Journal of Industrial and Systems Engineering
Vol. 10, special issue on Quality Control and Reliability, pp. 1 – 16
Winter (February) 2017

Cold standby redundancy optimization for non-repairable Series-parallel systems: Erlang time to failure distribution

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Abstract

In modeling a cold standby redundancy allocation problem (RAP) with imperfect switching mechanism, deriving a closed form version of a system reliability is too difficult. A convenient lower-bound on system reliability is proposed and this approximation is widely used as a part of objective function for a system reliability maximization problem in the literature. Considering this lower-bound does not necessarily lead to an optimal solution. In this study by assuming that working time of switching mechanism is exponentially distributed, exact value of system reliability is derived analytically through applying Markov process and solving a relevant set of differential-difference equations. The Runge-Kutta numerical scheme is also employed to verify the accuracy of the results. It is assumed that component time-to-failure follows an Erlang distribution which is appropriate for most engineering design problems by giving the possibility of modeling different increasing hazard functions. A new mathematical model is presented and its performance is evaluated through solving a well-known example in the literature. Results demonstrate that a higher level of system reliability is achievable through implementing the proposed model.

Keywords: Cold standby, Redundancy allocation problem, System reliability, Markov process, Differential-difference equations

1- Introduction

Reliability is a fundamental factor playing a crucial role in evaluating the performance of engineering systems. The necessity for enhancing reliability in many real world engineering systems including power plants, production, manufacturing and industrial systems has made the reliability optimization as one of the most important problems which has seized the interest of many researchers. For improving a system reliability, one of the following approaches may be chosen: (i) improving components reliabilities, (ii) utilizing redundant components in parallel, (iii) combining components reliabilities improvement and utilizing redundant components in parallel and (iv) reassigning interchangeable components (Kuo and Prasad, 2000). The second approach which is called redundancy allocation problem (RAP) is widely studied in the literature. RAP is categorized into two classes. In first one, discrete component choices with predetermined characteristics such as reliability, cost, weight, volume, etc. are available. The objective is to determine the types of utilized components and the associated redundancy levels.

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ISSN: 1735-8272,

In second one, component reliability is unknown and considered as a design variable, whereas the other component characteristics are defined as increasing functions of component reliability (Coit, 2003).

The two well-known types of redundancy strategies are called active and standby. In active redundancy, all components operate simultaneously at time zero, whereas in standby redundancy, redundant components are sequentially put into operation when the active one fails. Standby redundancy is classified into three types: cold, warm and hot standby. In cold standby redundancy, redundant component does not fail in standby mode and its failure rate is zero. In warm standby redundancy, the inactive component may fail before being put in operation and its failure rate is less than that of the same component in active mode. In hot standby redundancy, the failure rates of the standby component and the same component in active mode are equal.

In cold standby redundancy, detection of the failed component and activation of the redundant one are both performed by a detection and switching mechanism not necessarily perfect. Two scenarios, Cases 1 and 2, are identified for this switching mechanism. In Case 1, the switching mechanism monitors continuously the system functionality to detect any failure and to switch-on the redundant component, if one is available. It is assumed that the switching mechanism has a reliability function and may fail at any time. As long as the switching mechanism is in working condition, the failure of the active component is detected and is simultaneously replaced by the available standby component. In this case, the system does not fail necessarily due to the switching mechanism failure because no detection and replacement may be required during the remainder of the system mission time (Coit, 2001). In Case 2, switching mechanism fails with a constant probability in response to an active component failure.

A wide variety of exact optimization methods and meta-heuristic techniques such as dynamic programming (Bellman and Dreyfus, 1958, Fyffe et al., 1968, Nakagawa and Miyazaki, 1981, Yalaoui et al., 2005), integer programming (Coit, 2001, Coit, 2003, Misra and Sharma, 1991), branch and bound (Bulfin and Liu, 1985, Djerdjour and Rekab, 2001), geometric programming (Federowicz and Mazumdar, 1968), Lagrangean multipliers (Govil and Agarwala, 1983), genetic algorithm (GA) (Coit and Smith, 1996, Tavakkoli-Moghaddam et al., 2008, Ardakan and Hamadani, 2014), simulated annealing (SA) (Chambari et al., 2013), particle swarm optimization (PSO) (Chambari et al., 2012) and non-dominant sorting genetic algorithm (NSGA II) (Chambari et al., 2012, Safari, 2012) are applied to cope with RAPs. Comprehensive overviews on the models and methods in reliability optimization problems are also presented by Kuo and Prasad (2000) and Soltani (2014).

Coit (2001) formulated the RAP for the case of series-parallel systems with cold standby redundancy. He considered an Erlang distribution for the component time-to-failure under the two scenarios of imperfect switching mechanism. He stated that for Case 1, it is too difficult to obtain the exact value of system reliability. Therefore, a convenient lower-bound on system reliability was used to approximate the reliability. This approximation shed light on the modeling of the RAP in series-parallel systems containing components with Erlang distribution time-to-failure and imperfect switching mechanism. In many studies, this approximation is applied as a term of objective function to tackle the complexity and difficulty of modeling such problems. Some of these studies are addressed here.

Coit (2003) extended his formulation by assuming that either active or cold standby redundancy strategies can be selected for each subsystem. An equivalent problem formulation was introduced and integer programming method was applied to obtain the optimal solutions. Tavakolli-Moghaddam et al. (2008) and Chambari et al. (2013) proposed a GA and SA algorithm for solving this problem, respectively. Soltani et al. (2014) presented a nonlinear redundancy allocation model with the choice of redundancy strategy. In this model, the parameters such as scale parameter of Erlang distribution for components time-to-failure, cost and weight of components, available budget and allowable weight were considered as interval uncertainties. Considering the scale parameter of Erlang distribution as interval uncertainty, Sadjadi and Soltani (2015) formulated the problem through Min-Max regret criterion. They applied a Benders' decomposition method to deal with this problem. Chambari et al. (2012) and Safari (2012) formulated the problem as a multi-objective integer nonlinear programming. The two objective functions were maximizing the system reliability and minimizing the system cost. In these studies, NSGA II and PSO algorithms were proposed to solve the problem. Ardakan and Hamadani (2014) introduced mixed redundancy strategy, which is the combination of active and cold standby redundancy strategies in a particular subsystem. They formulated the problem and developed a GA.

