. So, the situations are investigated in which the SLs and target values of the response variable in each level of the explanatory variable are imprecise. In this way, we firstly extend two functional PCIs ($C_p(Profile)$ and $C_{pk}(Profile)$) proposed by Nemati et al. (2014a) based on fuzzy set theory. Then, since these two functional indices only focused on the process yield and cannot reflect departures of the process mean from the target value, we propose two fuzzy indices $\tilde{C}_{pm}(Profile)$ and $\tilde{C}_{pmk}(Profile)$ to overcome the deficiency of $\tilde{C}_p(Profile)$ and $\tilde{C}_{pk}(Profile)$, in considering the proximity of the process mean to the target. To do these, initially, we should ensure the critical assumptions of PCA which are stability of the profile process and normality of the response variable. In this regard, the fuzzy functional PCIs are calculated at the end of phase I of fuzzy profile monitoring and when the process is in-control; and in the case of the normality assumption of the quality characteristic, the response variable is normally distributed and expressed by FRV based on Kwakernaak (1978, 1979) approach. In this approach, FRV is defined as “a random variable whose values are not real numbers, but are fuzzy ones”. In other words, it is assumed that an imprecise value is assigned to a random variable with available and crisp values. Hence, the fuzzy observations of the response variable $\tilde{y}_{ij}$ are the observations of a crisp normal random variable Y_{ij}.

5-1- Description of fuzzy profile parameters, fuzzy SLs, and fuzzy target value

Assume that $\tilde{Y} = \tilde{A}_0 + \tilde{A}_1 X$ is the reference line of the fuzzy SLP obtained in phase I of fuzzy SLP monitoring. The $\hat{\tilde{A}}_o$, $\hat{\tilde{A}}_1$ and $\hat{\tilde{\sigma}}^2$ are given in equation (13). So, $\tilde{Y}$ is a fuzzy normal random variable with mean equal to $\tilde{\mu}_y(x) = \tilde{a}_0 \oplus (\tilde{a}_1 \otimes x)$ in $x \in [x_l, x_u]$, and variance $\hat{\tilde{\sigma}}^2$. The α-cut sets of $\tilde{\mu}_y(x)$ are as equation (14).

$$\tilde{\mu}_y(x)(\alpha) = \left[\tilde{\mu}_y(x)_l(\alpha), \tilde{\mu}_y(x)_r(\alpha)\right] = \left[\, \tilde{a}_{0_l}(\alpha) + \tilde{a}_{1_l}(\alpha)x,\ \tilde{a}_{0_r}(\alpha) + \tilde{a}_{1_r}(\alpha)x\right] \tag{14}$$

Also, to gain $\hat{\tilde{\sigma}}^2$, we use the α-cut and interval algebra approach. So, the α-cut intervals of $\hat{\tilde{\sigma}}^2$ can be calculated by equation (15).

$$\hat{\tilde{\sigma}}^2(\alpha) = \left[\hat{\tilde{\sigma}}^2{}_l(\alpha), \hat{\tilde{\sigma}}^2{}_r(\alpha)\right] = \left[\sum_{j=1}^{m}\sum_{i=1}^{n}\frac{\left(\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\otimes\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\right)_l(\alpha)}{mn}, \sum_{j=1}^{m}\sum_{i=1}^{n}\frac{\left(\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\otimes\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\right)_r(\alpha)}{mn}\right] \quad (15)$$

where

$$\left(\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\otimes\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\right)(\alpha) = \left[\left(\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\otimes\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\right)_l(\alpha), \left(\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\otimes\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\right)_r(\alpha)\right] = \left[min\left\{\left(Y_{ij}-\hat{Y}_{ij}\right)^2, \begin{array}{l} Y_{ij}\in\tilde{Y}_{ij}(\alpha)\\ \hat{Y}_{ij}\in\hat{\tilde{Y}}_{ij}(\alpha)\end{array}\right\}, max\left\{\left(Y_{ij}-\hat{Y}_{ij}\right)^2, \begin{array}{l} Y_{ij}\in\tilde{Y}_{ij}(\alpha)\\ \hat{Y}_{ij}\in\hat{\tilde{Y}}_{ij}(\alpha)\end{array}\right\}\right] \quad (16)$$

In equation (16) those answers are chosen that for each $\alpha_1 < \alpha_2$, α_1 and $\alpha_2 \in [0,1]$, we have $H_l(\alpha_1) \leq H_l(\alpha_2) \leq H_l(1) = H_r(1) \leq H_r(\alpha_2) \leq H_r(\alpha_1)$, where $\left(\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\otimes\left(\tilde{Y}_{ij}\ominus\hat{\tilde{Y}}_{ij}\right)\right) = H$. Now we place these α-cut intervals one on top of the other to obtain $\hat{\tilde{\sigma}}^2$. Then, we can define $\widetilde{LNTL}_y(x)$ and $\widetilde{UNTL}_y(x)$ as $\left(\tilde{a}_0\oplus(\tilde{a}_1\otimes x)\right)\ominus 3\hat{\tilde{\sigma}}$ and $\left(\tilde{a}_0\oplus(\tilde{a}_1\otimes x)\right)\oplus 3\hat{\tilde{\sigma}}$, respectively. Suppose that the SLs of $\tilde{Y}$ are linear functions of the fixed values of the explanatory variable X as $\widetilde{lSL}_y(x) = \tilde{a}'_0\oplus(\tilde{a}'_1\otimes x)$ an $\widetilde{USL}_y(x) = \tilde{a}''_0\oplus(\tilde{a}''_1\otimes x)$. Noted that $\tilde{a}'_0$ and $\tilde{a}'_1$ ($\tilde{a}''_0$ and $\tilde{a}''_1$) are the triangular fuzzy numbers of intercepts and slopes of $\widetilde{lSL}_y(x)$ ($\widetilde{USL}_y(x)$), respectively. These fuzzy linear relationships are obtained by the FLR model developed in (Hassanpour, Maleki and Yaghoobi, 2009). So, the α-cut intervals of $\widetilde{lSL}_y(x)$ and $\widetilde{USL}_y(x)$ are defined by equations (17) and (18), respectively.