In this study, a new redundancy allocation model for nonrepairable series-parallel systems with cold standby redundancy is presented. In this work, the component types and the redundancy levels are determined in order to maximize the system reliability. It is assumed that the component time-to-failure follows an Erlang distribution, appropriate for components with increasing hazard functions. The problem is formulated for Case 1 and Case 2 of imperfect switching mechanism. It is assumed that for Case 1, the switching mechanism time-to-failure follows an exponential distribution. An explicit expression of system reliability is obtained by means of Markov process and is applied as the coefficients of objective function. The rest of the paper is organized as follows: In Section 2, the system reliability function for two Cases of imperfect switching mechanism is obtained analytically. Section 3 describes the problem and presents the proposed mathematical model. In order to evaluate the performance of the proposed model, a well-known numerical example is solved in Section 4. Finally, conclusion and future research directions are presented in Section 5.

2- Reliability function

In this section, reliability function of a parallel cold standby subsystem consisting of n components with two cases of switching mechanism is derived analytically. It is assumed that time-to-failure of each component is distributed according to an Erlang distribution with shape parameter k and scale parameter λ. Moreover, for Case 1, the working time of switching mechanism is assumed to be exponentially distributed with parameter β. Hence, the reliability function of switching mechanism is represented by $\rho(t) = e^{-\beta t}$.

2-1- Case 1: Continuous monitoring and detection

Let $N(t)$ be the state of the non-repairable subsystem at time t. Then, $\{N(t); t \geq 0\}$ is a continuous-time Markov process whose states are indicated by (i, j, m) where i and j represent the number of failed components and the number of failures occurred for $(i+1)$ th active component, respectively and m represents the switching mechanism status. When the switching mechanism fails, $m = 0$ and while the switching mechanism is in working state, $m = 1$. The state transition diagram of this subsystem is illustrated in figure 1. The letter F represents the subsystem failure state.

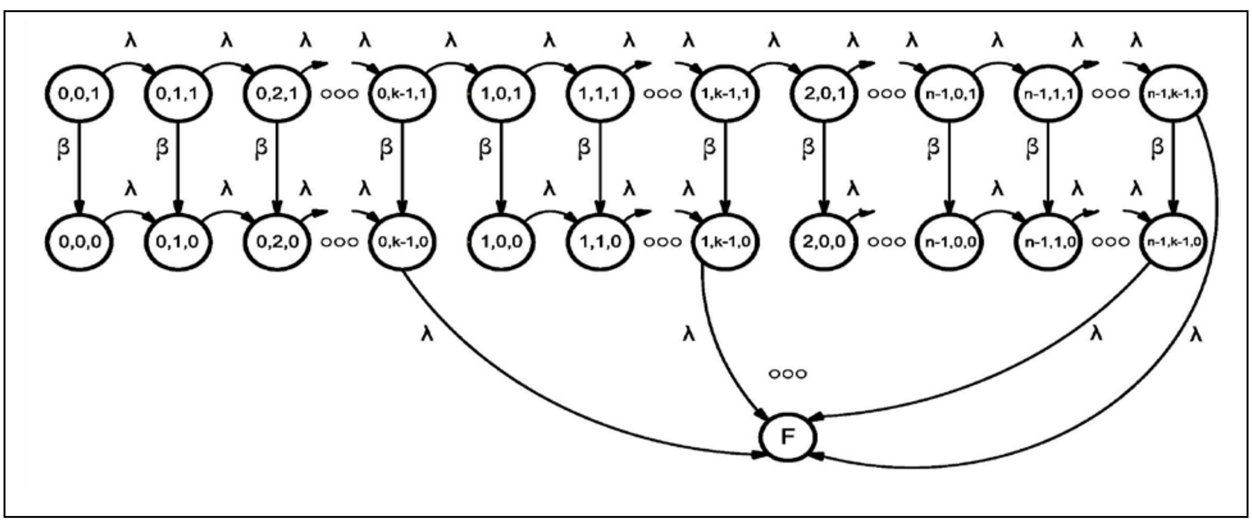

Figure 1. State transition diagram of cold standby subsystem with Case 1 of switching mechanism

Representing the probability of being in state (i, j, m) at time t by $P_{i,j,m}(t)$ for $i = 0,\cdots,n-1$, $j = 0,\cdots,k-1$ and $m = 0,1$, the set of differential-difference equations is attained by:

$$\frac{d}{dt}P_{0,0,1}(t) = -(\lambda+\beta)P_{0,0,1}(t) \tag{1a}$$

$$\frac{d}{dt}P_{i,j,1}(t) = -(\lambda+\beta)P_{i,j,1}(t) + \lambda P_{i,j-1,1}(t) \qquad i = 0,\cdots,n-1 \qquad j = 1,\cdots,k-1 \tag{1b}$$

$$\frac{d}{dt}P_{i,0,1}(t) = -(\lambda+\beta)P_{i,0,1}(t) + \lambda P_{i-1,k-1,1}(t) \qquad i = 1,\cdots,n-1 \tag{1c}$$

$$\frac{d}{dt}P_{i,j,0}(t) = -\lambda P_{i,j,0}(t) + \beta P_{i,j,1}(t) + \lambda P_{i,j-1,0}(t) \qquad i = 0,\cdots,n-1 \qquad j = 1,\cdots,k-1 \tag{1d}$$

$$\frac{d}{dt}P_{i,0,0}(t) = -\lambda P_{i,0,0}(t) + \beta P_{i,0,1}(t) \qquad i = 0,\cdots,n-1 \tag{1e}$$