$$\widetilde{LSL}_y(x)(\alpha) = \left[\widetilde{LSL}_y(x)_l(\alpha), \widetilde{LSL}_y(x)_r(\alpha)\right] = \left[\tilde{a}'_{0_l}(\alpha) + \tilde{a}'_{1_l}(\alpha)x, \tilde{a}'_{0_r}(\alpha) + \tilde{a}'_{1_r}(\alpha)x\right] \quad (17)$$

$$\widetilde{USL}_y(x)(\alpha) = \left[\widetilde{USL}_y(x)_l(\alpha), \widetilde{USL}_y(x)_r(\alpha)\right] = \left[\tilde{a}''_{0_l}(\alpha) + \tilde{a}''_{1_l}(\alpha)x, \tilde{a}''_{0_r}(\alpha) + \tilde{a}''_{1_r}(\alpha)x\right] \quad (18)$$

Moreover, target value of $\tilde{Y}$ is defined as $\tilde{T}_y(x) = \tilde{a}_{0T}\oplus(\tilde{a}_{1T}\otimes x)$ where $\tilde{a}_{0T}$ and $\tilde{a}_{1T}$ are the triangular fuzzy numbers of intercept and slope of the fuzzy functional target value, respectively. The α-cut sets of $\tilde{T}_y(x)$ can be defined similar to equations (17) and (18).

5-2- Fuzzy functional index $\tilde{C}_p(Profile)$

In this subsection, we introduce $\tilde{C}_p(Profile)$ that is based on the concept of fuzzy integral. To compute it, we obtain it's α–cuts that are calculated using equation (19).

$$\tilde{C}_p(Profile)(\alpha) = \left[\frac{\int_{x_l}^{x_k}\left[\widetilde{USL}_y(x)_l(\alpha) - \widetilde{LSL}_y(x)_r(\alpha)\right]dx}{\int_{x_l}^{x_k} 6\sqrt{\tilde{\sigma}_r^2(\alpha)}dx}, \frac{\int_{x_l}^{x_k}\left[\widetilde{USL}_y(x)_r(\alpha) - \widetilde{LSL}_y(x)_l(\alpha)\right]dx}{\int_{x_l}^{x_k} 6\sqrt{\tilde{\sigma}_l^2(\alpha)}dx}\right] \quad (19)$$

where the α-cut intervals of fuzzy variance and SLs are obtained based on equations (15), (17) and (18). Now place these α-cut intervals one on top of the other to produce $\tilde{C}_p(Profile)$.

5-3- Fuzzy functional index $\widetilde{C}_{pk}(Profile)$

Here we introduce $\tilde{C}_{pk}(Profile)$ which is minimum of $\tilde{C}_{pU}(Profile)$ and $\tilde{C}_{pL}(Profile)$ indices. To find $\tilde{C}_{pk}(Profile)$, first we obtain α-cut intervals of each item as equations (20) and (21).

$$\tilde{C}_{pU}(Profile)(\alpha)=\left[\frac{\int_{x_l}^{x_k}\left(\widetilde{USL}_y(x)_l(\alpha)-\tilde{\mu}_y(x)_r(\alpha)\right)dx}{\int_{x_l}^{x_k}3\sqrt{\tilde{\sigma}_r^2(\alpha)}dx},\frac{\int_{x_l}^{x_k}(\widetilde{USL}_y(x)_r(\alpha)-\tilde{\mu}_y(x)_l(\alpha))dx}{\int_{x_l}^{x_k}3\sqrt{\tilde{\sigma}_l^2(\alpha)}dx}\right] \tag{20}$$

$$\tilde{C}_{pL}(Profile)(\alpha)=\left[\frac{\int_{x_l}^{x_k}\left(\tilde{\mu}_y(x)_l(\alpha)-\widetilde{LSL}_y(x)_r(\alpha)\right)dx}{\int_{x_l}^{x_k}3\sqrt{\tilde{\sigma}_r^2(\alpha)}dx},\frac{\int_{x_l}^{x_k}(\tilde{\mu}_y(x)_r(\alpha)-\widetilde{LSL}_y(x)_l(\alpha))dx}{\int_{x_l}^{x_k}3\sqrt{\tilde{\sigma}_l^2(\alpha)}dx}\right] \tag{21}$$

where the α-cut intervals of fuzzy mean, variance and SLs are obtained by equations (14), (15), (17) and (18). Then, we apply the ranking method mentioned in definition 5 to get $\tilde{C}_{pk}(Profile)$.

5-4- Fuzzy functional index $\widetilde{C}_{pm}(Profile)$

To develop capability index $\tilde{C}_{pm}(x)$ for fuzzy SLP, the square form of the functional index C_{pm} for SLP must be firstly obtained. The index $\tilde{C}_{pm}^2(Profile)$ is defined based on the concept fuzzy integral. But, to gain $\tilde{C}_{pm}^2(Profile)$, we get it's α-cut intervals based on equation (22).

$$\begin{aligned}\tilde{C}_{pm}^2(Profile)(\alpha)&=\left[\widetilde{C}_{pm}^2(Profile)_l(\alpha),\widetilde{C}_{pm}^2(Profile)_r(\alpha)\right]\\&=\left[\frac{\widetilde{G}_l(\alpha)}{36\left(\int_{x_l}^{x_u}\tilde{\sigma}^2{}_r(\alpha)dx+\widetilde{F}_r(\alpha)\right)},\frac{\widetilde{G}_r(\alpha)}{36\left(\int_{x_l}^{x_u}\tilde{\sigma}^2{}_l(\alpha)dx+\widetilde{F}_l(\alpha)\right)}\right]\end{aligned} \tag{22}$$

In equation (22), we define $\tilde{G}=\int_{x_l}^{x_u}\tilde{Z}(\alpha)\,dx$ and $\tilde{F}=\int_{x_l}^{x_u}\widetilde{W}(\alpha)\,dx$, where $\tilde{Z}=\left(\left(\widetilde{USL}_y(x)\ominus\widetilde{LSL}_y(x)\right)\otimes\left(\widetilde{USL}_y(x)\ominus\widetilde{LSL}_y(x)\right)\right)$ and $\widetilde{W}=\left(\left(\tilde{\mu}_y(x)\ominus\tilde{T}_y(x)\right)\otimes\left(\tilde{\mu}_y(x)\ominus\tilde{T}_y(x)\right)\right)$. Therefore, the α-cut intervals of $\tilde{\sigma}^2$, $\tilde{G}$ and $\tilde{F}$ are given in equations (15), (23) and (24), respectively.

$$\tilde{G}(\alpha)=[\tilde{G}_l(\alpha),\tilde{G}_r(\alpha)]=\begin{bmatrix}min\left\{\left(\int_{x_l}^{x_u}\left(USL_y(x)-LSL_y(x)\right)^2dx\right),\begin{matrix}USL_y(x)\in\widetilde{USL}_y(x)(\alpha)\\LSL_y(x)\in\widetilde{LSL}_y(x)(\alpha)\end{matrix}\right\},\\max\left\{\left(\int_{x_l}^{x_u}\left(USL_y(x)-LSL_y(x)\right)^2dx\right),\begin{matrix}USL_y(x)\in\widetilde{USL}_y(x)(\alpha)\\LSL_y(x)\in\widetilde{LSL}_y(x)(\alpha)\end{matrix}\right\}\end{bmatrix} \tag{23}$$

$$\tilde{F}(\alpha)=[\tilde{F}_l(\alpha),\tilde{F}_r(\alpha)]=\begin{bmatrix}min\left\{\left(\int_{x_l}^{x_u}\left(\mu_y(x)-T_y(x)\right)^2dx\right),\begin{matrix}\mu_y(x)\in\tilde{\mu}_y(x)(\alpha)\\T_y(x)\in\tilde{T}_y(x)(\alpha)\end{matrix}\right\},\\max\left\{\left(\int_{x_l}^{x_u}\left(\mu_y(x)-T_y(x)\right)^2dx\right),\begin{matrix}\mu_y(x)\in\tilde{\mu}_y(x)(\alpha)\\T_y(x)\in\tilde{T}_y(x)(\alpha)\end{matrix}\right\}\end{bmatrix} \tag{24}$$

It should be noted that, in equation (23), the answers are chosen that for each $\alpha_1<\alpha_2$, α_1 and $\alpha_2\in[0,1]$, $\tilde{G}_l(\alpha_1)\le\tilde{G}_l(\alpha_2)\le\tilde{G}_l(1)=\tilde{G}_r(1)\le\tilde{G}_r(\alpha_2)\le\tilde{G}_r(\alpha_1)$. Also, in equation (24), the answers are chosen that for each $\alpha_1<\alpha_2,\alpha_1$ and $\alpha_2\in[0,1]$, $\tilde{F}_l(\alpha_1)\le\tilde{F}_l(\alpha_2)\le\tilde{F}_l(1)=\tilde{F}_r(1)\le\tilde{F}_r(\alpha_2)\le\tilde{F}_r(\alpha_1)$. Finally, the α-cut intervals of $\tilde{C}_{pm}(Profile)$ can be calculated by equation (25).

$$\tilde{C}_{pm}(Profile)(\alpha) = \left[\sqrt{\tilde{C}^2_{pm}(Profile)_l(\alpha)}, \sqrt{\tilde{C}^2_{pm}(Profile)_r(\alpha)}\right] \quad (25)$$

By placing these α-cut intervals one on top of the other, the $\tilde{C}_{pm}(Profile)$ will be obtained.

5-5- Fuzzy functional index $\tilde{C}_{pmk}(Profile)$

To develop the $\tilde{C}_{pmk}(x)$ index for fuzzy SLP, we follow similar steps in subsection 5-4. So, the index $\tilde{C}^2_{pmk}(Profile)$ is defined by equation (26).

$$\tilde{C}^2_{pmk}(Profile) = min\left\{\tilde{C}^2_{pmkU}(Profile), \tilde{C}^2_{pmkL}(Profile)\right\} \quad (26)$$

To obtain $\tilde{C}^2_{pmk}(Profile)$, first we define α-cut intervals of each item in equation (26), including $\tilde{C}^2_{pmkU}(Profile)$ and $\tilde{C}^2_{pmkL}(Profile)$ as equations (27) and (28).

$$\tilde{C}^2_{pmkU}(Profile)(\alpha) = \left[\tilde{C}^2_{pmkU_l}(Profile)(\alpha), \tilde{C}^2_{pmkU_r}(Profile)(\alpha)\right] = \left[\frac{\tilde{G}_{1_l}(\alpha)}{9\left(\int_{x_l}^{x_u} \tilde{\sigma}^2{}_r(\alpha)dx + \tilde{F}_r(\alpha)\right)}, \frac{\tilde{G}_{1_r}(\alpha)}{9\left(\int_{x_l}^{x_u} \tilde{\sigma}^2{}_l(\alpha)dx + \tilde{F}_l(\alpha)\right)}\right] \quad (27)$$

$$\tilde{C}^2_{pmkL}(Profile)(\alpha) = \left[\tilde{C}^2_{pmkL_l}(Profile)(\alpha), \tilde{C}^2_{pmkL_r}(Profile)(\alpha)\right] = \left[\frac{\tilde{G}_{2_l}(\alpha)}{9\left(\int_{x_l}^{x_u} \tilde{\sigma}^2{}_r(\alpha)dx + \tilde{F}_r(\alpha)\right)}, \frac{\tilde{G}_{2_r}(\alpha)}{9\left(\int_{x_l}^{x_u} \tilde{\sigma}^2{}_l(\alpha)dx + \tilde{F}_l(\alpha)\right)}\right] \quad (28)$$