Assuming that the process is initially in state $(0,0,1)$, the initial conditions are denoted by:

$$P_{0,0,1}(0) = 1$$

$$P_{i,j,m}(0) = 0 \qquad (i, j) \neq (0,0), m = 0,1$$

Thus, applying Laplace transforms to equation (1) leads to the following set of difference equations:

$$s\tilde{P}_{0,0,1}(s) - 1 = -(\lambda+\beta)\tilde{P}_{0,0,1}(s) \tag{2a}$$

$$s\tilde{P}_{i,j,1}(s) = -(\lambda+\beta)\tilde{P}_{i,j,1}(s) + \lambda\tilde{P}_{i,j-1,1}(s) \qquad i = 0,\cdots,n-1 \qquad j = 1,\cdots,k-1 \tag{2b}$$

$$s\tilde{P}_{i,0,1}(s) = -(\lambda+\beta)\tilde{P}_{i,0,1}(s) + \lambda\tilde{P}_{i-1,k-1,1}(s) \qquad i = 1,\cdots,n-1 \tag{2c}$$

$$s\tilde{P}_{i,j,0}(s) = -\lambda\tilde{P}_{i,j,0}(s) + \beta\tilde{P}_{i,j,1}(s) + \lambda\tilde{P}_{i,j-1,0}(s) \qquad i = 0,\cdots,n-1 \qquad j = 1,\cdots,k-1 \tag{2d}$$

$$s\tilde{P}_{i,0,0}(s) = -\lambda\tilde{P}_{i,0,0}(s) + \beta\tilde{P}_{i,0,1}(s) \qquad i = 0,\cdots,n-1 \tag{2e}$$

Solving equations. (2a), (2b) and (2c) recursively gives:

$$\tilde{P}_{i,j,1}(s) = \frac{\lambda^{ki+j}}{(s+\lambda+\beta)^{ki+j+1}} \qquad i = 0,\cdots,n-1 \qquad j = 0,\cdots,k-1 \tag{3}$$

On making the substitution of equation (3) into equation (2d), one can arrive at:

$$\tilde{P}_{i,j,0}(s) = \frac{\beta\lambda^{ki+j}}{(s+\lambda)(s+\lambda+\beta)^{ki+j+1}} + \frac{\lambda}{s+\lambda}\tilde{P}_{i,j-1,0}(s) \qquad i = 0,\cdots,n-1 \qquad j = 1,\cdots,k-1 \tag{4}$$

The prior equation is a nonhomogeneous difference equation whose initial condition is obtained through evaluating $\tilde{P}_{i,0,1}(s)$ from equation (3) and then substituting it into equation (2e) as:

$$\tilde{P}_{i,0,0}(s) = \frac{\beta\lambda^{ki}}{(s+\lambda)(s+\lambda+\beta)^{ki+1}} \qquad i = 0,\cdots,n-1 \tag{5}$$

Defining the z-transform of $\tilde{P}_{i,j,0}(s)$ as $\Pi(z) = \sum_{j=0}^{\infty}\tilde{P}_{i,j,0}(s)z^{j}$ and taking the z-transform of equation (4) results in:

$$\sum_{j=1}^{\infty}\tilde{P}_{i,j,0}(s)z^{j}=\tilde{P}_{i,0,0}(s)\sum_{j=1}^{\infty}\left(\frac{\lambda}{s+\lambda+\beta}z\right)^{j}+\frac{\lambda}{s+\lambda}\sum_{j=1}^{\infty}\tilde{P}_{i,j-1,0}(s)z^{j} \tag{6}$$

Therefore,

$$\Pi(z)-\tilde{P}_{i,0,0}(s)=\tilde{P}_{i,0,0}(s)\frac{\frac{\lambda}{s+\lambda+\beta}z}{1-\frac{\lambda}{s+\lambda+\beta}z}+\frac{\lambda}{s+\lambda}z\Pi(z) \tag{7}$$

By solving equation (7), $\Pi(z)$ is attained by:

$$\Pi(z)=\tilde{P}_{i,0,0}(s)\left(\frac{1}{1-\frac{\lambda}{s+\lambda}z}+\frac{\frac{\lambda}{s+\lambda+\beta}z}{(1-\frac{\lambda}{s+\lambda}z)(1-\frac{\lambda}{s+\lambda+\beta}z)}\right) \tag{8}$$

Equation (8) can be rewritten as:

$$\Pi(z)=\tilde{P}_{i,0,0}(s)\left(\frac{1}{1-\frac{\lambda}{s+\lambda}z}+\frac{\frac{s+\lambda}{\beta}}{1-\frac{\lambda}{s+\lambda}z}-\frac{\frac{s+\lambda}{\beta}}{1-\frac{\lambda}{s+\lambda+\beta}z}\right) \tag{9}$$

Taking the inverse z-transform of equation (9) yields:

$$\tilde{P}_{i,j,0}(s)=\tilde{P}_{i,0,0}(s)\left((\frac{\lambda}{s+\lambda})^{j}+\frac{s+\lambda}{\beta}(\frac{\lambda}{s+\lambda})^{j}-\frac{s+\lambda}{\beta}(\frac{\lambda}{s+\lambda+\beta})^{j}\right) \tag{10}$$

Substituting equation (5) into equation (10) results in the following equation:

$$\tilde{P}_{i,j,0}(s)=\frac{\beta\lambda^{ki+j}}{(s+\lambda)^{j+1}(s+\lambda+\beta)^{ki+1}}+\frac{\lambda^{ki+j}}{(s+\lambda)^{j}(s+\lambda+\beta)^{ki+1}}-\frac{\lambda^{ki+j}}{(s+\lambda+\beta)^{ki+j+1}} \tag{11}$$

for $i=0,\cdots,n-1,\ j=1,\cdots,k-1$.

The Laplace transform of the subsystem reliability is achieved by:

$$\tilde{R}(s)=\sum_{i=0}^{n-1}\sum_{j=0}^{k-1}\left(\tilde{P}_{i,j,1}(s)+\tilde{P}_{i,j,0}(s)\right)=\sum_{i=0}^{n-1}\sum_{j=0}^{k-1}\frac{\lambda^{ki+j}}{(s+\lambda)^{j+1}(s+\lambda+\beta)^{ki}} \tag{12}$$

Hence, the reliability function of subsystem by taking the inverse Laplace transform is derived as:

$$R(t)=\sum_{i=0}^{n-1}\sum_{j=0}^{k-1}\left((\frac{\lambda}{\beta})^{ki+j}\left[\begin{array}{l}e^{-(\lambda+\beta)t}(-1)^{j+1}\sum_{m=0}^{ki-1}\binom{ki-1+j-m}{j}\frac{(\beta t)^{m}}{m!}+\\ e^{-\lambda t}(-1)^{j}\sum_{m=0}^{j}\binom{ki-1+j-m}{j-m}\frac{(-\beta t)^{m}}{m!}\end{array}\right]\right) \tag{13}$$