In equations (27) and (28), we define $\tilde{G}_1 = \int_{x_l}^{x_u} \tilde{Z}_1(\alpha)\,dx$ and $\tilde{G}_2 = \int_{x_l}^{x_u} \tilde{Z}_2(\alpha)\,dx$, where $\tilde{Z}_1 = \left(\left(\widetilde{USL}_y(x) \ominus \tilde{\mu}_y(x)\right) \otimes \left(\widetilde{USL}_y(x) \ominus \tilde{\mu}_y(x)\right)\right)$ and $\tilde{Z}_2 = \left(\left(\tilde{\mu}_y(x) \ominus \widetilde{LSL}_y(x)\right) \otimes \left(\tilde{\mu}_y(x) \ominus \widetilde{LSL}_y(x)\right)\right)$. For obtaining the α-cut intervals of $\tilde{G}_1$ and $\tilde{G}_2$ in equations (27) and (28) we do similarly, and the α-cut intervals of $\tilde{\sigma}^2$and $\tilde{F}$ are given in equations (15) and (24), respectively. The α-cut intervals of $\tilde{C}_{pmkU}(Profile)$ and $\tilde{C}_{pmkL}(Profile)$ can be calculated by the square root of equations (27) and (28) and by placing these α–cut intervals one on top of other we gain $\tilde{C}_{pmkU}(Profile)$ and $\tilde{C}_{pmkL}(Profile)$. It should be noted that the sign of these two PCIs is determined based on the sign of $\tilde{C}_{pk}(Profile)$. Finally, to obtain $\tilde{C}_{pmk}(Profile)$ we apply the ranking method mentioned in definition 5.

5-6- Interpretation of fuzzy functional PCIs

A process is called "capable", if the $\tilde{C}_p(Profile)$, $\tilde{C}_{pk}(Profile)$, $\tilde{C}_{pm}(Profile)$ and $\tilde{C}_{pmk}(Profile)$ indices are at least approximately one. To make a decision, we should compare the fuzzy functional index with the fuzzy number 1. By using ranking method mentioned in Definition 5, we determine the minimum of the fuzzy functional index and $\tilde{1}$.

6- Performance evaluation

To evaluate the performance of the new indices, a simulation model is programmed in MATLAB environment. The model is given by Kang and Albin (2000), i.e. $Y_{ij} = 3 + 2X_i + \epsilon_{ij}$, $\epsilon_{ij} \sim N(0,1)$ with four fixed levels of the explanatory variable including 2, 4, 6, and 8 is used to generate necessary data. It is assumed that the process is in statistical control. Moreover, we consider other assumptions for modeling a situation where the observations of response variable at each level of the explanatory variable are fuzzy triangular numbers. The center of a triangular fuzzy number $\tilde{y}_{ij}$ is random and follows a normal distribution with a mean $\mu_i = 3 + 2X_i$ and variance $\sigma^2 = 1$ while its spread also follows a uniform distribution in the interval $(0,1)$. Fuzzy SLs of the response variable at each level of the explanatory variable are represented in table 2. In this example, fuzzy SLs are linear function of the explanatory variable and obtained by the FLR model proposed by Hassanpour, Maleki and Yaghoobi (2009). The results of fuzzy functional SLs and the midpoint of them which is assumed the target of process, are as $\widetilde{LSL}_y(x) = (-2.0833,0.5,0) \oplus ((2.2917,0,0) \otimes x)$, $\widetilde{USL}_y(x) = (5.4167,0,0.5) \oplus ((2.2917,0,0) \otimes x)$ and $\tilde{T}_y(x) = (1.4167,0.25,0.25) \oplus ((2.2917,0,0) \otimes x)$, respectively. $\tilde{C}_p(Profile)$, $\tilde{C}_{pk}(Profile)$, $\tilde{C}_{pm}(Profile)$ and $\tilde{C}_{pmk}(Profile)$ are obtained under different profile samples $m \epsilon \{25, 50,100, 200\}$ to investigate the effect of the sample size. The mean and variances of the profile are estimated from the random samples and then the fuzzy functional PCIs are calculated by the proposed method introduced in Section 5. Also, the true values of the PCIs are calculated based on the true values of the mean and variance of SLP.

Table 2. Fuzzy SLs for each level of the explanatory variable

i	x_i	$\widetilde{LSL}_i$	$\widetilde{USL}_i$
1	2	(2.5,0.5,0)	(10,0,0.5)
2	4	(6.85,0.5,0)	(14.35,0,0.5)
3	6	(11.25,0.5,0)	(18.75,0,0.5)
4	8	(16.25,0.5,0)	(23.75,0,0.5)

The simulations were repeated 10,000 times and the PCIs were calculated in each replicate. The mean and the true values (TVs) of the PCIs under different sample sizes are shown in table 3.

Table 3 shows that as the number of profile samples increases, the mean values of the PCIs tend to the corresponding true values regarding the $D_{2,\frac{1}{2}}$- distance between the estimated fuzzy PCIs and the true value. Figure 1, shows that despite the effect of the sample size on the estimates, for sample sizes greater than 100, this effect has an insignificant difference. Therefore, the number of profile samples equal to 100 is offered for remaining simulation studies.

Table 3. The simulation results of fuzzy functional PCIs for fuzzy SLP under different sample sizes ($p = 2, q = \frac{1}{2}$)

PCIs / m	$\hat{\tilde{C}}_p(Profile)$	$D_{p,q}\left(\hat{\tilde{C}}_p(Profile), TV\right)$	$\hat{\tilde{C}}_{pk}(Profile)$	$D_{p,q}\left(\hat{\tilde{C}}_{pk}(Profile), TV\right)$
25	(1.394, 0.646, 1.366)	0.607	(1.347, 0.709, 1.607)	0.712
50	(1.382, 0.636, 1.299)	0.574	(1.336, 0.698, 1.534)	0.675
100	(1.375, 0.630, 1.261)	0.555	(1.330, 0.693, 1.492)	0.654
200	(1.371, 0.626, 1.244)	0.546	(1.325, 0.689, 1.474)	0.644
TV	(1.250, 0, 0.167)		(1.208, 0, 0.167)	
PCIs / m	$\hat{\tilde{C}}_{pm}(Profile)$	$D_{p,q}\left(\hat{\tilde{C}}_{pm}(Profile), TV\right)$	$\hat{\tilde{C}}_{pmk}(Profile)$	$D_{p,q}\left(\hat{\tilde{C}}_{pmk}(Profile), TV\right)$
25	(1.192, 0.548, 0.841)	**0.361**	(1.163, 0.605, 1.021)	0.437
50	(1.191, 0.546, 0.824)	**0.355**	(1.163, 0.603, 1.004)	0.430
100	(1.190, 0.545, 0.814)	**0.351**	(1.162, 0.603, 0.994)	0.426
200	(1.189, 0.544, 0.810)	**0.349**	(1.160, 0.601, 0.989)	0.424
TV	(1.109, 0.051, 0.156)		(1.082, 0.050, 0.154)	

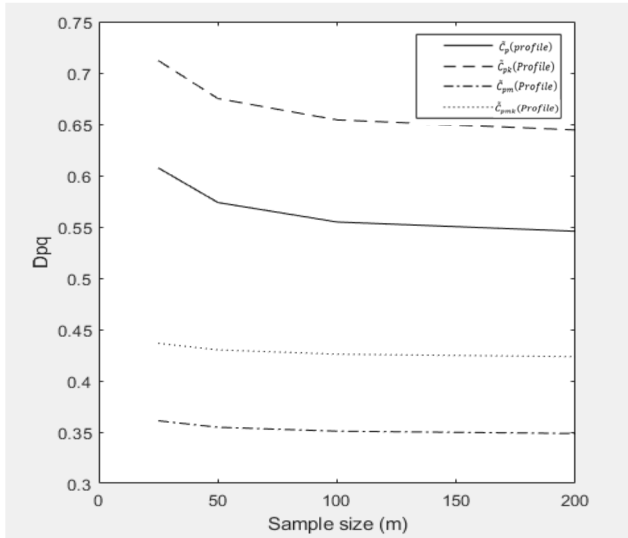

Fig 1. The values of $D_{2,\frac{1}{2}}$-distance between estimated PCIs and true value in $m\epsilon\{25, 50,100, 200\}$ for fuzzy functional PCIs

One of the most important features in performance evaluation of the estimated PCIs is the sensitivity of the index in terms of shift in process dispersion and the deviation of process mean from the target value. Hence, different profile variances, as well as the deviation in the intercept and slope parameters are tested under sample size equal to 100. Each run is replicated 10,000 times to yield the results. Figure 2 illustrates the performance of four fuzzy PCIs in terms of the $D_{2,\frac{1}{2}}$-distance between the estimated fuzzy PCIs and the true value, when variance of the profile changes. From figure 2, it can be seen that if variance of the profile increases, the values of the four fuzzy functional indices will decrease and vice versa. Furthermore, we can observe that for variance less than 2, the $D_{2,\frac{1}{2}}$-distance for $\hat{\tilde{C}}_{pm}(Profile)$ and $\hat{\tilde{C}}_{pmk}(Profile)$ is much lower than $\hat{\tilde{C}}_{p}(Profile)$ and $\hat{\tilde{C}}_{pk}(Profile)$, while the values of $D_{2,\frac{1}{2}}$-distance becomes very close to each other when variance increases.

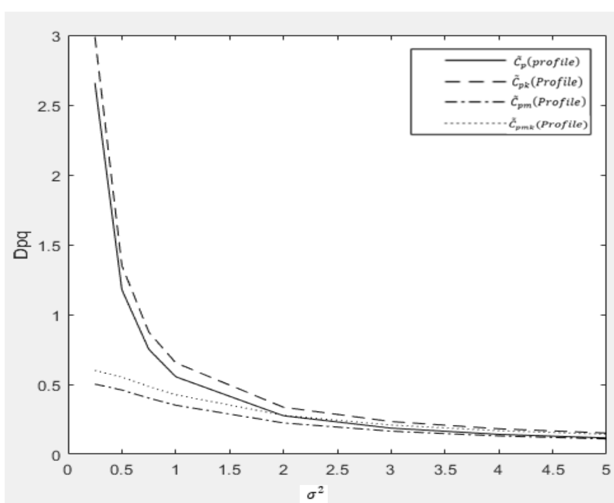

Fig 2. The values of $D_{2,\frac{1}{2}}$-distance between estimated PCIs and true value for fuzzy functional PCIs based on different values of $\sigma^2$with $m = 100$

From table 4, we found that when small shifts in coefficients of reference profile increases, the value of $\hat{\tilde{C}}_p(Profile)$ remains constant while other fuzzy functional indices decrease. This behavior of $\hat{\tilde{C}}_p(Profile)$ index is because of its structure which is insensitive to any shifts in the process mean, so, the values of $\hat{\tilde{C}}_p(Profile)$ are constant when shifts occur in the profile intercept and slope. Since $\hat{\tilde{C}}_{pmk}(Profile)$ is a combination of $\hat{\tilde{C}}_{pk}(Profile)$ and $\hat{\tilde{C}}_{pm}(Profile)$, it has the advantages of both of them, and can also give more information about the location of the process mean. In addition, it is more sensitive than other capability indices to any deviation in the process mean. But the values of $D_{2,\frac{1}{2}}$-distance in almost all simulated cases in table 4 reflect the superiority of the index $\hat{\tilde{C}}_{pm}(Profile)$ over $\hat{\tilde{C}}_{pmk}(Profile)$. We also conclude that the values of the indices decrease more severely when shifts in the slope increase but this does not happen for the intercept.