It should be noted that this equation is obtained by mathematical induction proof.
Following the work of Coit (2001), the lower-bound on subsystem reliability can be determined by

$$\widehat{R}(t) = e^{-\lambda t}\left(\sum_{l=0}^{k-1}\frac{(\lambda t)^l}{l!} + \rho(t)\sum_{l=k}^{kn-1}\frac{(\lambda t)^l}{l!}\right) \tag{14}$$

Assuming that the switching mechanism has an exponential time-to-failure distribution with parameter β, then equation (14) may be rewritten as

$$\widehat{R}(t) = e^{-\lambda t}\left(\sum_{l=0}^{k-1}\frac{(\lambda t)^l}{l!} + e^{-\beta t}\sum_{l=k}^{kn-1}\frac{(\lambda t)^l}{l!}\right) \tag{15}$$

In order to assess the accuracy and validity of the proposed reliability function given by equation (13), the Runge-Kutta numerical scheme is adopted to solve the set of equation (1). To this end, as a case study, the parameters are considered as $n=5$, $k=4$, $\lambda=0.05$ and $\beta=0.01$. As shown in figure 2, the proposed reliability function and the one obtained numerically are well the same. The lower-bound on reliability function presented in equation (15) is also included in this figure.

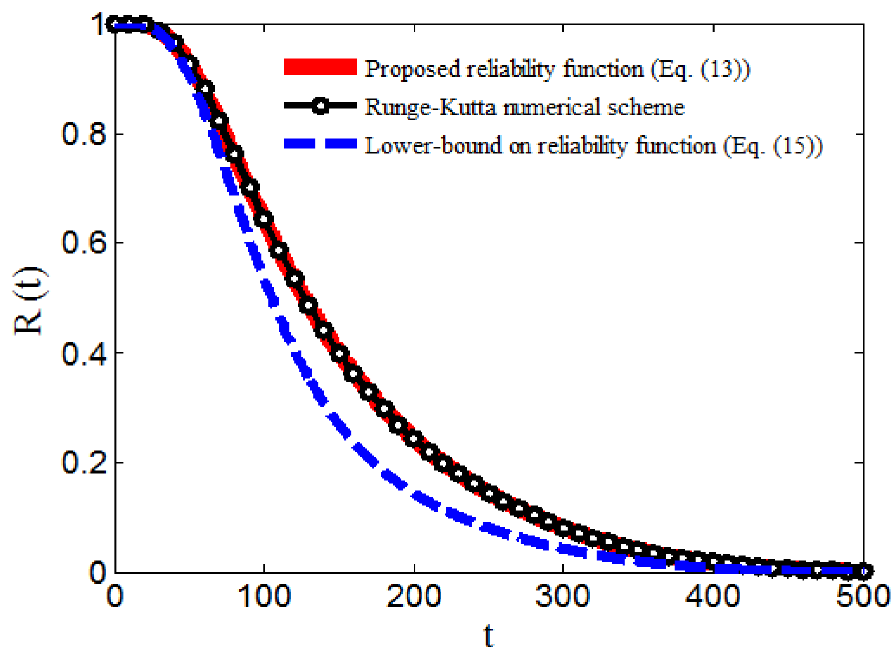

Figure 2. A comparison between the subsystem reliability functions obtained from Eq. (13), numerical scheme and the relevant lower-bound

Furthermore, mean time-to-failure of the cold standby subsystem can be achieved by evaluating $\widetilde{R}(s)$ at $s=0$. Thus, based on equation (12), MTTF is determined by:

$$MTTF = \frac{K}{\lambda}\sum_{x=0}^{n-1}\left(\frac{\lambda}{\lambda+\beta}\right)^{kx} \tag{16}$$

In case of $\beta=0$, the switching mechanism becomes reliable and the MTTF of subsystem is obtained as $\frac{nk}{\lambda}$ which is the expected value of the sum of n iid Erlang random variables with shape parameter k and scale parameter λ. As parameter β goes to infinity, the subsystem turns to be a non-redundant system and hence the MTTF of subsystem approaches $\frac{k}{\lambda}$. Considering a series-parallel system consisting of s subsystems, the reliability function and MTTF are also obtained as the followings:

$$R(t) = \prod_{i=1}^{s} R_i(t) \tag{17}$$

$$MTTF = \int_0^\infty R(t)dt \tag{18}$$

2-2- Case 2: detection and switching only at failure time

In this case, the probability of successful detection and replacement is symbolized by ρ. The states of the relevant continuous-time Markov process are indicated by (i, j) where i and j represent the number of failed components and the number of failures occurred for $(i+1)$ th active component, respectively. The state transition diagram of this subsystem is illustrated in figure 3 in which the subsystem failure state is symbolized by F.