As a result, regarding $D_{2,\frac{1}{2}}$-distance in almost all simulated cases are shown in tables 3 and 4 and figures 1 and 2, the index $\hat{\tilde{C}}_{pm}(Profile)$ is more efficient than other fuzzy functional indices. Thus, we suggest using the index $\hat{\tilde{C}}_{pm}(Profile)$ to measure the process capability of fuzzy SLP processes.

Table 4. The simulation results for shifts in intercept and slope ($m = 100, p = 2, q = \frac{1}{2}$)

Simulated case (A_0, A_1, σ^2)		$\hat{\hat{C}}_p(\boldsymbol{Profile})$	$D_{p,q}\left(\hat{\hat{C}}_p(\boldsymbol{Profile}), \boldsymbol{TV}\right)$	$\hat{\hat{C}}_{pk}(\boldsymbol{Profile})$	$D_{p,q}\left(\hat{\hat{C}}_{pk}(\boldsymbol{Profile}), \boldsymbol{TV}\right)$
(3.1, 2, 1)	Estimated	(1.373, 0.628, 1.258)	0.553	(1.364, 0.708, 1.514)	0.665
	TV	(1.250, 0, 0.167)		(1.242, 0, 0.167)	
(3.2, 2, 1)	Estimated	(1.3745, 0.629, 1.261)	0.555	(1.347, 0.701, 1.503)	0.660
	TV	(1.250, 0, 0.167)		(1.225, 0, 0.167)	
(3.3, 2, 1)	Estimated	(1.374, 0.629, 1.257)	0.553	(1.309, 0.683, 1.474)	0.644
	TV	(1.250, 0, 0.167)		(1.192, 0, 0.167)	
(3.4, 2, 1)	Estimated	(1.374, 0.629, 1.261)	0.554	(1.273, 0.667, 1.452)	0.632
	TV	(1.250, 0, 0.167)		(1.158, 0, 0.167)	
(3.5, 2, 1)	Estimated	(1.374, 0.629, 1.257)	0.553	(1.237, 0.650, 1.424)	0.618
	TV	(1.250, 0, 0.167)		(1.125, 0, 0.167)	
(3, 2.1, 1)	Estimated	(1.375, 0.630, 1.261)	0.555	(1.238, 0.651, 1.426)	0.619
	TV	(1.250, 0, 0.167)		(1.125, 0, 0.167)	
(3, 2.2, 1)	Estimated	(1.375, 0.629, 1.261)	0.555	(1.054, 0.567, 1.300)	0.551
	TV	(1.250, 0, 0.167)		(0.958, 0, 0.167)	
(3, 2.3, 1)	Estimated	(1.374, 0.629, 1.261)	0.554	(0.870, 0.482, 1.174)	0.483
	TV	(1.250, 0, 0.167)		(0.792, 0, 0.167)	
(3, 2.4, 1)	Estimated	(1.375, 0.630, 1.262)	0.555	(0.688, 0.399, 1.047)	0.417
	TV	(1.250, 0, 0.167)		(0.625, 0, 0.167)	
(3, 2.5, 1)	Estimated	(1.373, 0.628, 1.258)	0.554	(0.504, 0.315, 0.919)	0.349
	TV	(1.250, 0, 0.167)		(1.242, 0, 0.167)	
Simulated case (A_0, A_1, σ^2)		$\hat{\hat{C}}_{pm}(\boldsymbol{Profile})$	$D_{p,q}\left(\hat{\hat{C}}_{pm}(\boldsymbol{Profile}), \boldsymbol{TV}\right)$	$\hat{\hat{C}}_{pmk}(\boldsymbol{Profile})$	$D_{p,q}\left(\hat{\hat{C}}_{pmk}(\boldsymbol{Profile}), \boldsymbol{TV}\right)$
(3.1, 2, 1)	Estimated	(1.196, 0.540, 0.785)	**0.346**	(1.199, 0.613, 0.978)	0.428
	TV	(1.115, 0.032, 0.149)		(1.118, 1.032, 0.148)	
(3.2, 2, 1)	Estimated	(1.195, 0.537, 0.765)	**0.336**	(1.182, 0.606, 0.954)	0.416
	TV	(1.113, 0.042, 0.151)		(1.101, 0.041, 0.149)	
(3.3, 2, 1)	Estimated	(1.181, 0.535, 0.754)	**0.324**	(1.138, 0.589, 0.926)	0.394
	TV	(1.102, 0.059, 0.162)		(1.061, 0.005, 0.160)	
(3.4, 2, 1)	Estimated	(1.159, 0.528, 0.757)	**0.313**	(1.086, 0.566, 0.912)	0.375
	TV	(1.084, 0.073, 0.181)		(1.015, 0.069, 0.177)	
(3.5, 2, 1)	Estimated	(1.128, 0.616, 0.768)	**0.304**	(1.028, 0.537, 0.907)	0.360
	TV	(1.058, 0.084, 0.199)		(0.963, 0.076, 0.193)	
(3, 2.1, 1)	Estimated	(1.202, 0.576, 1.011)	**0.399**	(1.087, 0.593, 1.159)	0.459
	TV	(1.118, 0.097, 0.218)		(1.011, 0.088, 0.211)	
(3, 2.2, 1)	Estimated	(0.981, 0.434, 1.317)	**0.476**	(0.754, 0.396, 1.303)	**0.475**
	TV	(0.934, 0.108, 0.257)		(0.717, 0.083, 0.229)	
(3, 2.3, 1)	Estimated	(0.758, 0.286, 0.836)	0.278	(0.480, 0.234, 0.757)	**0.255**
	TV	(0.735, 0.080, 0.206)		(0.466, 0.051, 0.171)	
(3, 2.4, 1)	Estimated	(0.597, 0.189, 0.470)	0.143	(0.301, 0.140, 0.404)	**0.126**
	TV	(0.586, 0.055, 0.153)		(0.295, 0.028, 0.120)	
(3, 2.5, 1)	Estimated	(0.487, 0.130, 0.294)	0.083	(0.185, 0.086, 0.241)	**0.070**
	TV	(0.480, 0.039, 0.116)		(0.182, 0.015, 0.086)	

7- Illustrative example

In this section, we consider the real case discussed in (Moghadam, Ardali and Amirzadeh, 2018) to calculate the capability of a specific process in tile and ceramic industry by the proposed method. In tiles and ceramics industry; the process of grinding to produce the slurry is a key process. To produce the slurry, a combination of materials mixed according to a predetermined formula and transferred to a ball mill device. The particle size distribution of the slurry is a key quality characteristic that should be kept in-control over time. The results of the slurry roughness tests by operators at given intervals (after 3, 6, and 9 h ball mill device operation) are reported in linguistic expressions. The mass distribution of particles of different sizes is an exponential function of the grinding time (Monov, Sokolov and Stoenchev, 2012). The Phase I fuzzy profile monitoring of a set of historical data in (Moghadam, Ardali and Amirzadeh, 2018) showed that based on 29 in-control samples, the stable model for fuzzy profile function of the process is $\tilde{y} = (11.42,2.16,2.12)\exp((-0.18,\ 0.05, 0.03)t)$.This function can be transformed into a fuzzy SLP, by the transformation described in (Moghadam, Ardali and Amirzadeh, 2015). So, proposed methods can be applied for evaluating the PC of the mentioned profile. The average of estimated parameters in 29 remained fuzzy SLPs leads to $\tilde{a}_0 = (2.4354, 0.2097, 0.1702)$, $\tilde{a}_1 = (-0.18, 0.05, 0.03)$.

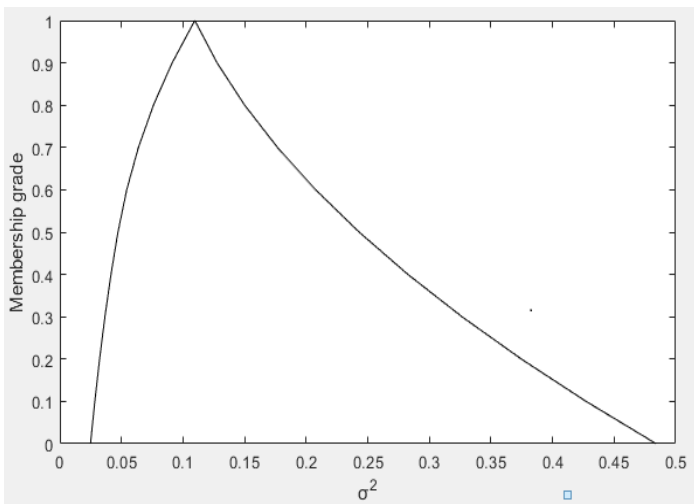

Fig 3. The membership function of $\tilde{\sigma}^2$

Figure 3, depict the membership function of $\tilde{\sigma}^2$ and it can be seen that the variance changes between 0.0249 and 0.4841with different membership values. In this case, we set SLs of the slurry roughness at different grinding time based on the natural tolerance limits of the process. The FLR model in (Hassanpour, Maleki and Yaghoobi, 2009) is used to construct fuzzy functional SLs and the target of the process. The methods proposed in Section 5 can now be used to assess PC for the grinding process. The $\tilde{C}_{pm}(Profile)$ is obtained as (0.2533, 0.9591, 4.5178), i.e. 'approximately 0.9591'. Figure 4 depicts the membership function of this index.

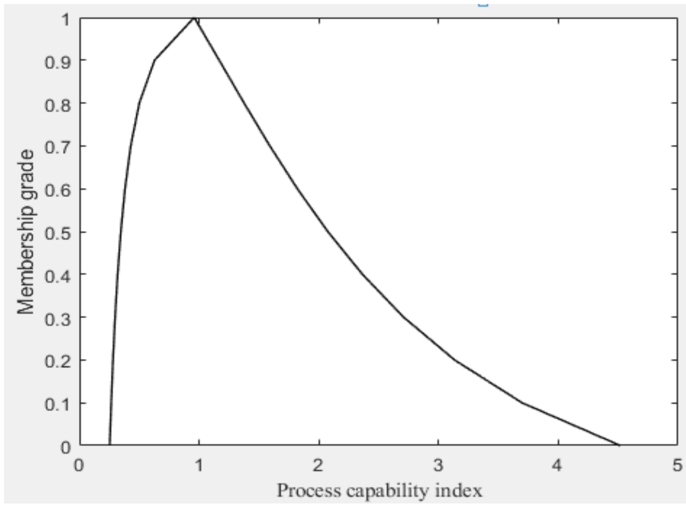

Fig 4. The membership function of the index $\tilde{C}_{pm}(Profile)$

To compare $\hat{\tilde{C}}_{pm}(Profile) = (0.2533, 0.9591, 4.5178)$ and $\tilde{1} = (0.5, 1, 1.5)$, we use the ranking method in Definition 5. By setting $\tilde{C} = (5.5, 6, 6.5)$, i.e. approximately 6, we get $D_{2,\frac{1}{2}}\left(\hat{\tilde{C}}_{pm}(Profile), \tilde{C}\right) = 6.8914 > D_{2,\frac{1}{2}}(\tilde{1}, \tilde{C}) = 5$. So, $\hat{\tilde{C}}_{pm}(Profile)$ is less than $\tilde{1}$, and we conclude the grinding process is incapable. It seems the difference between the process mean and the target that obtained as (0.0274, 0.2359, 2.7097), i.e. 'approximately 0.2359' makes the process incapable. Thus, corrective adjustments for the process mean are needed.