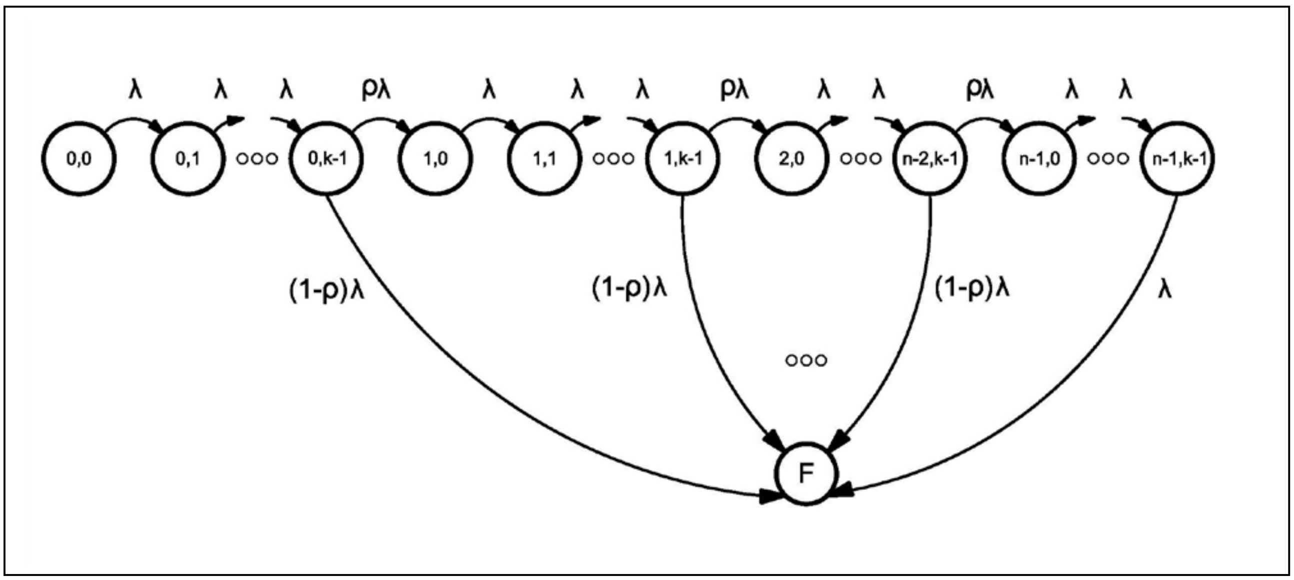

Figure 3. State transition diagram of cold standby subsystem with Case 2 of switching mechanism

Representing the probability of being in state (i, j) at time t by $P_{i,j}(t)$ for $i = 0,\cdots,n-1$ and $j = 0,\cdots,k-1$, the set of differential-difference equations is expressed by:

$$\frac{d}{dt}P_{0,0}(t) = -\lambda P_{0,0}(t) \tag{19a}$$

$$\frac{d}{dt}P_{i,j}(t) = -\lambda P_{i,j}(t) + \lambda P_{i,j-1}(t) \qquad i = 0,\cdots,n-1 \qquad j = 1,\cdots,k-1 \tag{19b}$$

$$\frac{d}{dt}P_{i,0}(t) = -\lambda P_{i,0}(t) + \rho\lambda P_{i-1,k-1}(t) \qquad i = 1,\cdots,n-1 \tag{19c}$$

If it is assumed that the process is initially in state $(0,0)$, the initial conditions are given by:

$$P_{0,0}(0) = 1$$

$$P_{i,j}(0) = 0 \qquad (i, j) \neq (0,0)$$

Therefore, applying Laplace transforms to equation (19) results in the following set of difference equations:

$$s\tilde{P}_{0,0}(s) - 1 = -\lambda\tilde{P}_{0,0}(s) \tag{20a}$$

$$s\tilde{P}_{i,j}(s) = -\lambda\tilde{P}_{i,j}(s) + \lambda\tilde{P}_{i,j-1}(s) \qquad i = 0,\cdots,n-1 \qquad j = 1,\cdots,k-1 \tag{20b}$$

$$s\tilde{P}_{i,0}(s) = -\lambda\tilde{P}_{i,0}(s) + \rho\lambda\tilde{P}_{i-1,k-1}(s) \qquad i = 1,\cdots,n-1 \tag{20c}$$

The preceding equation can be solved recursively to obtain $\tilde{P}_{i,j}(s)$ as:

$$\tilde{P}_{i,j}(s) = \rho^i \frac{\lambda^{ki+j}}{(s+\lambda)^{ki+j+1}} \qquad i = 0,\cdots,n-1 \qquad j = 0,\cdots,k-1 \tag{21}$$

The Laplace transform of the subsystem reliability is stated by:

$$\tilde{R}(s) = \sum_{i=0}^{n-1}\sum_{j=0}^{k-1}\tilde{P}_{i,j}(s) = \sum_{i=0}^{n-1}\sum_{j=0}^{k-1}\rho^i \frac{\lambda^{ki+j}}{(s+\lambda)^{ki+j+1}} \tag{22}$$

Accordingly, one can readily arrive at the reliability function of subsystem by taking the inverse Laplace transform as follows:

$$R(t) = e^{-\lambda t}\sum_{i=0}^{n-1}\sum_{j=0}^{k-1}\rho^i \frac{(\lambda t)^{ki+j}}{(ki+j)!} \tag{23}$$

Equation (23) can be rewritten by:

$$R(t) = e^{-\lambda t}\left(\sum_{l=0}^{k-1}\frac{(\lambda t)^l}{l!} + \sum_{x=1}^{n-1}\rho^x\sum_{l=0}^{k-1}\frac{(\lambda t)^{kx+l}}{(kx+l)!}\right) \tag{24}$$

Replacing $kx+l$ by l in Eq. (24) yields:

$$R(t) = e^{-\lambda t}\left(\sum_{l=0}^{k-1}\frac{(\lambda t)^l}{l!} + \sum_{x=1}^{n-1}\rho^x\sum_{l=kx}^{k(x+1)-1}\frac{(\lambda t)^l}{l!}\right) \tag{25}$$

This equation is exactly same as the reliability function derived by Coit (2001). An approximation for subsystem reliability extracted by Coit (2003) is also given in equation (26).

$$\begin{aligned}\hat{R}(t) &= e^{-\lambda t}\left(\sum_{l=0}^{k-1}\frac{(\lambda t)^l}{l!} + \rho^{n-1}\sum_{x=1}^{n-1}\sum_{l=kx}^{k(x+1)-1}\frac{(\lambda t)^l}{l!}\right) \\ &= e^{-\lambda t}\left(\sum_{l=0}^{k-1}\frac{(\lambda t)^l}{l!} + \rho^{n-1}\sum_{l=k}^{kn-1}\frac{(\lambda t)^l}{l!}\right)\end{aligned} \tag{26}$$

In addition, mean time-to-failure of the cold standby subsystem can be obtained by evaluating $\tilde{R}_s(s)$ at $s = 0$. Thus, using Eq. (22), MTTF may be calculated from:

$$MTTF = \frac{k}{\lambda}\sum_{x=0}^{n-1}\rho^x \tag{27}$$

In case of $\rho = 1$, the switching mechanism becomes reliable and the MTTF of subsystem is expressed as $\frac{nk}{\lambda}$ which is the expected value of the sum of n iid Erlang random variables with shape parameter k and scale parameter λ. When $\rho = 0$, the subsystem turns to be a non-redundant system and thus the MTTF of subsystem approaches $\frac{k}{\lambda}$.

3- Problem statement and formulation

In this section, systems with serial parallel structures are considered. The purpose is to maximize the system reliability at its mission time by allocating redundant components with cold standby strategy in some subsystems. It is assumed that component time-to-failure follows an Erlang distributions with two parameters shape and scale. This distribution gives the possibility of modeling a wide variety of increasing hazard functions which makes it appropriate for versatile engineering design problems. Moreover, when the shape parameter is equal to one, the Erlang distribution transforms to the exponential distribution. The system has imperfect switching mechanism according to Cases 1 and 2 introduced in section 2. Time-to-failure in Case 1 is assumed to be exponentially distributed. In addition, there is no repair or preventive maintenance and the replacement time is negligible. In each subsystem, component mixing is not allowed.

The mathematical model and its parameters and decision variables for a series-parallel system with s subsystems and two linear constraints on cost and weight under the condition of cold standby redundancy are presented as the following integer programming model.

3-1- Parameters and decision variables

Parameters

λ_{ij}	Scale parameter of an Erlang distribution for component j in subsystem i
k_{ij}	Shape parameter of an Erlang distribution for component j in subsystem i
β_i	Parameter of an exponential distribution for switching mechanism of Case 1 in subsystem i
ρ_i	The probability of successful detection and replacement for Case 2 in subsystem i
c_{ij}	Cost corresponding to the component type j available for subsystem i
w_{ij}	Weight corresponding to the component type j available for subsystem i
C	Available budget
W	Allowable weight
$n_{\max,i}$	Maximum number of allowable components in subsystem i
s	Number of subsystems
m_i	Number of component types available for subsystem i
t	Mission time
$R_{i,j,q_1}(t)$	Reliability of subsystem i at time t when q_1 of the component type j are allocated under cold standby strategy

Decision variables

$n_{i,j}$	Number of components of type j utilized in subsystem i under cold standby strategy
X_{i,j,q_1}	A binary variable which is equal to one if q_1 of the components of type j is utilized in subsystem i

3-2- Proposed mathematical model

The proposed mathematical model is as follows:

$$Max \ln R(t) = \sum_{i=1}^{s}\sum_{j=1}^{m_i}\sum_{q_1=1}^{n_{\max,i}} \left(\ln R_{i,j,q_1}(t)\right) X_{i,j,q_1} \tag{28}$$

$$\sum_{j=1}^{m_i}\sum_{q_1=1}^{n_{\max,i}} X_{i,j,q_1} = 1 \qquad i = 1,\cdots,s \tag{29}$$

$$n_{i,j} = \sum_{q_1=1}^{n_{\max,i}} q_1 \times X_{i,j,q_1} \qquad i = 1,\cdots,s,\ j = 1,\cdots,m_i \tag{30}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{m_i} c_{ij} \times n_{i,j} \le C \tag{31}$$

$$\sum_{i=1}^{s}\sum_{j=1}^{m_i} w_{ij} \times n_{i,j} \le W \tag{32}$$

$$X_{i,j,q_1} \in \{0,1\} \qquad i = 1,\cdots,s,\ j = 1,\cdots,m_i,\ q_1 = 1,\cdots,n_{\max,i} \tag{33}$$

The objective function (28) maximizes the series-parallel system reliability with cold standby redundancy strategy in which the natural logarithm function and the binary variable X_{i,j,q_1} are applied to linearize the problem. $R_{i,j,q_1}(t)$ is achieved by equations (34) and (35) for Case 1 and 2 of imperfect switching mechanism, respectively. The derivations of these two equations are provided in Section 2.

$$R_{i,j,q_1}(t) = \sum_{q_2=0}^{q_1-1}\sum_{q_3=0}^{k_{ij}-1}\left(\left(\frac{\lambda_{ij}}{\beta_i}\right)^{k_{ij}q_2+q_3} \left[\begin{array}{l} e^{-(\lambda_{ij}+\beta_i)t}(-1)^{q_3+1}\sum_{l=0}^{k_{ij}q_2-1}\binom{k_{ij}q_2-1+q_3-l}{q_3}\frac{(\beta_i t)^l}{l!} + \\ e^{-\lambda_{ij}t}(-1)^{q_3}\sum_{l=0}^{q_3}\binom{k_{ij}q_2-1+q_3-l}{q_3-l}\frac{(-\beta_i t)^l}{l!} \end{array} \right] \right) \tag{34}$$

$$i = 1,\cdots,s,\ j = 1,\cdots,m_i,\ q_1 = 1,\cdots,n_{\max,i}$$

$$R_{i,j,q_1}(t) = e^{-\lambda_{ij}t}\sum_{q_2=0}^{q_1-1}\sum_{q_3=0}^{k_{ij}-1}\rho_i^{\,q_2}\frac{(\lambda_{ij}t)^{k_{ij}q_2+q_3}}{(k_{ij}q_2+q_3)!} \tag{35}$$

$$i = 1,\cdots,s,\ j = 1,\cdots,m_i,\ q_1 = 1,\cdots,n_{\max,i}$$

Constraint set (29) indicates that for each subsystem only one type of component can be selected and the number of the allocated components ranges from 1 to $n_{\max,i}$. The redundancy level of each subsystem is determined using constraint set (30). Constraints (31) and (32) respectively restrict system's cost and weight and constraint set (33) defines the binary variables of the problem.

4-Numerical example

In order to evaluate the performance of the proposed model, a well-known example taken from Coit (2001) is solved. In this example, a series-parallel system consisting of 14 subsystems is considered in which each subsystem has three or four component choices. Table 1 tabulates component cost, weight and Erlang distribution parameters. The objective is to maximize system reliability at a 100 hours mission time subject to constraints $C = 130$ for system cost and $W = 170$ for system weight. For each subsystem, cold standby redundancy can be used. The switching mechanism is imperfect and Case 1 with exponential time-to-failure distribution and reliability equal to 0.99 at 100 hours for all subsystems is considered, i. e. $\beta_i = -0.01\ln(0.99)\forall i$. Also, the maximum number of allowable components in each subsystem is taken to be 6.