8- Conclusions and future researches

In this study, we introduced fuzzy PCIs based on the functional approach for a linear profile where the response variable is imprecise. These fuzzy functional PCIs can evaluate fuzzy SLPs in circumstances that the SLs and target values of the response variable in each level of the explanatory variable are imprecise. Using the fuzzy functional PCIs, it is possible to estimate PC of fuzzy SLPs in each interval of the profile. To investigate the performance of the indices, simulations were done. We found that PCIs performs well with large sample sizes and it is a proper choice to use 100 profiles for measuring the PC of fuzzy SLPs in real applications. Furthermore, The simulation results indicate that $\tilde{C}_{pm}(Profile)$ outperforms all other fuzzy functional PCIs and have greater performance than the others regarding the smallest values of the $D_{2,\frac{1}{2}}$-distance in almost all simulation runs. Finally, a practical example, based on a real data set, is presented and analyzed to illustrate the applicability of the proposed approach. Consequently, developing fuzzy approaches for evaluating the capability of fuzzy profiles in more complex structures such as polynomial, nonlinear, multivariate and multiple, can be areas for future research.

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Appendix

A. Estimating the coefficients of j^{th} fuzzy simple linear profile

To estimate the coefficients of j^{th} fuzzy SLP model in equation (11), the FLR model developed by Hassanpour, Maleki and Yaghoobi (2009), is used. So, the observed responses and regression coefficients of j^{th} profile in equation (11), are asymmetric triangular fuzzy numbers $\tilde{y}_i = (y_i, \lambda_i, \beta_i), i = 1, 2, \dots, n$ and $\tilde{a}_r = (a_r, \lambda_r, \beta_r), r = 0,1$, respectively. Without loss of generality, assume that $x_{ri} > 0 \; \forall \, r, i$. Therefore, the estimated value of response variable can be calculated using the arithmetic of fuzzy numbers as follows:

$$\hat{\tilde{Y}}_i = \left(\sum_{r=0}^{1} a_r x_{ri}\, , \sum_{r=0}^{1} \lambda_r x_{ri}\, , \sum_{r=0}^{1} \beta_r x_{ri}\right) \quad i = 1, 2, \dots, n, \qquad x_{0i} = 1, x_{1i} = x_i. \tag{29}$$

The aim of this approach is to close the membership function of each estimated response to the membership function of corresponding observed response as much as possible. Hence, the parameters a_r, λ_r, and β_r for $r = 0,1$, are estimated in such a way that the deviation of centers and the spreads of the response variable from its estimated value is minimized. So, we use following goal programming (GP) model (Hassanpour, Maleki and Yaghoobi, 2009) .

$$(GP1): Min\ Z = \sum_{i=1}^{n} (n_{iL} + p_{iL} + n_{iC} + p_{iC} + n_{iR} + p_{iR}) \tag{30}$$

$$s.t.\ \sum_{r=0}^{1} a_r x_{ri} + n_{iC} - p_{iC} = y_i, \ \ i = 1, 2, \dots, n \tag{31}$$

$$\sum_{r=0}^{1} \lambda_r x_{ri} + n_{iL} - p_{iL} = \lambda_i, \qquad i = 1, 2, \dots, n \tag{32}$$

$$\sum_{r=0}^{1} \beta_r x_{ri} + n_{iR} - p_{iR} = \beta_i, \qquad i = 1, 2, \dots, n \tag{33}$$

$$n_{iK} p_{iK} = 0, i = 1, 2, \dots, n, K = L, C, R \tag{34}$$

$$a_r \in \mathbb{R}, \lambda_r, \beta_r \geq 0, r = 0, 1 \tag{35}$$

$$n_{iK}, p_{iK} \geq 0, i = 1, 2, \dots, n, K = L, C, R \tag{36}$$

In GP1, for each i, n_{iC} and p_{iC} are the negative and positive deviations between the centers of observed and estimated response, respectively. Also, n_{iL} and p_{iL} (n_{iR} and p_{iR}) are the negative and positive deviations between the left (right) spreads of them, respectively. Furthermore, the constraints (34) are removable and the GP1 can be solved by linear programming methods. Since the variables in constraints (31), (32) and (33) are separate, the problem GP1 can be decomposed into the following three separate GP problems.

$$(GP2): Min\ Z = \sum_{i=1}^{n} (n_{iC} + p_{iC}) \tag{37}$$

$$s.t.\quad \sum_{r=0}^{1} a_r x_{ri} + n_{iC} - p_{iC} = y_i, \ i = 1, 2, \dots, n \tag{38}$$

$$n_{iC}p_{iC} = 0, \qquad i = 1, 2, \dots, n \tag{39}$$

$$a_r \in \mathbb{R}, \qquad r = 0, 1 \tag{40}$$

$$n_{iC}, p_{iC} \geq 0, \qquad i = 1, 2, \dots, n \tag{41}$$

$$(GP3): Min\ Z = \sum_{i=1}^{n} (n_{iL} + p_{iL}) \tag{42}$$

$$s.t: \quad \textstyle\sum_{r=0}^{1} \lambda_r x_{ri} + n_{iL} - p_{iL} = \lambda_i, \; i = 1, 2, \dots, n \tag{43}$$

$$n_{iL}p_{iL} = 0, \qquad i = 1, 2, \dots, n \tag{44}$$

$$\lambda_r \geq 0, \qquad r = 0, 1 \tag{45}$$

$$n_{iL}, p_{iL} \geq 0, \qquad i = 1, 2, \dots, n \tag{46}$$

$$(GP4): Min\ Z = \sum_{i=1}^{n} (n_{iR} + p_{iR}) \tag{47}$$

$$s.t: \quad \textstyle\sum_{r=0}^{1} \beta_r x_{ri} + n_{iR} - p_{iR} = \beta_i, \; i = 1, 2, \dots, n \tag{48}$$

$$n_{iR}p_{iR} = 0, \qquad i = 1, 2, \dots, n \tag{49}$$

$$\beta_r \geq 0, \qquad r = 0, 1 \tag{50}$$

$$n_{iR}, p_{iR} \geq 0, \qquad i = 1, 2, \dots, n \tag{51}$$

Similar to GP1, by removing the constraints (39), (44), and (49) the models GP2 to GP4 can be solved using linear programming methods. Note that if both positive and negative x_{ri} are considered, the above decomposition is no longer valid. In this case, it can be decomposed into two models, one of which finds the centers a_r and the other one finds the spreads λ_r and β_r.