For this example, the proposed mathematical model with 288 binary decision variables is solved on a PC with a 3.00 GHz processor and 8 GB memory using General Algebraic Modeling System (GAMS) version 23.5.1. The optimal solution of the proposed model and the one presented by Coit (2001) are compared in Table 2. As demonstrated in this table, the optimal solution of the model obtained by Coit (2001) corresponds to a system with system reliability of 0.9896, a system cost of 123 and a system weight of 170. However, the system reliability increases to 0.9898 by solving the proposed mathematical model. System cost and system weight are 116 and 170, respectively. As expected, by applying the proposed model which considers the exact reliability value instead of its convenient lower-bound as the coefficients of objective function, higher level of system reliability can be obtained. It should be remarked that the convenient lower-bound on system reliability corresponding to the optimal solution presented by Coit (2001) is 0.9863. According to table 2, one can observe that the system configuration obtained by the two models only differs for subsystems 8, 9 and 12. The corresponding values of subsystems reliability are also depicted in figure 4.

Table 1. Component data for example

i	Choice 1(j=1)				Choice 2 (j=2)				Choice 3 (j=3)				Choice 4 (j=4)			
	λ_{ij}	k_{ij}	c_{ij}	w_{ij}	λ_{ij}	k_{ij}	c_{ij}	w_{ij}	λ_{ij}	k_{ij}	c_{ij}	w_{ij}	λ_{ij}	k_{ij}	c_{ij}	w_{ij}
1	0.00532	2	1	3	0.000726	1	1	4	0.00499	2	2	2	0.00818	3	2	5
2	0.00818	3	2	8	0.000619	1	1	10	0.00431	2	1	9	-	-	-	-
3	0.0133	3	2	7	0.0110	3	3	5	0.0124	3	1	6	0.00466	2	4	4
4	0.00741	2	3	5	0.0124	3	4	6	0.00683	2	5	4	-	-	-	-
5	0.000619	1	2	4	0.00431	2	2	3	0.00818	3	3	5	-	-	-	-
6	0.00436	3	3	5	0.00567	3	3	4	0.00268	2	2	5	0.000408	1	2	4
7	0.0105	3	4	7	0.00466	2	4	8	0.00394	2	5	9	-	-	-	-
8	0.0150	3	3	4	0.00105	1	5	7	0.0105	3	6	6	-	-	-	-
9	0.00268	2	2	8	0.000101	1	3	9	0.000408	1	4	7	0.000943	1	3	8
10	0.0141	3	4	6	0.00683	2	4	5	0.00105	1	5	6	-	-	-	-
11	0.00394	2	3	5	0.00355	2	4	6	0.00314	2	5	6	-	-	-	-
12	0.00236	1	2	4	0.00769	2	3	5	0.0133	3	4	6	0.0110	3	5	7
13	0.00215	2	2	5	0.00436	3	3	5	0.00665	3	2	6	-	-	-	-
14	0.0110	3	4	6	0.000834	1	4	7	0.00355	2	5	6	0.00436	3	6	9

Table 2. Comparison between the optimal solution of the proposed mathematical model and the one given by Coit (2001)

i	Proposed model		Coit (2001)	
	j	$n_{i,j}$	j	$n_{i,j}$
1	3	3	3	3
2	1	2	1	2
3	4	3	4	3
4	3	3	3	3
5	2	3	2	3
6	2	2	2	2
7	1	2	1	2
8	1	3	3	2
9	1	2	2	2
10	2	3	2	3
11	3	2	3	2
12	1	4	4	2
13	2	2	2	2
14	3	2	3	2
System reliability (present study)	0.9898		0.9896	
Lower-bound	-		0.9863	
Cost	116		123	
Weight	170		170	

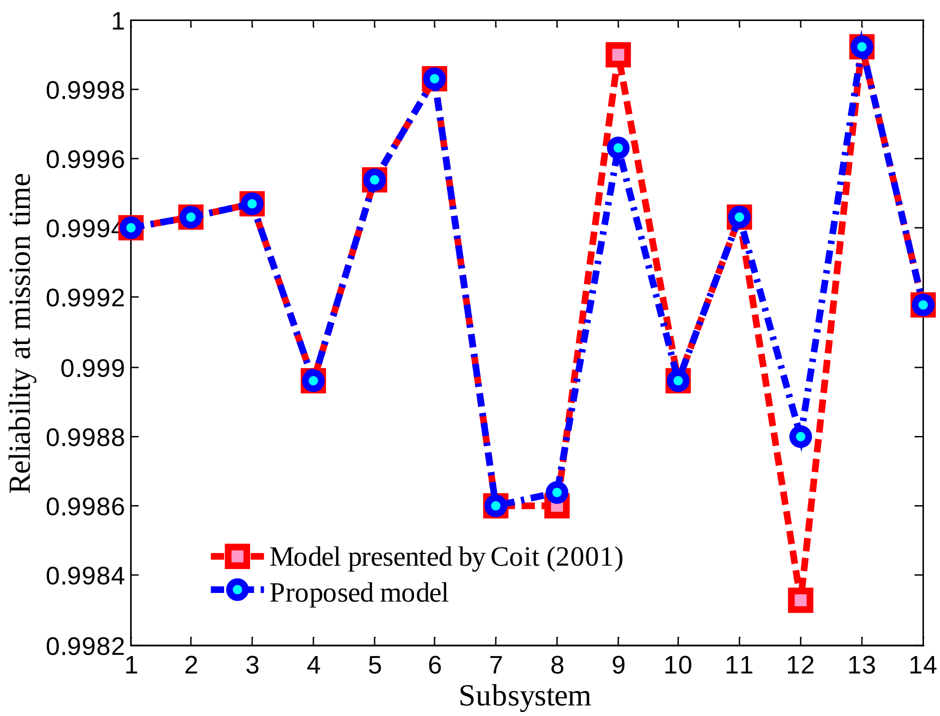

Figure 4. Subsystems reliability at mission time

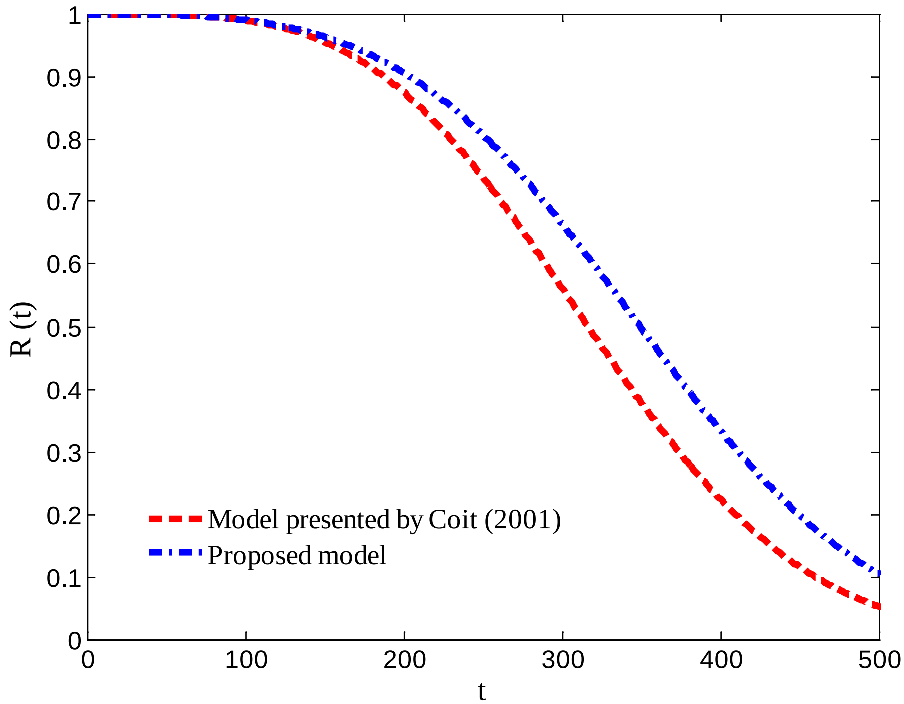

Figure 5. The system reliability function

The system reliability associated with the optimal solution of the two models is displayed in figure 5. To generate the results, using the data of tables 1 and 2, by substituting the associated value of optimal solution into equation (13), the reliability function of each subsystem is first obtained and then the system reliability, which is the product of subsystem reliability values, is calculated as a function of time. In order to investigate the influence of the switching mechanism reliability on the system reliability, figure 6 is displayed for three values of $\rho(100)$. As depicted, the system reliability is sensitive to the changes of switch reliability and increases as this parameter intensifies.

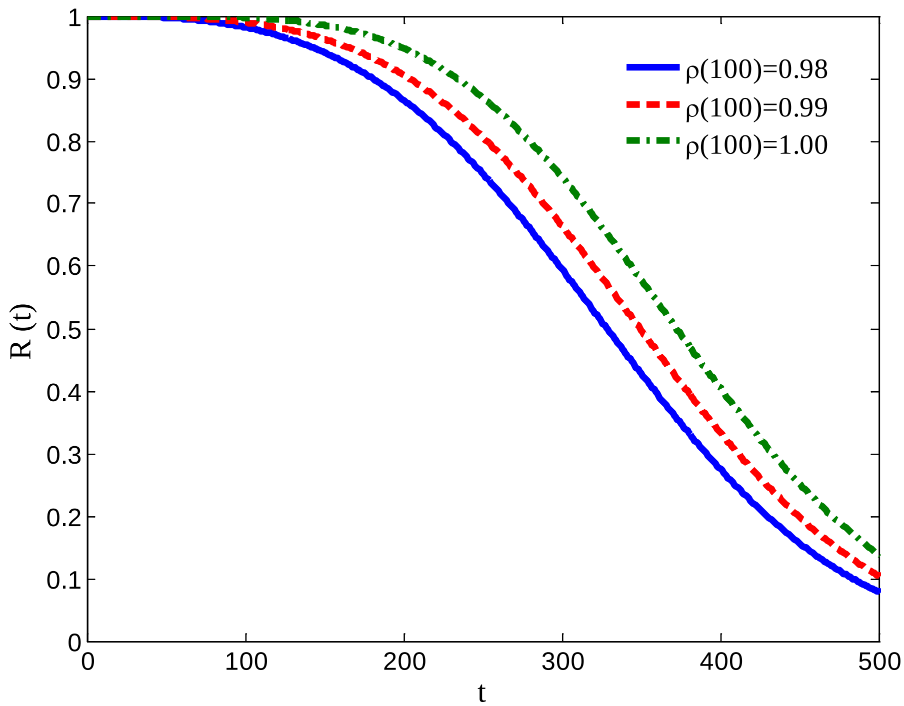

Figure 6. The effect of switch reliability on system reliability

Moreover, the MTTF of each subsystem and that of the entire system are calculated in table 3. As shown, increasing the switching mechanism reliability causes the subsystem MTTF to increase resulting in higher values for system MTTF.

Table 3. The effect of switch reliability on the MTTF of system and its subsystems

Subsystem	$\rho(100) = 0.98$	$\rho(100) = 0.99$	$\rho(100) = 1.00$
1	1112.9856	1156.0073	1202.4048
2	707.6118	720.3037	733.4963
3	1185.6180	1234.5112	1287.5536
4	829.6651	853.4262	878.4773
5	1273.7946	1330.3314	1392.1114
6	1005.4473	1031.0341	1058.2011
7	555.5514	563.3788	571.4286
8	576.7050	588.1781	600.0000
9	1391.5791	1439.5635	1492.5373
10	829.6651	853.4262	878.4773
11	1199.2059	1234.9888	1273.8854
12	1504.7692	1593.8668	1694.9153
13	1288.7223	1330.6762	1376.1468
14	1067.7238	1096.1663	1126.7606
System MTTF	331.1866	352.0931	376.5041

5- Conclusions

Since deriving the exact system reliability function in cold standby redundancy allocation problem is too difficult, this problem is usually formulated by considering a convenient lower-bound on system reliability as the objective function. For some particular cases, deriving the exact value of system reliability is possible. In this study, assuming that component time-to-failure is distributed according to an Erlang distribution and working time of switching mechanism follows an exponential distribution, the system reliability is derived analytically through solving a set of differential-difference equations. Utilizing the exact system reliability as the objective function, a new mathematical model is presented. A well-known example is solved and it is demonstrated that for the particular case, implementing this model results in a meaningful increase in system reliability. For future research, the formulation can be extended to allow selection of either active, cold standby or mixed redundancy strategies for each subsystem. Moreover, assuming that time-to-failure of switching mechanism follows an Erlang distribution is an interesting and applicable generalization. Various forms of uncertainty can be also incorporated into the parameters of the proposed model.

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